Optical Potentials for the Enhanced and Absorptive Scattering of Rossby Waves by Shear Layers

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Geophysical Rossby waves and their scattering phenomena in simple geometries are discussed in regard to the definition of the wave equation in the presence of the so-called critical layers. A feasible way of approximation is proposed and tested for the stationary scattering problems. Inferences are also made on the interrelation of the stability theory of parallel flows and the quantum mechanical scattering.

§1. Introduction

Let there be a zonal, parallel flow of atmosphere, and assume that the small disturbances superimposed on it obey the inviscid-limit, barotropic and divergence-free dynamics\(^1\) (Eq. (1) below), whose essential features have been discussed in a separate paper.\(^2\) The aim of the present note is to discuss the problems of definition of the Rossby wave modes\(^3\)\(^{-5}\) of disturbance at the existence of critical layers.\(^6\)\(^{-8}\)\(^\)\(^{-9}\) More specifically, the note is an attempt to extend the results of 2) on the wave packet cases to extended mode problems. A Rossby wave mode is a stationary scattering amplitude of free Rossby waves at distant spaces by shear layers of the zonal flow. The analysis will reveal that the sole possible definition of the wave equation is the one adopted in 3)\(^{-5}\), which agrees with the definition of Lin\(^7\) on the inviscid limit of a damping mode and which shows a natural coincidence with the quantum mechanical scattering.\(^8\)\(^{-9}\)

The basic equation is,\(^2\)
\[
\frac{i\phi}{\partial t} = H(U)\phi, \quad H(U) = aU(y) + a[U''(y) - \beta](-\Delta)^{-1},
\]
\[
(-\Delta)^{-1}\phi(y) = \left(\frac{1}{2|a|}\right)^{1/2} \int_{-\infty}^{\infty} e^{-i|y-y'|} \phi(y') dy', \quad a \neq 0, \quad \beta > 0, \quad (1)
\]
where \(\phi(y, t)e^{i\alpha x}\) is the vorticity of the two-dimensional, divergence-free disturbance velocity field on the \(xy\)-plane which simulates the surface of the earth, \(x\)-axis is the equator and \(y\)-axis is directed north, \(\beta\) represents the effect of Coriolis’ force,\(^1\) and \(U(y)\) is a zonal (main) flow in \(x\)-direction whose stability is to be sought in the usual usage of \((1, (1), (7), (10))\).

As clarified in 2), Eq. (1) gives a well-defined, (singular) limit for \(\nu \downarrow 0\) of the Cauchy problem
\[
i\partial \phi(y, t)/\partial t = H(U)\phi + iv\Delta \phi, \quad \Delta = -a^2 + \partial^2/\partial y^2, \quad \phi(y, 0) \in \mathcal{H}, \quad (2)
\]
where the function class \(\mathcal{H}\) is characterized as
\[
\mathcal{H} \equiv \{\phi(y); \quad \hat{\phi}(k) = (2\pi)^{-1/2} \int_{-\infty}^{\infty} e^{-iky} \phi(y) dy, \quad 2E(\phi) = \int_{-\infty}^{\infty} \phi^*(y)(-\Delta)^{-1}\phi(y) dy < \infty\}. \quad (3)
\]
Namely, the solution of (2) converges, in the sense of the norm \(2E(\phi)^{1/2}\), to the solution of Eq. (1) at respective \(t\) as \(\nu \downarrow 0\), if \(U(y)\) has uniformly bounded derivatives \(U'(y)\) and \(U''(y)\). See §2 of Ref. 2) for more details on the class \(\mathcal{H}\) that contains distributions such as \(\delta(y-y_0)\) as its members.

The scope of the present note is in the wave-type motion of \(\phi(y, t)\) of (1) for the geophysical problem \(\beta > 0\). Since \(\phi(y) \propto \exp(ik_0y)\) implies \(\tilde{\phi}(k) \propto \delta(k-k_0)\) and \(\phi(y) \in \mathcal{H}\), this is an extension of the dynamics of (1); we have to define the (Rossby) wave modes in terms of the generator \(H(U)\) of (1). Its non-triviality is discussed in §§2 and 3. The note also aims at giving insights into the extension of the dynamics to the cases of more singular (broken-line type) \(U(y)\).

§ 2. Problems

We assume the existence of finite

\[ U_{\pm} = \lim_{y \to \pm \infty} U(y). \tag{4} \]

The asymptotic form \(\phi(y, t) \propto e^{i(k_0y - \omega t)}\) for \(y \sim \pm \infty\) in (1) gives the dispersion relation of free Rossby waves at \(y \sim \pm \infty\),

\[
\omega = \omega_{\pm}(k) = aU_{\pm} - a\beta/(\alpha^2 + k^2), \quad k \in \mathbb{R} = (-\infty, \infty). \tag{5}
\]

A more general form \(\phi(y, t) = \phi(y)e^{-i\omega t}\) satisfies (1) if there holds

\[
[aU(y) - \omega]\phi(y) + aU''(y) - \beta(-\Delta)^{-1}\phi(y) = 0, \tag{6}
\]

which may also be taken as a second order ordinary differential equation for \(\Psi(y)\) that gives \(\phi(y) = (-\Delta)\Psi(y)\).

Just as in the well-known problems of the scattering in one-dimensional Schrödinger equations, we need to solve (6) with the wavy asymptotic forms for \(\phi(y)\),

\[
\phi(y) \propto \begin{cases} e^{\pm ik_0y}, & y \sim +\infty, \quad \omega_{\pm}(k) = \omega, \\ e^{\pm ik'y}, & y \sim -\infty, \quad \omega_{\pm}(k') = \omega. 
\end{cases}
\]

Nontriviality arises with the critical point \(^{(13)}\) \(y = y_c\) that satisfies

\[ aU(y_c) - \omega = 0, \quad y_c \in \mathbb{R}. \tag{7} \]

The critical point is a regular singular point \(^{(11)}\) of (6); in terms of \(\Psi(y)\) the highest, second order term (see (8) below) vanishes at \(y = y_c\). There exists a local regular solution of \(\Psi(y)\) at \(y = y_c\) as well as a singular solution\(^{(12)}\)

\[ \Psi(y) \propto (y-y_c)\log(y-y_c) + O((y-y_c)^2 \log(y-y_c)). \]

How should the branch of \(\log(y-y_c)\) for \(y < y_c\) be continued to that on \(y > y_c\)?

\(^{(1)}\) Hereafter we shall make this distinctive use of the notation for the vorticity \(\phi\) and the corresponding stream function \(\Psi\).

\(^{(12)}\) We assume, discarding the generality, that \(y = y_c\) is a simple zero of \(aU(y) - \omega\), so that \(aU'(y_c) \neq 0\) holds at every critical point.
§ 3. Eigenfunctions

As to Eq. (2) with \( \nu > 0 \) the dynamics is clearly known\(^3\)\(^{\text{--5}}, \)\(^{\text{12}} \) even for the case \( \phi(y) \propto e^{iky} \). By its construction\(^2\) of (1) as the limit \( \nu \downarrow 0 \) of (2), therefore, the natural ideas are to take the asymptotically wavy modes (a \( \beta = 0 \) case of which was discussed by Grosch and Salwen\(^3\)) as the candidates for the eigenfunctions for \( \nu > 0 \), and to consider their inviscid limits. The plot is in fact successful, but only to a level that erases all except one definition inappropriate. We resort almost completely on the result of Wasow.\(^6\)

Throughout this section we assume that \( U(y) \) is regular on real \( y \). As the first step, consider (6) for \( \Psi(y) \),

\[
[aU(y) - \omega](\nu - \mathcal{A})\Psi(y) + a[U''(y) - \beta]\Psi(y) = 0, \quad \omega = \omega_+(k). \tag{8}
\]

Any solution \( \Psi(y) \) of (8) can have singularities only at the roots of \( aU(y) - \omega_+(k) = 0 \).

**Definition 1.** Let \( \{y_c\} \) be the real zeros of \( aU(y) - \omega_+(k) \) for a given \( k \), and call each \( y_c \) a critical point for \( k \). Assume that \( \{y_c\} \) are finite in their number and \( U'(y_c) \neq 0 \) holds at any \( y_c \). Let \( C(k) \) be the path along the real line from \( y = -\infty \) to \( +\infty \) which is deformed, case by case, in a small neighborhood of each \( y_c \) so that it circumvents \( y_c \) counterclockwise or clockwise according to \( aU'(y_c) > 0 \) or \( < 0 \), respectively.

As the second step we substitute the form \( \phi(y, t) = \exp(-i\mathcal{A}t) \times \phi(y) \) into (2). We have

\[
-N(y) + M(y)/\nu = 0, \quad N(y) = -i(\nu - \mathcal{A})^2\Psi(y) + [aU(y) - \lambda_\nu](\nu - \mathcal{A})\Psi(y) + a[U''(y) - \beta]\Psi(y) = 0,
\]

\[
(\nu - \mathcal{A})\Psi(y) = \phi(y). \tag{9}
\]

The value of \( \lambda_\nu \) is fixed so that it gives the asymptotic form \( \Psi_\nu(y) \propto e^{\pm iky} \) at \( y \to +\infty \) with the same real \( k \) as in (8). This gives

\[
\lambda_\nu = \omega_+(k) - i\nu(a^2 + k^2), \tag{10}
\]

so that the mode \( \Psi_\nu(y) \) is always damping.\(^{13}\)

What we obtain with generality on the convergence of \( \Psi_\nu(y) \), is summarized below. This is nothing but a quotation of Theorems 2\(^\text{--}4\) of Wasow.\(^6\)

**Proposition 2.** Let \( \Psi(y) \) be any piece of a solution of (8) at sufficiently distant \( |y| \),\(^*) \) and let \( \Psi_{\text{phys}}(y) \) be the analytic continuation of \( \Psi(y) \) along \( C(k) \) of Definition 1. At each point \( y \in C(k) \) there exists a (small) closed domain \( D(y) \supseteq \bar{y} \) and also a local solution \( \Psi_{\nu}(y) \) of (9) such that \( \Psi_{\nu}(y) \) converges to \( \Psi_{\text{phys}}(y) \) uniformly on \( D(y) \) as \( \nu \downarrow 0 \).

(Comments on the proof) Rewrite (9) in the form of Wasow,\(^6\)

\[
N(y) + M(y)/\nu = 0, \quad N(y) = -i(\nu - \mathcal{A})^2\Psi(y) + i(a^2 + k^2)(\nu - \mathcal{A})\Psi(y),
\]

\[
M(y) = [aU(y) - \omega_+(k)](\nu - \mathcal{A})\Psi(y) + a[U''(y) - \beta]\Psi(y). \tag{11}
\]

Theorems 2\(^\text{--}4\) of Wasow tell us the assertion of Proposition 2, together with one restriction that, a vicinity of a critical point \( y_c \) is divided into three sectors \( \{S_1, S_2, S_3\} \) with the vertical angle 120° as in Figs. 1(a) and (b) and the mentioned local solutions

\(^*\) That is, there should be no critical points on \( (-\infty, -|y|) \) or \( [|y|, \infty) \).
\begin{align*}
\Psi_\nu(y) \text{ are classified into such three types that}^* \text{ they converge in these sectors except } S_k, \quad k=1, 2, 3. \\
The path } C(k) \text{ of Definition 1 exactly avoids the sector } S_3 \text{ in Fig. 1(a) or (b), which is the sole possible way to connect real } y<y_c \text{ and } y>y_c \text{ circumventing one sector.}
\end{align*}

It will be worth noting here a few points of difficulty. One is that the nature of the fundamental solutions of (9). At } y \rightarrow +\infty \text{ (9) was so constructed to have two solutions } e^{\pm k y}; \text{ other two are of the form } e^{i k y} \text{ with}
\begin{equation}
\chi = \pm e^{-i\pi/4}(a\beta/\nu)^{1/2} + O(\nu^0).
\end{equation}

These viscous solutions diverge or vanish very rapidly as } \nu \downarrow 0. \text{ At } y \rightarrow -\infty \text{ we have } \Psi(y) \sim e^{i k y} \text{ with two viscous solutions } \chi \sim \pm \nu^{-1/2} \text{ present invariably, and other two that have two alternative limits as } \nu \downarrow 0:
\begin{align*}
[a] & \text{ Two real } \pm k' \text{ given by } \omega_+(k) = \omega_-(\pm k'), \\
[\text{b}] & \text{ two pure imaginary } \pm k' \text{ given by (12)}.
\end{align*}

What is certain in Proposition 2 is that, at each point } y \in C(k) \text{ there exists one convergent piece of local solution } \Psi_\nu(y) \text{ which may not be connected analytically to } \Psi_\nu(y') \text{ for } y \neq y'. \text{ It seems that, in order to give a definite conclusion whether a global solution } \Psi_\nu(y) \text{ of (9) can be chosen so that it converges at every point of } C(k) \text{ to } W_{\text{phys}}(y) \text{ as } \nu \downarrow 0, \text{ we have to go into the analysis for respective forms of } U(y), \text{ as executed by Morawetz.}^{15}
\text{ However, it is certain that no other choice of the path } C(k) \text{ is allowed in order for any such convergence of } \Psi_\nu(y) \text{ can occur.}

The choice of the path } C(k) \text{ for the analytic continuation of } \Psi_{\text{phys}}(y) \text{ of the inviscid wave mode is in agreement with Lin's definition}\text{) of the inviscid limit of a stable mode for } \nu \downarrow 0. \text{ We shall see another confirmation for the path } C(k) \text{ in } \S 5. \text{ To this end we finally note that, in view of the choice of the path } C(k), \text{ the solution } \Psi_{\text{phys}}(y) \text{ may be written to satisfy the following, obvious form:}
\begin{equation}
\lim_ {\nu \rightarrow 0} [aU(y)-\omega_+(k)-i\varepsilon](-\Delta)\Psi_{\text{phys}} + [aU''(y)-\beta]\Psi_{\text{phys}} = 0.
\end{equation}

For } \phi(y), \psi(y) \in H \text{ the inner product in } H \text{ is transferred to the stream functions } \Phi(y), \Psi(y) \text{ with } (-\Delta)\Phi = \phi, \text{ as}
\begin{equation}
\int_{-\infty}^{\infty} \phi^*(y)(-\Delta)^{-1}\psi(y) dy = \int_{-\infty}^{\infty} \Phi^*(y)(-\Delta)\Psi(y) dy.
\end{equation}

\text{*}) \text{ Detailed discussions on the viscous solutions in the mentioned, exceptional sectors is given in Tatsumi and Gotoh.}^{14}
Therefore, the space of the stream functions is same as Sobolev’s space $H^1$. Constructing a triple $\mathcal{D} \subset H^1 \subset \mathcal{D}^*$, where $\mathcal{D}$ is Schwartz’s space and $\mathcal{D}^*$ is the dual of $\mathcal{D}$ w.r.t. $H^1$, the construction of $\Psi_{\text{phys}}(y)$ at once justifies that (13) may be taken in the sense of $\mathcal{D}^*$.

§ 4. Examples

In order to see the basic features of the scattering of Rossby waves, we now turn to a few examples. Equation (13) is also written as

$$-a^2 \frac{d^2 \Psi}{dy^2} + V(y) \Psi = -a^2 \Psi,$$

$$V(y) = \lim_{\varepsilon \to 0} \alpha \left[ U''(y) - \beta \right] / \left[ a U(y) - \omega - i \varepsilon \right], \quad \omega = \omega_p(k) = \omega_-(k').$$  \hspace{1cm} (14)

Problems of wave scattering with Eq. (14) have been discussed in detail by Lindzen and Tung,3) Yamada4) and Yamada and Gotoh.5) Two points of interest in these works are the occurrence of the so-called over-reflection3) $|R| > 1$ and/or the over-transmission4,5,17) $|T| > 1$ for the coefficients of reflection and transmission in the one-dimensional Schrödinger problem (14). The result of the present §3 gives a sufficient physical and mathematical basis for the construction rule, i.e., the sign of $\varepsilon$ in (14) which, of course, has been adopted by these authors correctly.

The very form of (14) suggests at once a local form of the potential $V(y)$ around any critical point $y_c$,

$$V(y) \approx \chi(y) \left[ \frac{\varphi}{y-y_c} + i \pi \delta(y-y_c) \right], \quad \alpha = \text{sign}[a U'(y_c)],$$

$$\chi(y) = \chi_0 + \chi_1(y-y_c) + \cdots, \quad \chi_0 \equiv [U''(y_c) - \beta] / U'(y_c).$$ \hspace{1cm} (15)

This type of potentials is known in the Kronig-Penney model for electrons in solid lattices and, upon integration of (14) from $y_c - \varepsilon$ to $y_c + \varepsilon$, gives the well-known connection rule of $\Psi(y)$ across $y_c$:

**Prescription.** $\Psi(y)$ must be continuous at $y = y_c$ with

$$\lim_{\varepsilon \to 0} \left[ \Psi'(y_c + \varepsilon) - \Psi'(y_c - \varepsilon) \right] = i \pi \chi_0 \sigma \Psi'(y_c).$$ \hspace{1cm} (16)

This rule may be confirmed with the precise forms of the two independent, local solutions of (14) at $y \sim y_c$:

$$\Psi_1(y) = (y-y_c) + \left( \chi_0 / 2 \right) (y-y_c)^2 + \cdots,$$

$$\Psi_2(y) = \chi_1(y) \log(y-y_c) + 1/\chi_0 + \cdots,$$

with Definition 1 for the path $C(k)$ of connection from $y < y_c$ to $y > y_c$.

Since the one-dimensional Schrödinger scattering is well-known to our intuition, it is suggested that we could approximate (14) with more feasible types of $V(y)$, and construct solutions analytically with Prescription. Specifically, form (15) of the potential and Prescription do not require the regularity of $U(y)$. A square-well type $V(y)$ may well be tested to grasp qualitative features of the scattering for realistic $U(y)$ profiles. The aim
of this section is to push this idea forward by examples, giving comparisons with the numerical results of Okamura.\(^{17}\)

[Example 1] Consider the equation with given constants \(U_0\) and \(\chi_0\sigma\),

\[
-\Psi''(y) + \{-a\beta/ [aU_0 - \omega(k)] + i\pi\chi_0\sigma\delta(y)\} \Psi(y) = -a^2 \Psi(y). \tag{17a}
\]

The dispersion relation is taken as usual, \(\omega(k) = aU_0 - a\beta/ (a^2 + k^2)\) with \(U_0\) for \(U(\pm\infty)\). Sorting out the cases of signs of \(k\) and the group velocity \(d\omega(k)/dk = 2a\beta k/ (a^2 + k^2)\) (cf., § 5 for the use of group velocity) for \(-\infty < k < \infty\), and posing the form

\[
\Psi(y) = \begin{cases} 
  e^{-iky} + R e^{iky}, & y > 0, \\
  T e^{-iky}, & y < 0,
\end{cases} \quad s = \text{sign}(\alpha \beta), \tag{17b}
\]

we obtain from Prescription,

\[
R = \pi\chi_0\sigma/(2sk), \quad T = 1 + R. \tag{18}
\]

The model is crude, but certainly explains some basic features of the potentials of the type of \(V(y) \approx \chi_0\pi\sigma \delta(y)\) for (15).

[Example 2]* The simplest case of geophysical interest\(^{31-33,17}\) is \(U(y) = \tanh(y)\). We assume \(a > 0\) and \(\beta/2 - a^2 > 0\). The potential \(V(y)\) may be written in this case as

\[
V(y) = -\{2aU(y)[1 - U^2(y)] + a\beta\}/ [aU(y) - \omega - i0]. \tag{19}
\]

The form of \(V(y)\) varies complicatedly with \(\omega \in (-a, a)\). The schematic graphs of \(V(y)\) for various \(\omega\) suggest a model of this problem,

\[
\tilde{V}(y) = i\pi\chi_0\sigma\delta(y) + \begin{cases} 
  a\beta/ (a + \omega), & y < 0, \\
  -a\beta/ (a - \omega), & y > 0,
\end{cases} \quad a^2 > 0. \tag{20}
\]

The form \(\Psi(y) \propto e^{\pm iky}\) for \(y > 0\) gives \(\omega = \omega(k) = a - a\beta/ (a^2 + k^2)\) or \(k = [a\beta/ (a - \omega) - a^2]^{1/2}\). As \(k\) varies from \((\beta/2 - a^2)^{1/2} > 0\) to \(\infty\), \(\omega\) varies from \(-a\) to \(a > 0\) with positive group velocity. We consider only in this wave number range. Assuming the form \(\Psi(y) = e^{-iky} + Re^{iky}\) for \(y > 0\) and \(\Psi(y) = Te^{\omega y}\) for \(y < 0\), we have the solution \(T = 1 + R\) with

\[
x = [a^2 + a\beta/ (a + \omega)]^{1/2} > 0, \quad R = (k - ix + \pi\chi_0\sigma)/(k + ix - \pi\chi_0\sigma). \tag{21}
\]

In Fig. 2 \(|R|\) is plotted against \(k\) for a few values of \(\beta > 0\) with \(\alpha = 0.2\), together with the numerical result of Okamura\(^{19}\) (Fig. 2 in the second reference of 5). The agreement is excellent for the

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\* Examples 2~4 below, with all of their conclusions and with Example 1 for the pure Schrödinger case \(-\Psi'' + i\pi\chi_0\delta(y)\Psi = E\Psi (E > 0)\), were submitted on March 26, 1983 as a Progress Letter. They were incorporated into the present paper by suggestion of a referee.
enhanced reflection at small $\gamma$. However, the asymptotic behavior $\lim_{k \to \infty} |R| = 1$ of (21) fails to reproduce the analytical result of Yamada\(^4\) (see also 5), $|R| \sim e^{-\pi k}$ for $k \gg 1$. This will be discussed below.

[Example 3] Take $U(y) = \tanh(y)$ and assume $a > \omega > 0$ again. Define $\tanh(y_c) = c = \omega/\alpha$ and $\xi = \exp(y - y_c)$. We supplement an analytical result for the case $k = \gamma \to \infty$ on the results of Yamada\(^4\) for this specific profile of $U(y)$.

Define $\eta = \exp(-2y_c) = (1-c)/(1+c) < 1$, $y_c > 1$. For $y > y_c/2$ (14) is approximated by $-\Psi''(y) + \left[\beta/(2\eta)\right]z^2(1-\xi^3)\Psi \equiv 0$. The WKB solution\(^4\) $\Psi = \exp\left[\eta^{1/2}e^{i\phi(y)/dy}\right]$ for this equation with the continuation by Prescription, may be expressed as follows:

$$
\Psi(y) \approx A \exp\{i k \log[e^{-y_c} + i(1 - e^{2y_c-2y_c})^{1/2}]\} + B \exp\{-i k \log[e^{-y_c} + i(1 - e^{2y_c-2y_c})^{1/2}]\}, y_c/2 < y < y_c, \quad (22a)
$$

$$
\approx A \exp\{i k \log[e^{-y_c} - (e^{2y_c-2y_c} - 1)^{1/2}]\} + B \exp\{-i k \log[e^{-y_c} - (e^{2y_c-2y_c} - 1)^{1/2}]\}, y_c < y, \quad (22b)
$$

$$
= A e^{-i\theta - i k y} + B e^{i \theta + i k y}, \quad \theta = k(\log 2 - y_c), \quad y_c < y. \quad (22c)
$$

Solution (22a) must be continued to $y \ll -1$ in order to be matched with $\Psi(y) \approx T e^{\nu y}$, $x = [\beta/(1+c) + \alpha^2]^{1/2} \equiv (\beta/2)^{1/2}$. This continuation remains unaccomplished in 4). However, the following simple argument gives a sufficient substitute. Replace $V(y)$ for $-y_c/2 < y < y_c/2$ by $\bar{V}(y) = V_0$. On this interval the solution is then $\bar{\Psi}(y) = C e^{\mu y} + D e^{-\mu y} = (V_0 + \alpha^2)^{1/2}$. Matching of $\bar{\Psi}$ with $T e^{\nu y}$ at $y = -y_c/2$ and $\bar{\Psi}$ with (22a) at $y = y_c/2$ yields

$$
\begin{align*}
(A) &= T \begin{pmatrix} 1/M, & (e^{y_c} - 1)^{1/2} / (2kM) \end{pmatrix} (1, 0)^T, \\
(B) &= \frac{4\mu}{M} \begin{pmatrix} - (e^{y_c} - 1)^{1/2} M / (2k), \mu, -\mu \end{pmatrix} (1, 0, -\Lambda)^T, \\
M &= \exp(-\pi k/2 - i \kappa - y_c]), \quad \Lambda = \exp(-\mu y_c). \quad (23)
\end{align*}
$$

For $k \to \infty$ or $c-1-0$ we have, irrespective of $V_0$ and the place $y = y_c/2$ of the matching, $|B/A| = |R| = M^2 = e^{-\pi k}$.

[Example 4] The idea in Example 3 may be applied to other profiles, e.g., $U(y) = \sech^2(y)$. In order to make descriptions short we take $U(y) = |\tanh(y)|$ with $c = \omega/\alpha = 1 - 0$. There are now two critical points $y = \pm y_c$, and WKB solutions are $\Psi(y)$ of (22a) for $y > y_c/2$ and $\Phi(y) = \Psi(-y)$ for $y < -y_c/2$; Prescription is automatically satisfied by $\Phi(y)$. The potential $V(y)$ is replaced with $\bar{V}(y) = V_0$ for $-y_c/2 < y < y_c/2$. The procedure is lengthy (the method of transfer matrices such as (23) will be adequate), but matching can be done consecutively with $\Phi(y) = T \exp\{-i k \log[e^{-y_c} + i(1 - e^{2y_c-2y_c})^{1/2}]\}$ and $\bar{\Psi}(y)$ of Example 3 at $y = -y_c/2$ and at $y = y_c/2$ between $\bar{\Psi}(y)$ and $\Psi(y)$ of (22a). We obtain for $k \to \infty$,

$$
|R| = e^{-\pi k}, \quad T = (4\mu e^{-\pi k}/k) \left[\beta/(2\alpha^2)^{1/2}\right]^{(2\mu-1)/2}, \quad (24)
$$

where $\mu = (V_0 + \alpha^2)^{1/2}$.

\(^{\ast}\) If $\phi(y) \in \mathfrak{H}$ is transformed to $J\phi(y) = (-A)^{1/2} \phi(y) \in L^2(dy)$, Eq. (1) reads $i \partial (J\phi)/\partial t = (a(-A)^{-1/2} U(y)(-A)^{1/2} + a(-A)^{1/2} U''(y) - \beta(-A)^{-1/2})(J\phi)$. Thus $a(-A)^{-1/2} U(y)(-A)^{1/2}$ is non-Hermitian and unbounded on $L^2(dy)$ if $U(y)$ varies with $y$. 


§ 5. Comments

The generator $H(U)$ of (1) is generally non-Hermitian and not bounded\(^{2,4}\) if $U(y)$ varies with $y$ (i.e., if the shearing motion exists), and only known to generate a semigroup $e^{-iH(U)t}$ for $t > 0$.

There exists a further complication that the free dynamics generally has two distinct generators

$$H_{\pm} = aU_{\pm} - a\beta(-\Delta)^{-1}. \quad (25)$$

Though $H_{\pm}$ are bounded and self-adjoint, these complications put obstacles against a direct application of the general formalisms of quantum scattering theory.\(^{3,9,18}\) However, the very points of difficulty seem to provide us with valuable insights. The aim of this closing section is to give comments on them.

Take the case $U_+ = U_-$ with $H_+ = H_- = H_0$. With some regularizing assumptions\(^{**}\) it is expected that any wave packet $\phi(y) \in \mathcal{M}$ at $t = 0$, evolving with $e^{-iH(U)t}$, will stay only for a finite duration in the shearing $y$-region, and will eventually flee to the regions of free dynamics $e^{-iH_0t}$. Thus a certain $\phi(y) \in \mathcal{M}$ will give

$$\|e^{-iH(U)t}\phi(y) - e^{-iH_0t}\phi(y)\| \to 0 \quad (26)$$
as $t \to \infty$, where $\|\cdots\|$ is the norm in $\mathcal{M}$. This implies the existence of a wave operator.\(^{9}\)

$$\mathcal{Q} = s\text{-lim}_{t \to \infty} A_t, \quad A_t = e^{iH_0t}e^{-iH(U)t}. \quad (27)$$

Consider now a hypothetical case that $H(U)$ is self-adjoint. Then $e^{-iH(U)t}$ is invertible, and (26) is likewise expected for $t \to -\infty$ with other $\phi(y)$; there exists another wave operator,\(^{3,9}\)

$$\mathcal{Q} = s\text{-lim}_{t \to -\infty} e^{iH(U)t}e^{-iH_0t} = s\text{-lim}_{t \to -\infty} A_t e^{iH_0t} \quad (28)$$
with the well-known property $H(U)\mathcal{Q} = \mathcal{Q}H_0$. Thus, if $\phi_s(y) = \mathcal{Q}e^{i\epsilon y}$ makes sense, it should fulfil

$$H(U)\phi_s(y) = \omega(k)\phi_s(y), \quad \omega(k) = \omega_+(k) = aU_+ - a\beta/(a^2 + k^2). \quad (29)$$
The well-known argument (cf. p. 98 of 9)) that uses the invertibility of $e^{-iH(U)t}$ yields the Lippmann-Schwinger equation for $\phi_s(y)$,

$$\phi_s(y) = e^{i\epsilon y}\lim_{\epsilon \to 0} [H_0 - \omega(k) - i\epsilon]^{-1} W\phi_s(y), \quad W \equiv H(U) - H_0. \quad (30)$$

Fourier transform readily shows for a suitable class of $\chi(y)$,

$$\left[H_0 - \omega(k) - i0\right]^{-1}\chi(y) = (a^2 + k^2)(\alpha\beta)^{-1}[\chi(y) - i(a^2 + k^2)(2|k|)^{-1} \times \int_{-\infty}^{\infty} e^{i\epsilon(k|y-y'|)}\chi(y')dy'], \quad s = \text{sign}(\alpha\beta). \quad (31)$$
The sign of $\epsilon$ in (30) is related to the direction of time $t$ in (28), and assures the r.h.s. of (31) to be outgoing, in the sense of group velocity, for $|y| > 1$.

In the problem at our hand the operator $e^{-iH(U)t}$ is not invertible. However, (28) and

\(^{2}\) See the footnote on p. 919.
\(^{**}\) For example, $H(U)$ does not allow bound states, shearing $y$-region with $U'(y) \neq 0$ is bounded in its extension,\(^\cdots\).
$H(U)\mathcal{Q} = \mathcal{Q}H_0$ do not require this invertibility, and there exists another argument\(^8\) for (30) that uses only the resolvents of $H(U)$ and $H_0$. Further, the sign of $\varepsilon$ in (30) agrees with that of (13); (30) with (31) picks out what was called the physical solution of the inviscid limit of §3, if it has a solution $\phi_*(y)$. Appearance of the group velocity\(^8\) is natural, since the primary role expected on $\phi_*(y)$ would be the eigenfunction expansion,\(^9\) typically of $\mathcal{Q}e(y) \approx \int (\mathcal{Q}e^{ikx}) \tilde{\chi}(k) dk$ with the Fourier transform $\tilde{\chi}(k)$ of $\chi(y)$.

The above arguments show the physical significance of the Lippmann-Schwinger equation (30) with (31) for the case $U_+=U_-$. The existence of its solution $\phi_*(y)$ is, of course, still putative in general, but we have examples of §4 as well as a natural expectation that the realistic Rossby wave problems might well be approached by the cases with better-behaved $W$’s, i.e. bounded ones. Gustavsson\(^{12}\) initiated the inquiry on the completeness problem of the continuous modes in the linear stability theory of fluid motions for $\nu > 0$ and $\beta = 0$. The initial value problem was taken under the Laplace transform, and the representability of arbitrary forms of disturbance was asked. The subject was extended by Yamada\(^9\) (see also 5)) to the case $\nu > 0$, $\beta \neq 0$, aiming at the inviscid limit with the Rossby wave modes. The problem of exchangeability of the inverse Laplace transform and the inviscid limit was given an affirmative answer for packet-type disturbances.\(^2\) A necessary task now is to introduce clear specifications of the class of disturbances into the types of works as 4) in view of 2).

It is natural\(^9\) to start the problem of completeness with the inviscid limit dynamics compressed into (30) and (31). Needless to say, the solvability of the problem is still open, and if $U_\neq U_-$ is the case, even the notion of $\mathcal{Q}$ or (30) loses their sense.\(^*\) In this case we shall have to recede to Prescription (which uses (13) only locally at $y \cong y_c$) and the outgoing condition, as done in Example 2 of §4. This, however, will never diminish the value of the investigations on the case $U_- = U_+$; they must be done before any more complicated phenomena be understood.

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References

4) M. Yamada, thesis (Kyoto University 1983), Part III.

\(^*\) $\mathcal{Q}$ may survive in this case, if we introduce projection operators\(^8\) that sort $\mathcal{M}$ into subspaces of packets that go to the right and the left, respectively, under $\exp[-iH(U)t]$ as $t \to \infty$.\(^9\)