Equation of State of Supernova Matter

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We have calculated the equation of state of the supernova matter for various values of lepton fraction $Y_L$ and entropy per nucleon $s$ by using the finite temperature Thomas-Fermi model of nuclei.

The pressure is dominated by the degenerate pressure of leptons and the adiabatic index $\gamma$ stays around $4/3$. However, $\gamma$ decreases by the effect of the nuclear force at the very high density region.

Although the nuclei melt at rather low density, about $10^{13}$ gcm$^{-3}$ in this calculation, the comparison with a Hartree-Fock calculation reveals that the further investigation should be done to clarify the phase transition from nuclear phase of nucleon to homogeneous phase.

§ 1. Introduction

The properties of matter at subnuclear densities and at very high temperature ($\sim 10$ MeV) play an essential role in understanding gravitational collapses of stars at their final stage to result in supernova explosions which leave neutron stars and/or black holes. In such a high density and/or temperature region, the matter consists of nucleons, leptons and some other heavy particles. During the collapse, the neutrinos produced by electron capture on nuclei and/or protons in the collapsing stellar core become trapped in the core in dynamical sense, i.e., it takes longer time than the free fall time of the core for neutrinos to escape from the core. This is the reason why the lepton number $Y_L$ and the entropy per baryon $s$ stay almost constant during the collapse. We call this matter of high density and/or temperature with degenerate neutrino supernova matter.

In literature, there are so many works about the equation of state of the supernova matter. The point is how to treat the nucleons, i.e., protons and neutrons. In lower density region where the neutrino can escape through the core ($\rho \lesssim 10^{12}$ gcm$^{-3}$), a simple approach with mass-formula of nuclei is a good approximation because the mean separation of nuclei is large enough compared to the radius of a nucleus. On the contrary, at high density region, the effect of neighbouring nuclei and the nucleon vapour consisting of nucleons dripped out of nuclei should not be neglected and a more microscopic treatment is required.

Bonche and Vautherin have carried out a Hartree-Fock calculation by a spherical Wigner-Seitz approximation, and obtained the equation of state for a fixed electron fraction, $Y_e$. Hillebrandt et al. have obtained an equation of state by a Hartree-Fock method, and tabulated it for various values of $Y_e$. Though these Hartree-Fock calculations seem to be most reliable, they need a large amount of computational time.

Marcos et al. have done a Thomas-Fermi calculation with a trial function approach to obtain an equation of state for the matter of $Y_e=0.31$. Suraud and Vautherin have carried out a Thomas-Fermi calculation adopting an imaginary time step method and also obtained an equation of state for the matter of $s=1$ and $Y_e=0.25$. However, since the electron capture hardly occurs because of shell blocking effect, the value of $Y_e$ stays around $0.30 \sim 0.35$. Besides, the supernova matter should be labeled with the lepton fraction $Y_L$. The adiabatic index $\gamma$ decreases by the effect of the nuclear force at the very high density region. Although the nuclei melt at rather low density, about $10^{13}$ gcm$^{-3}$ in this calculation, the comparison with a Hartree-Fock calculation reveals that the further investigation should be done to clarify the phase transition from nuclear phase of nucleon to homogeneous phase.
fraction $Y_L$ rather than $Y_e$.

Ogasawara and Sato\cite{13,14} have investigated the properties of nuclei in supernova matter with a special attention paid to the neutrino degeneracy, and have shown that there exist huge nuclei in high density and/or temperature region close to the nuclear matter density. In this paper, we present the equation of state of supernova matter calculated with the finite temperature Thomas-Fermi model of nuclei.\cite{15}

This paper is organized as follows. We briefly sketch the method of computation in §2 and present the results in §3. The comparison with Hartree-Fock result is done in §4. Finally, conclusion is given in §5.

§2. Basic equations and numerical computations

The details of basic equations and methods of computations have already been given in a previous paper.\cite{14} Therefore we make a brief comment here.

We have adopted the usual Wigner-Seitz approximation. We divide the supernova matter into an ensemble of identical cells. Each cell has spherical symmetry and the matter consists of nucleons (protons and neutrons) and leptons (electrons and electron neutrinos). We are mainly interested in the temperature region where nuclei exist, and the other particles such as pions, muons are neglected. We consider one kind of nuclide in the matter when the temperature and chemical potentials of particles are given.

The basic equations are derived from the variational principle to minimize the free energy per unit volume of the cell,

$$
\frac{F}{V} = \frac{1}{V} \int \{ \varepsilon [n_i; \nabla n_i] - TS \} dV \quad (2.1)
$$

under the β-equilibrium condition; where $n_i (i=n, p, e, \nu)$ denotes the particle number density, $\varepsilon$ is the energy density as a functional of $n_i$, $V$ is the volume of the cell, $n=n_n + n_p$ and $S$ is the entropy per unit volume. The chemical potentials, $\mu_i (i=n, p, e, \nu)$ satisfy the β-equilibrium condition,

$$
\mu_n - \mu_p = \mu_e - \mu_\nu \quad (2.2)
$$

Then, the Euler-Lagrange equations to be integrated become

$$
\mu_i = k_B T \phi_i + \frac{\partial \varepsilon}{\partial n_i} - \nabla \cdot \nabla \nabla n_i \quad \phi_i
$$

where $\phi_i$ is the degeneracy parameter of $i$-th particle species and $k_B$ is the Boltzmann constant.

The boundary conditions are as follows: The derivatives with respect to the radial coordinate vanish at the center of the cell for symmetry reason and at the cell boundary because of periodicity condition implied by Wigner-Seitz approximation.

We have directly integrated these equations coupled with Poisson's equation for Coulomb interaction among charged particles (protons and electrons) for several values of cell radius to find a minimum of the free energy per unit volume, Eq. (2.1). This procedure defines the pressure of the supernova matter self-consistently. For a fixed value of temperature $T$, chemical potentials of neutron $\mu_n$ and that of neutrino $\mu_\nu$, we obtain a unique solution. The integration was done by Runge-Kutta method with the
interval of radial coordinate of $\Delta r=0.02$ fm.

As has been mentioned above, we have adopted the energy density formalism\textsuperscript{15} together with the finite temperature Thomas-Fermi model of nuclei.\textsuperscript{16} We have chosen Lombard's parameters\textsuperscript{16} for nuclear force; the binding energy of nuclear matter is $-15.9$ MeV, the density $2.8 \times 10^{14} \text{gcm}^{-3}$ and the bulk compressibility $180\text{MeV}$. As for the parameter representing the nuclear symmetry energy, we have used the value pointed out by Barranco et al.\textsuperscript{17}

§ 3. Results

In order to integrate Eq. (2·3) with Eq. (2·2) and Poisson's equation, we choose the temperature $T$, chemical potentials of neutron $\mu_n$ and of neutrino $\mu_\nu$ as parameters.

In stellar collapse, especially in high density region where neutrinos are trapped within the core, the matter is usually labeled with entropy per nucleon $s$ in units of Boltzmann constant $k_B$, lepton number per nucleon $Y_L$ and density $\rho$. Therefore we

Table I. The equation of state of supernova matter with adiabatic index $\gamma$ defined as $\gamma = d\log p/d\log \rho$. The tables (a), (b) and (c) correspond to the matter of $Y_L=0.3$, $Y_L=0.35$ and $Y_L=0.40$, respectively.

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(b)

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(c)

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Fig. 1. The equation of state for the matter of $Y_L = 0.35$ and $s = 1.5$. The line marked with N corresponds to the nuclear phase and the line marked with H denotes the homogeneous phase. The abscissa denotes the matter density and the ordinate does the pressure of the matter both in cgs unit with logarithmic scale. The transition from the nuclear phase to the homogeneous phase occurs around the crossing point, $\rho = 10^{13.9} \text{ g cm}^{-3}$.

Fig. 2. The adiabat for the matter of Fig. 1. The ordinate denotes the temperature of the matter in units of MeV with logarithmic scale. H and N have the same meaning as in Fig. 1.

We need some numerical transformation from computational space $(T, \mu_n, \mu_v)$ to physical space $(s, Y_L, \rho)$.

In computational space $(T, \mu_n, \mu_v)$ we have calculated $0.5 \leq T/\text{MeV} \leq 9.5$ with interval of 0.5 and $-8 \leq \mu_n/\text{MeV} \leq 8$ with interval of 0.25 and $20 \leq \mu_v/\text{MeV} \leq 120$ with interval of 1. In some cases, we have not obtained a nuclear solution. This is due to the following two reasons. (1) The nuclei melt at high densities and/or temperatures. (2) Too large cell radius ($\geq 150\text{ fm}$) to find a correct minimum of the free energy per unit volume in low density region.

We have obtained 7600 nuclear solutions. These solutions cover the range $0.5 \leq s \leq 3$, $0.2 \leq Y_L \leq 0.45$ and $12.0 \leq \log \rho (\text{g cm}^{-3}) \leq 14.2$. The transformation to $(s, Y_L, \rho)$ space is done by the gaussian weighted linear interpolation. This interpolation implies numerical error of a few percent. Since we have interpolated in logarithmic values of the density and the pressure, this error is not so small. The equation of state is obtained by the least mean square fit to the transformed values of $\log \rho$ and $\log \rho$ with quadratic functions of $\log \rho$. The fitting error is less than $10^{-2}$.

We present the equation of state for various values of $Y_L$ and $s$ with the adiabatic index in Table I. Here, we only show the equation of state for the matter containing nuclei.

As for an example of nuclear melting process, we display the equation of state for the matter of $Y_L = 0.35$ and $s = 1.5$ in Fig. 1. The line marked with N denotes the equation of state for nuclear solutions and the one marked with H corresponds to uniform solutions. The transition from nuclear phase to uniform matter will occur around the crossing point of these lines.

In Fig. 2, we present an adiabat corresponds to the matter of Fig. 1. This line expresses approximately a trajectory of matter during the collapse.
§ 4. Discussion

In the lower density region where the radius of a cell becomes very large compared with the radius of the nucleus contained in the cell, the free energy per unit volume is dominated by the contribution of dilute vapour of nucleons because of geometrical reason: The volume fraction of this dilute vapour in the Wigner-Seitz cell becomes very large compared with that of nuclei. Therefore, we treat very carefully this dilute vapour of nucleons. It needs long time for integration. Thus the direct integration method becomes not so good in the low density region ($\rho \lesssim 10^{12.5} \text{g cm}^{-3}$). This is the reason why we did not show the result at the lower densities. However a simple liquid drop model is valid in these low density region, hence it is not necessary to consider such a region in this calculation.

As for the adiabatic index $\gamma$ defined as

$$\gamma = \left( \frac{d \log p}{d \log \rho} \right)_{s}$$

which plays an essential role in numerical simulations of stellar collapses,\(^{19}\) it stays around 4/3, i.e., the pressure is dominated by degenerate leptons and the fine structure of nucleon cluster — so to say, nucleus — does not affect the equation of state drastically when the entropy per nucleon is larger than the unity. For the case of low entropy per nucleon, $\gamma$ takes a rather low value. And $\gamma$ decreases as the density increases in all cases. The behaviour of $\gamma$ is understood as a reflection of the softness of the nuclear force which we have used here: When the fraction of neutron exceeds 1/2, the pressure of the nuclear matter obtained by the nuclear force becomes even negative as the density exceeds the saturation density. It has been pointed out that the ejection of the matter during the stellar collapse occurs effectively when the adiabatic index is larger than some critical value ($\approx 1.3$).\(^{19,22}\) Therefore, it is hard to blow off the matter by this soft nuclear force, especially in the case of low entropy.

The melting point of nuclei is determined by the phase equilibrium condition between nuclear phase ($N$) and homogeneous phase ($H$),

$$p^{(n)} = p^{(H)}, \quad \mu_{i}^{(n)} = \mu_{i}^{(H)}, \quad (i = n, \nu)$$

where $p$ is the pressure of the matter.\(^{20}\) In our formalism, since the computations are carried out in $(T, \mu_n, \mu_\nu)$ space, these equations are solved by interpolation for the matter labeled with $(s, Y_L, \rho)$. Because of the numerical error accompanied by this interpolation, it is hard to solve Eq. (4·1) accurate enough to show the nuclear melting point. However, the crossing of two lines ($H$ and $N$) suggests that the melting of nuclei is likely to occur at a rather low density, $\rho \approx 10^{13.9} \text{g cm}^{-3}$ in the case of $Y_L = 0.35$ and $s = 1.5$.

For cold matter with degenerate neutrino, we have obtained bubble solutions,\(^{13}\) in which the number density of nucleon is lower at the center of the cell than that at the cell boundary. We believe there also exist bubble solutions in high temperature region. Unfortunately, due to the restricted computational time, we have not succeeded in finding bubble solutions for initial values used in the present calculation to integrate the Euler-Lagrange equations. Another method of integration which consumes much less
computational time should be adopted to study the phase transition including a possible bubble phase in detail.

A Hartree-Fock calculation\textsuperscript{9) has reported a ‘jump’, i.e., sudden increase in temperature for an adiabat around the density of $10^{14}\text{gcm}^{-3}$ induced by the melting of nuclei. In Fig. 2, we do not find such a ‘jump’ around the melting point, and the temperature changes smoothly. This discrepancy may be explained by the following reason. Hillebrandt et al.\textsuperscript{9) used the Skyrme type force,\textsuperscript{20) while we used a simpler force.\textsuperscript{16)} These two forces differ mainly in the values of bulk compressibility and the critical temperature for bulk equilibrium; the Skyrme type force gives a higher value (14 MeV) of the critical temperature, while Lombard’s force gives a lower value (10MeV). Hence the former gives the higher values of nuclear melting density. The higher critical temperature and the higher melting density will result in more entropy stored in excited states of nuclei just before they melt. This excess entropy might be used to heat up the uniform matter after nuclear melting. Thus, the existence of ‘jump’ in the Hartree-Fock calculation could be understood qualitatively.

However, since there is no Thomas-Fermi calculation which has ever reported such a ‘jump’ at the melting point of nuclei, the discrepancy may lead to the limitation of Thomas-Fermi model if we believe the Hartree-Fock result. In order to clarify the validity limit of the Thomas-Fermi model, it is important to study the details of this phase transition by the same potential with different methods;\textsuperscript{11) semi-analytical approach with the liquid drop model, Thomas-Fermi calculations and Hartree-Fock calculations, or by different potentials with the same method.

§ 5. Conclusion

We have presented the equation of state of the supernova matter obtained by the finite temperature Thomas-Fermi calculation in high density region. The structure of nucleon system does not affect the pressure of the matter drastically, and the pressure is mainly determined by degenerate pressure of leptons, but the softness of the nuclear force causes to soften the matter in the very high density region.

We do not find a sudden increase in temperature along an adiabat near the melting point of nuclei though it has been reported by a reliable Hartree-Fock calculation. This discrepancy shows that we need more investigation about the phase transition associated with the nuclear melting in the supernova matter in order to clarify (i) the validity limit of the Thomas-Fermi model in the supernova matter, (ii) sensitivity to nuclear forces adopted in model calculations.

The details of the phase transition through the bubble phase may provide us with the information about the fine structure in the adiabatic index, which may change our picture of the dynamical structure of the stellar collapse.\textsuperscript{22) We are now searching for a more adequate method of integration to obtain the bubble solutions. We hope to refine the equation of state and will apply it to some dynamical simulation of stellar core collapse in future.

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