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Toast sliding off a table

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Toast always lands butter-side down. But before trying to explain *why*, I should check *whether*: Is this old saying, often cited as the canonical example of Murphy's Law, even true?

King Charles II once called together the Fellows of the Royal Society and asked them, "My Lords and Gentlemen. Why is it that if I place the same amount of water into each of two pails, and then put a four-pound fish into one of them, this pail will not weigh more than the other?" After hearing several learned explanations, he told them that his claim was false and that he was testing whether they followed facts or blindly followed authority.[1]

So, it is prudent to check claims from authority. In my experience, toast usually does land butter-side down. A few home experiments that I tried just now also support the old saying.[2]

Turning now to why the saying is true: The standard reason is that the butter side is heavier, a reason that I had accepted for years. But having become paranoid about non-Newtonian thinking, especially my own, I worry. For the standard reason subtly uses the $F = mv$ misconception about motion: that the slightly heavier, buttered side (more F) wants to fall slightly faster (more v), so it eventually orients itself downward.

A Newtonian analysis—one respecting $F = ma$—would be preferable. Thus, let's look at the forces. While the toast, which I assume to be frictionless,[3] slides off the corner of the table, it experiences a gravitational and a normal force (Fig. 1). The normal force's direction is known: perpendicular to the toast. However, because this force is a constraint force, its magnitude is hard to determine.

To avoid this problem, we could analyze the system using torques and place the origin at the table corner. Then, no matter the magnitude of the normal force, this force would exert no torque and would disappear from the analysis. Alternatively, we can use the Lagrangian. A convenient coordinate system is polar (Fig. 2): θ gives the rotation of the toast relative to horizontal, and r gives the distance of the center of mass from the table corner (how far the toast has slid).

Then, with m as the toast's mass and I as its moment of inertia (about its center of mass), the Lagrangian is

$$L = \underbrace{\frac{1}{2}m\left[\dot{r}^2 + (r^2 + I/m)\dot{\theta}^2\right]}_{T} + \underbrace{mgr\sin\theta}_{-V}. \tag{1}$$

In the kinetic energy (T), the term with the I/m factor is the energy of the toast's rotation about its center of mass. The other kinetic-energy terms constitute the toast's translational energy. The resulting Euler–Lagrange equations are

$$\ddot{r} = r\dot{\theta}^2 + g\sin\theta, \tag{2}$$

$$\ddot{\theta} = \frac{r(g\cos\theta - 2\dot{r}\dot{\theta})}{r^2 + I/m}. \tag{3}$$

Alas, these differential equations are coupled and nonlinear.

Terrible as they are, they apply only until the toast leaves the table, which happens when the normal force is zero. But how do we find the normal force? In a Lagrangian analysis, constraint forces like the normal force can be hard to resurrect. Fortunately, here there's a trick. In the center-of-mass frame—the nonrotating frame with origin at the toast's center of mass—only the normal force produces a torque (the gravitational force

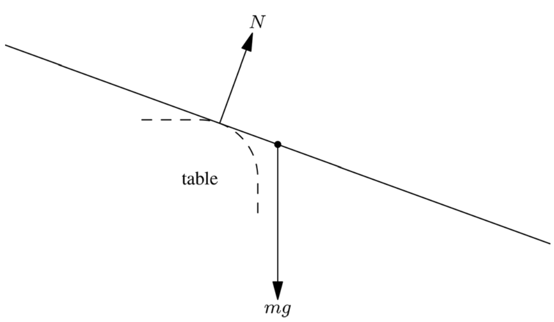

Fig. 1. Freebody diagram of the toast as it slides off the table (ignoring friction). It experiences the gravitational force at its center of mass and a normal force where it touches the table. The normal force is perpendicular to the toast and to the table. The table corner is drawn with exaggerated roundness to show how the normal force can be perpendicular to both toast and table.

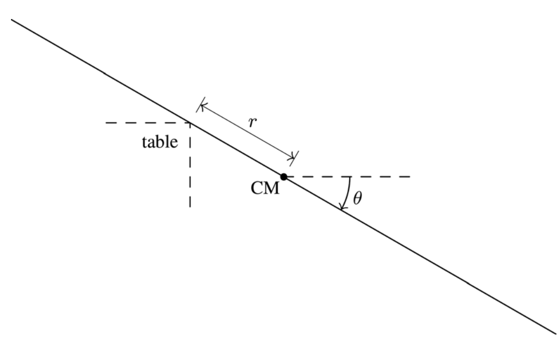

Fig. 2. Coordinate system for the Lagrangian analysis. It locates the toast's center of mass (CM) using polar coordinates whose origin is the table corner: r measures how far the toast has slid past the corner, and θ measures how much the toast has rotated (clockwise) about its center of mass.

acts at the origin). This torque, T_{CM}, produces the toast's angular acceleration ($\ddot{\theta}$) according to

$$T_{CM} = I\ddot{\theta}. \tag{4}$$

(This equation holds even though the frame is noninertial!) So, T_{CM} and, therefore, the normal force are zero when $\ddot{\theta} = 0$. Using the differential equation for $\ddot{\theta}$, Eq. (3), the condition for the toast leaving the table is then

$$g\cos\theta = 2\dot{r}\dot{\theta}. \tag{5}$$

Once the toast leaves the table, the only torque about the center of mass has vanished, so the toast rotates with constant angular velocity ($\ddot{\theta} = 0$) until one of its ends hits the ground.

The while-in-contact differential equations, Eqs. (2) and (3), cannot be solved, at least not by me. So, I simulated them using a simple energy-preserving numerical integrator and treating the toast as a thin, uniform ruler (the toast is 15 cm long, and the table is a standard 75 cm high). During the toast's fall, it flips onto its back, meaning that it lands butter-side down (Fig. 3). Thus, the sad truth of landing butter-side down is due only to the dynamics of a thin, symmetric ruler and is independent of buttering the toast.

However, Murphy's law can be violated in the right circumstances. Based on the preceding dynamics, roughly how high should a table be so that toast lands butter-side up: 1.5 m, 3 m, or 6 m? For the most enjoyment, write down your educated guess and reasoning before reading on.

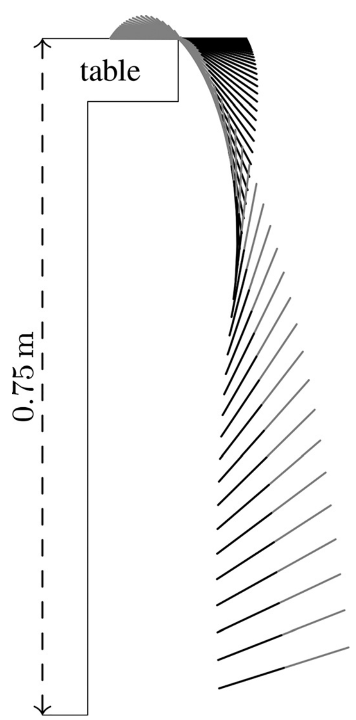

Fig. 3. Simulated fall of the toast showing the roughly 180° rotation. The simulated toast, shown every 10 ms, is 15 cm long (with zero thickness) and falls from standard table height. It starts at rest with its center of mass a hair beyond the table. To make its orientation clear, I have drawn it with the (original) left half in gray and the right half in black. When it has reached the ground, the gray half is on the right, meaning that the toast lands butter-side down.

A back-of-the-envelope answer begins with the idea that the toast needs to rotate twice as much as it does when falling from standard table height (0.75 m). While falling, it rotates at constant angular speed, so rotating twice as much means falling twice as long. In free fall, twice as long means four times as far. So, the table should be roughly 4 $\times 0.75\,\text{m} = 3\,\text{m}$ high. This uncomfortable table height indeed solves the butter-side-down problem (Fig. 4).

Returning to the regular-height simulation (Fig. 3): A drawback of any simulation, as compared to a closed-form analysis, is the difficulty of extracting meaning from the results. The simulation shows us that neither butter nor $F = mv$ is needed to explain the old saying, so we have extracted some meaning. But the results still seem random. For example, when the toast leaves the table,

$$\dot{\theta}^2 \approx 45.3928\,\text{s}^{-2}, \tag{6}$$

$$r^2 \approx 0.0014066\,\text{m}^2. \tag{7}$$

These results become less random when we use dimensionless equations, using a unit system in which $I = m = g = 1$. This choice is equivalent to measuring lengths in units of the toast's radius of gyration

$$r_g \equiv \sqrt{\frac{I}{m}} \tag{8}$$

and measuring times in units of $\sqrt{r_g/g}$ (the rough free-fall time from a height of r_g). The dimensionless variables are

$$\bar{r} \equiv \frac{r}{r_g}, \tag{9}$$

$$\bar{t} \equiv \frac{t}{\sqrt{r_g/g}}. \tag{10}$$

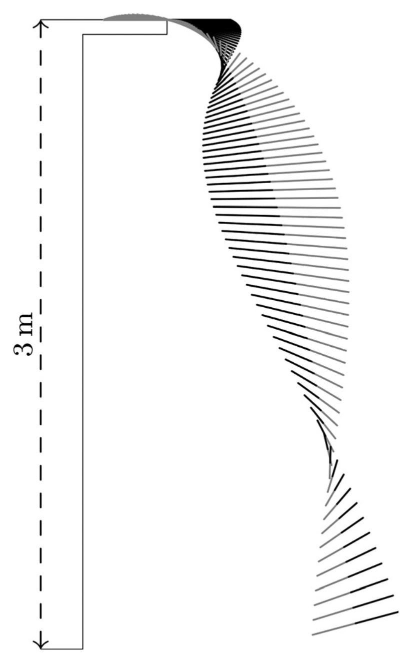

Fig. 4. Simulated fall of the toast from a quadruple-height table. When the toast reaches the ground, it has completed almost a full rotation (rotating 316.6°), as shown by the gray half being on the left side, as it was when resting on the table at the start of its fall. (To save space, this figure has been shrunk vertically by factor of 4 relative to the true, 1:1 aspect ratio of Fig. 3, but the toast is 15 cm long in both simulations.)

The dimensionless differentiation operator, replacing $\dot{}$ or d/dt, is

$$' \equiv \frac{d}{d\bar{t}}. \tag{11}$$

With these definitions, the nondimensionalized equations are

$$\bar{r}'' = \bar{r}\theta'^2 + \sin\theta, \tag{12}$$

$$\theta'' = \frac{\bar{r}(\cos\theta - 2\bar{r}'\theta')}{1+\bar{r}^2}. \tag{13}$$

Now the toast leaves the table with much more polite values

$$\theta'^2 \approx 0.200087, \tag{14}$$

$$\bar{r}^2 \approx 0.75012. \tag{15}$$

The θ'^2 value has several zeros after the 0.2. Perhaps the tiny nonzero piece that follows the zeros is an artifact of simulating the solution approximately, and—as a conjecture—the true value are $\theta'^2 = 1/5$ and $\bar{r}^2 = 3/4$. I haven't been able to solve the differential equations in closed form to know for sure (and salve my ego by blaming their nonlinearity). But perhaps a sharp-eyed and sharp-thinking reader will spot an invariant that decides this conjecture (without needing to solve the equations)?[4]

Sanjoy Mahajan is interested in the art of approximation and physics education and has taught varying subsets of physics, mathematics, electrical engineering, and mechanical engineering at MIT, the African Institute for Mathematical Sciences, and the University of Cambridge. He is the author of Street-Fighting Mathematics *(MIT Press, 2010)*, The Art of Insight in Science and Engineering *(MIT Press, 2014)*, and A Student's Guide to Newton's Laws of Motion *(Cambridge University Press, 2020).*

AUTHOR DECLARATIONS

Conflict of Interest

The author has no conflicts to disclose.

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[1]Ralph E. Oesper, "A royal practical joke," J. Chem. Educ. **25**, 93 (1948).

[2]The saying also gets overall, although not dispositive, support from the extensive home experiments organized by Robert A. J. Matthews, winner of the 1996 Ig Nobel Prize in Physics for his paper "Tumbling toast, Murphy's law and the fundamental constants," Eur. J. Phys. **16**, 172–176 (1995). Matthews pressed into service 1000 schoolchildren doing 21,000 drops, of which 62% landed butter-side down with "[s]ome experiments show[ing] an even higher rate of buttered floor[.]" See Esther Inglis–Arkell, "An experiment that solves the world's most important question: How to keep toast from landing buttered-side down," Gizmodo (December 13, 2011) <https://gizmodo.com/an-experiment-that-solves-the-worlds-most-important-que-5867322>.

[3]The analysis in Darryl Steinert, "It's not Murphy's law, it's Newton's," Phys. Teach. **34**, 288–289 (1996), assumes nonzero static friction and (more questionably) that, once the toast begins sliding, it also leaves the table. But the resulting equations can be solved in closed form. Meanwhile, in my home experiments, oiled pens and rulers also landed butter-side down. So, an analysis assuming zero friction still seems valuable, even though it leads to difficult equations.

[4]For progress toward an invariant: Riccardo Borghi, "On the tumbling toast problem," Eur. J. Phys. **33**, 1407–1420 (2012). Borghi uses angular-momentum and energy conservation to eliminate the time variable, getting (nonlinear) differential equations for the trajectory itself.