On the Scaling Law of the $SU(5)$ Lattice Gauge Model with the Mixed Action

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Creutz’s Monte Carlo data of the string tension for $SU(5)$ lattice gauge model is studied using the universality of the mixed action. For a certain negative $r = \beta_A / \beta$, the scaling behavior, which is not seen for Wilson’s action, seems to be present with the correlation length smaller than 3 lattice constant. This observation implies that, when the correlation length is a few lattice constant, the scaling behavior is very sensitive to the action type even in the class of the single-plaquette action.

Recently, Creutz et al. reported that the result of the Monte Carlo calculation for the $SU(5)$ lattice gauge model with Wilson’s action does not show the scaling behavior of string tension. It is expected that the string tension or other physical length should obey the scaling law along the trajectory of the renormalization group transformation in the low temperature region. Generally speaking, R.G.T. does not go across the phase transition point. Since Wilson’s action for $SU(5)$ lattice gauge model has the first order phase transition at certain temperature, we expect that, in the vicinity of this point, the change of the temperature of Wilson’s action does not approximate the trajectory of the R.G.T. for $SU(5)$ lattice gauge theory. On the other hand, if there is no phase transition along a line which connects low and high temperature region, this line has a chance to approximate the trajectory of R.G.T. The $SU(5)$ fundamental-adjoint (or mixed) action has such a property for some parameter region. Fortunately, it is known that, in the large $N$ limit, the average value $R(\beta, \beta_A)$ of any physical quantity for the mixed action can be deduced from the corresponding value for Wilson’s action $R(\tilde{\beta})$. More precisely, $R(\beta, \beta_A)$ is equal to $R(\tilde{\beta})$, if $\tilde{\beta}$ is determined by the formula

$$\tilde{\beta} = \beta + 2\beta_A W(\tilde{\beta}),$$

where $W(\tilde{\beta}) = \langle (1/N) \text{Re} \text{tr} U_p \rangle$ for Wilson’s action. This is the universality of the mixed action. Using this universality we can explicitly obtain the lines of the constant physics, i.e., the lines on which the string tension is constant on the $\beta$-$\beta_A$ plane. The finite $N$ corrections to this formula will induce the numerical changes to these lines. We expect however that formula (1) will give us some insight to the qualitative features of the string tension for $\beta_A \neq 0$. In this paper, using Creutz’s data and this universality, we discuss the scaling behavior of the string tension of the mixed action. We concentrate on the negative $\beta_A$, where the first order phase transition ceases to exist.

The mixed action is given in the form

$$-S = \frac{\beta}{N} \sum \text{Re} \text{tr} U_p + \frac{\beta_A}{N^2} \sum \text{tr} U_p^2,$$

where $N = 5$. For convenience, we fix the ratio $\beta_A / \beta = r$ and study the string tension along
this line. The string tension of Wilson's action $\chi(\bar{\beta})$ with $\bar{\beta}$ is then equal to the one of the mixed action with $(\beta, r\beta)$, where $\beta$ is given by

$$\beta = \frac{\bar{\beta}}{1 + 2rW(\bar{\beta})}. \quad (3)$$

For numerical computations we need the value $\chi(\bar{\beta})$ and $W(\bar{\beta})$ for several $\bar{\beta}$. From Figs. 1 and 2 in Ref. 1), we take the set of values given in the table. At this point, we should admit that these values have some ambiguities because of the drastic change of $\chi(\bar{\beta})$ and $W(\bar{\beta})$ around $\bar{\beta} = 17$. However the qualitative aspect of the following argument will not change. We will discuss this point later. We introduce here the value $C$ defined by

$$C = \chi\left(\frac{275}{24\pi^2\beta^*}\right)^{102/121} \exp\left(\frac{24\pi^2}{275\beta^*}\right). \quad (4)$$

where $\beta^* = (1 + 2r)\beta$. $C$ should be constant, if $\chi$ behaves according to the weak coupling renormalization group equation. We should note here that the data of $\chi(\bar{\beta})$ is obtained by a $3 \times 3$ Wilson loop. This means that the correlation length should be smaller than 3 lattice constants to obey the scaling law, i.e., $\chi$ should be larger than $(3 \times 3)^{-1} = 0.11$. Therefore, we restrict ourselves to the region $16.4 < \bar{\beta} < 17.0$ or $0.15 < \chi < 0.33$ to study the change of $C$. Figure 1 shows the dependence of $C$ for several $r$. The drastic decrease of $C$ for $r = 0$ indicates that in this $\beta$-region the $\chi$ of Wilson's action decreases more rapidly than the scaling formula. In the region $-0.3 < r < -0.2$, the deviation of $C$ from its average decreases to ten percent. The ambiguities of the value $W$ and $\chi$ correspond to the horizontal and vertical movement of the points respectively. The small horizontal movement of these points does not change the statement concerning the deviation of $C$ from constant value. On the other hand, the change of $\chi$ linearly change the amplitude of $C$. In the process of reading the numerical values $\log_{10}\chi$, they seem to get an ambiguity of order $10^{-2}$. Then $C$ has an ambiguity of the same order. Therefore the value smaller

<table>
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<th>$\beta$</th>
<th>$W(\beta)$</th>
<th>$\log_{10}\chi(\beta)$</th>
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<tr>
<td>16.4</td>
<td>0.48</td>
<td>-0.48</td>
</tr>
<tr>
<td>16.5</td>
<td>0.51</td>
<td>-0.57</td>
</tr>
<tr>
<td>16.6</td>
<td>0.52</td>
<td>-0.63</td>
</tr>
<tr>
<td>16.8</td>
<td>0.535</td>
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<td>30</td>
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Fig. 1. $C(\beta)$. Width/ (the smallest value) is 34% for $r = 0.0$, 19% for $r = -0.1$, 10% for $r = -0.2$, $\sim 0.3$, 14% for $r = -0.35$, 28% for $r = -0.4$.
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than a few percent is meaningless in Fig. 1. Taking into account these points, this figure indicates that the scaling behavior seems to be present in the region $-0.3 < r < -0.2$. The value $-0.3$ or $-0.2$ should not be taken seriously because of the ambiguities of $\chi$ mentioned above. Following the usual method to get the string tension by the MC calculation, we assume, for the time being, that $C$ is independent of $\beta$ in this region, and study the $r$ dependence of $C$. To clarify the calculations, we regard the value $C$ from $\bar{\beta} = 17$ as the constant value $C$. Figure 2 shows the $r$-dependence of $\ln C(r)$ obtained from Fig. 1. This $r$-dependence can also be obtained by the following consideration. In the region $-0.3 < r < -0.2$, $\beta$-dependence of $\chi$ is given by

$$\chi = C(r) \left( \frac{275}{24\pi^2}\beta^* \right)^{-102/121} \exp \left( -\frac{24\pi^2}{275} \beta^* \right).$$  \hspace{1cm} (5)

$\beta^*$ can be rewritten in the form

$$\beta^* = (1+2r)\beta$$

$$= \frac{(1+2r)\bar{\beta}}{1+2rW(\beta)}$$

$$= \bar{\beta} + f(\bar{\beta}, r),$$  \hspace{1cm} (6)

where

$$f(\bar{\beta}, r) = \frac{2r\bar{\beta}(1-W(\bar{\beta}))}{1+2rW(\beta)}.$$  \hspace{1cm} (7)

Therefore for Wilson's action, $\chi$ is given by

$$\chi = C(r) \left( \frac{275}{24\pi^2(\bar{\beta} + f(\bar{\beta}, r))} \right)^{-102/121} \exp \left( -\frac{24\pi^2}{275} (\bar{\beta} + f(\bar{\beta}, r)) \right).$$  \hspace{1cm} (8)

Then we require that $\chi$ is independent of $r$ at $\bar{\beta} = 17$. Expanding the $r$-dependence in the logarithm, we get

$$\ln C_0(r) = \left( \frac{24\pi^2}{275} \frac{102}{121} \frac{1}{17} \right) f(17, r) + C_0$$

$$= 0.811 \times \frac{15.3r}{1+1.1r} + C_0,$$  \hspace{1cm} (9)

which is in agreement with Fig. 2 when $C_0 = 10.5$.

The above argument was given in the narrow region in the intermediate temperature. Now, in formula (5), we let $\beta$ become large. $\bar{\beta}$ in (8) also becomes large and the function $f(\bar{\beta}, r)$ tends to $f(\infty, r)$ which is given by $12r/(1+2r)$ where we use $W(\bar{\beta}) = 1 - (6/\beta)$.\(^6\) If we require again that $\chi$ in (8) is independent of $r$ for the region $-0.3 < r < -0.2$, $C(r)$

Fig. 2. $r$-dependence of $C$ for $\bar{\beta} = 17$ (solid line) and $\bar{\beta} = \infty$ (broken line).
must have the form

$$\ln C_\omega(r) = \frac{24\pi^2}{275} \frac{12r}{1+2r} + C_\omega, \quad (10)$$

which is not equal to (9). This function is also depicted in Fig. 2 with $C_\omega=10.5$. Thus at this point, we are forced to admit that $C(r)$ must depend upon $\beta$. This situation, which may sound unusual, is the result of the nonlinearity of formula (1) in the intermediate temperature region. On the contrary, in the very low temperature region, this formula reduces to the linear map, which assures the scaling behavior of the string tension for arbitrary $r$, if it behaves according to the scaling law for a given $r$. Now we introduce the function $C(r, \beta)$, which is almost constant in the narrow region of $\beta$. It is possible that, at some $r$, $C(r, \beta)$ is approximately independent of $\beta$. Figure 2 is consistent with this possibility, since $C_0(r=-0.15)$ is nearly equal to $C_\omega(r=-0.15)$. This value $r=-0.15$ can be shifted when $C_0 \neq C_\omega$. To discuss this possibility, we need to know the properties of the function $C(r, \beta)$ for large but finite $\beta$. Here we only give some speculations. We study the behavior of $f(\bar{\beta}, r)$ since $r$-dependence of $f(\bar{\beta}, r)$ should be cancelled by $C(r, \beta)$ in (8). Figure 3 represents $\bar{\beta}$-dependence of $f(\bar{\beta}, r)/f(\infty, r)$ for several $r$. In the region $16.5 < \bar{\beta} < 17.0$, $f(\bar{\beta}, r)$ changes rapidly, which accounts for the drastic change of the string tension of Wilson's action in this region. By the numerical computations, we notice that $f(16.5, r) - f(17, r)$ is almost independent of $r$. This indicates that $f(\bar{\beta}, r)$ can be expressed in the form $f_1(\bar{\beta}) + f_2(r)$ for $16.5 < \bar{\beta} < 17.0$, and the latter term cancels the $r$-dependence of $C(r)$. For $\bar{\beta} > 18$, $f(\bar{\beta}, r)/f(\infty, r)$ converge slowly to the asymptotic value 1. We should note that the function $f(\bar{\beta}, r)$ at $r=-0.15$ is almost constant and equal to the asymptotic value even for $\bar{\beta} \sim 18$. $r$-dependence of $\ln C(r, \beta)$ should be given by $(24\pi^2/275)f(\bar{\beta}, r)$ for $\bar{\beta} \geq 18$. Then for $\bar{\beta} \geq 18$, we can assume

$$\ln C(r, \beta) = \frac{24\pi^2}{275} f(\bar{\beta}, r) + 10.5 + g(\beta). \quad (11)$$

In this formula, the unknown function $g(\beta)$ determined whether the scaling law is seen in Fig. 1, or not. If we simply assume $g(\beta)=0$, the scaling behavior is achieved near $r=-0.15$ because of the good behavior of $f(\bar{\beta}, r=-0.15)$. This value is independent of the ambiguities of $\chi$ mentioned before. If $g(\beta)$ is chosen suitably, $r$ may be shifted to another value, but $r=-0.15$ is a reasonable one since this line is near the trajectory of the R.G.T. in the following sense. By the Migdal-Kadanoff R.G.T., the action transforms along Manton’s action $tr(F)^2$ in the low temperature region, where $U_p=exp(iF)$.\(^{7,8}\) For $SU(2)$ and $SU(3)$, the mixed action with suitable ratio $\beta_0/\beta$ is equal to Manton’s action to the fourth order of $F$. For $SU(N)$ ($N \geq 4$), this does not happen. However, the contributions to R.G.T. from the terms of order of $F^4$ are proportional to the form,\(^{9}\)

\(^{9}\) The derivation is essentially equal to the one performed by the background field method. See Ref. 9).
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\[
\left(\frac{N^2+1}{8N} - \frac{r}{1+2r} + \frac{1}{24N} - \frac{1}{16N}\right)\text{tr} F^2 ,
\]

which is zero for $r \approx -0.19$, when $N=5$. This means that the mixed action with $r = -0.19$ is effectively equal to Manton's action to the fourth order of $F$. Therefore we can say that the assumption $g(\beta) \approx 0$ is consistent with the prediction by the Migdal-Kadanoff R.G.T.

To summarize, we studied the properties of $\chi$ along the line without the first order phase transition in the $SU(5)$ mixed action. We have explicitly observed that even if $C(r, \beta)$ looks constant in the narrow region of $\beta$, it can change gradually to the asymptotic value when $r$ is not chosen suitably, or worse, there is no such $r$ in the intermediate temperature region. These observations may also be true when the gauge group is $SU(2)$ or $SU(3)$. A few years ago, the $SU(2)$ mixed action was studied near the end point of the first order phase transition line. It was found that the naive low temperature scaling is inconsistent with the Monte Carlo data. Using the universality of the mixed action, Makeenko et al. showed explicitly that the scaling law with the $SU(2)$ Wilson's action leads naturally to the scaling violation with the mixed action near the first order phase transition line. This observation inspired us the fundamental idea of this paper. During our study we noticed Ref. 11, which studied the $SU(3)$ case intensively using the universality of the mixed action. They assumed that, at some negative $r$, the string tension obeys the scaling law and investigate the scaling violation of the string tension of Wilson's action. Their results imply that the considerable improvement of the scaling behavior can be achieved even in the class of single-plaquette actions (see also Ref. 12). In the $SU(5)$ case, this improvement looks really considerable, since the $\chi$, which apparently violates the scaling law for Wilson's action, gets a chance to obey the scaling law with the correlation length smaller than 3 lattice constant. To affirm this point we need information deep in the low temperature region.

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References

8) A. Migdal, Sov. Phys. JETP 42 (1976), 413, 743.