Progress of Theoretical Physics, Vol. 73, No. 3, March 1985

Generation and Evolution of the Density Fluctuation in the Evolving Universe. II

— The Radiation-Dust Universe —

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(Received September 25, 1984)

The generation of density perturbations from stress perturbations is investigated in the radiation-dust universe, in which the universe changes smoothly from radiation-dominated to dust-dominated. It is shown that the amplitude of the generated density perturbations is of the order of the amplitude of the original stress perturbations, when the density perturbations come within the Hubble horizon. It is also shown that the generation predominantly occurs while the universe is radiation-dominated even if the stress perturbations continue to work after the universe has become dust-dominated. These results hold irrespective of the type of the stress perturbation, isotropic or anisotropic.

§ 1. Introduction

It is generally believed that the large scale structure of the present universe, ranging from galaxies to superclusters and the recently found void-cell structure, is formed through gravitational instability from density fluctuations presented in the very early universe with very small amplitudes. In the previous paper (referred to as paper I hereafter) the possibility of these primordial density fluctuations having been generated from stress perturbations possibly associated with weak transition phenomena in the very early universe was investigated. There, when the density perturbation comes within the horizon, the amplitude of a density perturbation generated from a stress perturbation was shown to be of order of the strength of the seed stress perturbation, if the change in the equation of state of the background cosmic matter (referred to as B-EOS in the following) is small, when the stress perturbation is working.

In the present paper we investigate the generation and evolution of density perturbations in the radiation-dust universe, namely in the universe in which the cosmic matter is composed of the non-interacting mixture of radiation and pressure-free particles (dust), in order to see whether the result obtained in paper I also holds in the case B-EOS changes largely when stress perturbations are working. The analysis is based on the gauge-invariant formalism as in paper I. We follow the formulation given in paper I and do not repeat it here. We use the same notation as in the review article, which is the same as in paper I except for some unimportant changes. As for the detailed account of the gauge-invariant formalism for cosmological perturbations see the original paper by Bardeen and the extensive review article. We use the units $c=8\pi G=1$.

§ 2. The fundamental equation and its general solution

We only consider a spatially flat Robertson-Walker spacetime as the unperturbed background,
and a density perturbation with scale $l = a/k$ which is larger than the Hubble horizon size $1/H = a/\dot{a}$. Then the Einstein equations are reduced to the equation for the cosmic scale factor

$$\frac{\dot{a}}{a} = \rho/3,$$  \hspace{1cm} (2.2)

and the energy equation

$$\rho = -3(\dot{a}/a)(\rho + p),$$  \hspace{1cm} (2.3)

where $\rho$ is the proper energy density, $p$ is the pressure and a dot denotes the differentiation with respect to the cosmic time $t$. The fundamental equation describing the time evolution of the gauge-invariant amplitude of a density perturbation, $A$, is given by

$$\frac{d^2 A}{d\xi^2} - \frac{\mu}{\xi} \frac{dA}{d\xi} - \frac{2 + \nu}{\xi^2} A = \mathcal{S},$$  \hspace{1cm} (2.4)

where

$$\mathcal{S} = -\frac{w}{f^2}\left(\Gamma - \frac{2}{3} \Pi\right) + \frac{2}{\xi^2}(3w^2 + 3c_s^2 - 2w)\Pi - \frac{2w}{\xi} \frac{d\Pi}{d\xi},$$ \hspace{1cm} (2.5)

$$\mu = -\frac{5}{2}(1 - 3w) + 1 - 3c_s^2,$$ \hspace{1cm} (2.6)

$$\nu = -\frac{1}{2}(1 - 3w)(7 - 3w) + 3(1 - 3c_s^2),$$ \hspace{1cm} (2.7)

$$f = Hl\xi,$$ \hspace{1cm} (2.8)

and we have used a variable proportional to the cosmic scale factor $a$ denoted by $\xi$ as the independent time variable instead of $t$.* We have omitted the term $c_s^2/f^2$ on the left-hand side of Eq. (2.4) since it is negligibly small for $IH \gg 1$. From Eqs. (2.2) and (2.3), $f$ is expressed in terms of $w$ as

$$f = f_* \exp\left[\frac{1}{2} \int_{\xi_*}^{\xi} (1 - 3w) d\xi / \xi\right],$$ \hspace{1cm} (2.9)

where $f_* = f(\xi_*)$ and $\xi_*$ is a value of $\xi$ at some reference time. In Eq. (2.5) $\Gamma$ and $\Pi$ represent the gauge-invariant amplitudes of the isotropic and the anisotropic stress perturbation, respectively. As Eq. (2.4) shows, they act as a source for density perturbations. Regarding $\mathcal{S}$ as a fixed source and assuming that it does not vanish only for a finite time, we solve Eq. (2.4) and estimate the amplitude of the generated density perturbation.

We normalize $\xi$ to unity at the equal time $t = t_*$, that is, when the energy densities of radiation and dust coincide. Then from the assumption on the composition of the background cosmic matter, the energy density and pressure of the unperturbed universe are given by

$$\rho = \frac{1}{2} \xi^{-4} + \frac{1}{2} \xi^{-3},$$ \hspace{1cm} (2.10)

* $\xi$ in the present paper corresponds to $\xi$ in paper 1.
\[ p = \frac{1}{6} \zeta^{-4}. \quad (2.11) \]

From these equations it immediately follows that
\[ w = \frac{p}{\rho} = \frac{1}{3z}, \quad (2.12) \]
\[ c_s^2 = \frac{\dot{\rho}}{\rho} = \frac{4}{3(1+3z)}, \quad (2.13) \]
where
\[ z = 1 + \zeta. \quad (2.14) \]

Do not confuse \( z \) in the present paper with the redshift factor. From Eqs. (2.6), (2.7), (2.12) and (2.13), Eq. (2.4) is written in terms of the new variable \( z \) as
\[ \frac{d^2A}{dz^2} + \left( \frac{5}{2} \frac{1}{z} - \frac{3}{3z+1} \right) \frac{dA}{dz} + \left[ -\frac{2}{(z-1)^2} + \frac{3}{4} \frac{1}{z-1} + \frac{1}{2z^2} - \frac{3}{z} + \frac{27}{4} \frac{1}{3z+1} \right] A = S. \quad (2.15) \]

To solve this equation with a given \( S \), we need the general solution of the homogeneous equation associated with this equation. In contrast to the general cases as discussed in paper I, we can find a closed expression for it in the present case (cf., Ref. 5):
\[ A(z) = A U_d(z) + B U_c(z), \quad (2.16) \]
where
\[ U_d(z) = z^{-1/2}(z-1)^{-1}, \quad (2.17a) \]
\[ U_c(z) = (X(z) + 10/9) U_d(z) \quad (2.17b) \]
with
\[ X(z) = z^{-1/2} \left( z^2 - \frac{25}{9} z^3 + \frac{5}{3} z - \frac{5}{3} \right), \quad (2.18) \]
and \( A \) and \( B \) are arbitrary constants. For \( \zeta \gg 1(z \gg 1) \) which corresponds to the dust-dominated stage, \( U_d \) and \( U_c \) behave as
\[ U_d \sim z^{-3/2} \sim \zeta^{-3/2}, \quad (2.19a) \]
\[ U_c \sim z \sim \zeta. \quad (2.19b) \]
Hence \( U_d \) and \( U_c \) represent the decaying mode and the growing mode in the dust-dominated stage, respectively. On the other hand in the radiation-dominated stage \( \zeta \ll 1(z \sim 1) \), their asymptotic behavior is
\[ U_d = \frac{1}{\zeta} - \frac{1}{2} + \frac{3}{8} \zeta - \frac{5}{16} \zeta^2 + O(\zeta^3), \quad (2.20a) \]
\[ U_c = \frac{10}{9} \zeta^2 + O(\zeta^3). \quad (2.20b) \]
Thus \( U_c \) continues to grow and \( U_d \) continues to decay throughout both the radiation-dominated and the dust-dominated stages (see for details Ref. 3)).
Now let us specify the situation. Consider as follows: There exists no density perturbation before some time \( t_i (\Delta = d\Delta/\partial \xi = 0) \) and then a stress perturbation is provoked during a finite time interval \( t_1 < t < t_2 \) and vanishes thereafter. The universe is assumed to be described by the radiation-dust universe at least until \( t_2 \). However, as for the behavior of the universe after \( t_2 \) we consider two cases: The case the universe remains radiation-dust dominated throughout and the case the universe undergoes a discrete change from the radiation-dust stage to the pure radiation-dominated stage at some time \( t_*(>t_2) \).

In this situation, with the aid of the Green function of the differential operator on the left-hand side of Eq. (2.15)

\[
G(z, z') = \left[ U_c(z) U_D(z') - U_D(z) U_c(z') \right] W(z')^{-1} \theta(z - z'), \tag{2.21}
\]

where

\[
W(z) = U_D \frac{dU_c}{dz} - U_c \frac{dU_D}{dz} = \frac{5}{6} z^{-5/2} (3z + 1), \tag{2.22}
\]

Eq. (2.15) can be easily solved:

\[
\Delta(z) = A(z) U_c(z) + B(z) U_D(z), \tag{2.23}
\]

where

\[
A(z) = \int_{z_i}^{z} \frac{U_D(z')}{W(z')} \mathcal{S}(z') dz', \tag{2.24a}
\]

\[
B(z) = - \int_{z_i}^{z} \frac{U_c(z')}{W(z')} \mathcal{S}(z') dz', \tag{2.24b}
\]

for \( z > z_1 \). In expressions (2.21)~(2.24) \( U_D \) and \( U_c \) are assumed to be appropriately matched to the solutions of the homogeneous equation associated with Eq. (2.4) in the purely radiation-dominated stage, if the universe changes to radiation-dominated. The precise matching condition will be discussed later. Since \( \mathcal{S} \) vanishes for \( z > z_2 \), \( A(z) \) and \( B(z) \) become constant for \( z > z_2 \), which we write as \( A \) and \( B \), respectively.

\section{3. Estimation}

First we consider the isotropic stress perturbation; \( \mathcal{S} = -f^{-2} w \Gamma \). From Eqs. (2.9) and (2.12) \( f \) is given by

\[
f = \left( z/z_2 \right)^{1/2} z. \tag{3.1}
\]

Hence the constants \( A \) and \( B \) are expressed as

\[
A = \frac{6}{5} a \int_{z_1}^{z_2} \frac{\Gamma(z)}{(3z + 1)(z - 1)} dz, \tag{3.2a}
\]

\[
B = \frac{6}{5} a \int_{z_1}^{z_2} \frac{\Gamma(z)}{(3z + 1)(z - 1)} \left( X(z) + \frac{16}{9} \right) dz, \tag{3.2b}
\]

where

\[
a = z_2/3f_2^2. \tag{3.3}
\]
To estimate $A$ and $B$, we must specify the $\zeta$-dependence of $\Gamma$. We consider the three cases: $\Gamma = \text{const}$, $\Gamma = \gamma(z-1)$ and $\Gamma = \gamma(z-1)^n$ ($n \geq 2$), during $z_1 < z < z_2$ for all the cases.

(i) $\Gamma = \text{const}$

In this case the explicit integration yields

$$
A = -\frac{3}{10} a \Gamma \left[ \ln \frac{z_2 - 1}{z_1 - 1} - \ln \frac{3 z_2 - 1}{3 z_1 - 1} \right],
$$

$$
B = \frac{6}{5} a \Gamma \left[ \frac{2}{9} z^{3/2} - \frac{32}{27} z^{-1/2} + \frac{40}{27 \sqrt{3}} \tan^{-1}(\sqrt{3} z) + \frac{4}{9} \ln \left( \frac{\sqrt{z} + 1}{\sqrt{z} - 1} \right) \right],
$$

(3.4a)

(3.4b)

In this form we cannot yet see whether the amplitude of the generated density perturbation is large or not. What we must do is to estimate the amplitude of the perturbation when it comes within the Hubble horizon (that time is denoted by $t_H$). First let us assume that the universe stays in the radiation-dust stage until the time $t_H$. Then from Eq. (3.1) $f/z^{1/2}$ remains constant until $t = t_H$. Since $f = H\xi = \zeta_H$ at $t = t_H$, $a$ is written as

$$
a = z_H / 3 \zeta_H^2 \approx 1 / 3 \zeta_H \quad \text{for } \zeta_H \gg 1.
$$

(3.5)

In order to estimate $\Delta(t_H)$, let us consider two limiting cases. First in the limit $\xi_1 < \xi_2 \ll 1 < \xi_H$ we obtain

$$
\Delta(t_H) \approx -\frac{1}{10} \Gamma \frac{\xi_2}{\xi_1} + \frac{1}{30} \Gamma \xi_2^{1/2} \left( \frac{\xi_2}{\xi^2_H} \right)^{5/2} \left[ 1 - \left( \frac{\xi_1}{\xi_2} \right)^3 \right].
$$

(3.6)

The first term comes from the growing mode and the second term from the decaying mode; the latter is negligibly small compared to the former. Equation (3.6) shows that the amplitude of the generated density perturbation when it comes within the horizon, is of the order of the strength of the original stress perturbation. Next in the limit $\xi_1 \ll 1 < \xi_2$, we obtain

$$
\Delta(t_H) \approx -\frac{1}{10} \Gamma \left[ -\ln \xi_1 + \ln \frac{4}{3} \left( \frac{1}{\xi_2} \right) \right] + \frac{4}{45} \Gamma \frac{1}{\xi_2} \left( \frac{\xi_2}{\xi_H} \right)^{3/2}.
$$

(3.7)

Thus the amplitude of $\Delta(t_H)$ is again of the order of $\Gamma$. From the comparison of Eqs. (3.6) and (3.7) we find that the amplitude increases with $\xi_2$ when $\xi_2 < 1$ but it saturates after $\xi_2$ becomes greater than 1. This means that the generation of density perturbations during the matter-dominated stage is negligible if $\Gamma(\xi)$ is constant. The main reason of this is that the decrease in $w$ at the matter-dominated stage depresses the effect of $\Gamma$ because $\Gamma$ acts as the source of density perturbations in the combination $w\Gamma$.

Now we estimate the amplitude of the generated density perturbation in the case the universe suddenly changes from the radiation-dust to the purely radiation-dominated at $t = t_* (> t_2$ and $< t_H)$. The general solution of the source-free density perturbation equation in the purely radiation-dominated stage is given by

$$
\Delta(\xi) = C U_+(\xi) + D U_-(\xi),
$$

(3.8)

where

$$
U_+(\xi) = \xi^2,
$$

(3.9a)

$$
U_-(\xi) = \xi^{-1},
$$

(3.9b)
and $C$ and $D$ are arbitrary constants. In order to investigate the behavior of the density perturbation after the universe has changed to purely radiation-dominated, we must find the combinations of $U_+ \text{ and } U_-, \text{ } CU_+ + DU_-$, which are matched to the general solution $AU_b + BU_a$ in the radiation-dust dominated stage. The junction condition is the continuity of $\Delta$ and $V$ (the gauge-invariant amplitude of the velocity perturbation) at $\zeta = \zeta_*$ (see for details Ref. 3):

\begin{equation}
\Delta_b = \Delta_a,
\end{equation}

\begin{equation}
\frac{1}{1+w_b} \frac{dA}{d\zeta}|_b - \frac{3w_b}{1+w_b} \Delta_b = \frac{1}{1+w_a} \frac{dA}{d\zeta}|_a - \frac{3w_a}{1+w_a} \Delta_a, \quad (3.10b)
\end{equation}

where the suffices $a$ and $b$ denote the values of quantities in the radiation-dust stage just before $\zeta_*$ and in the purely radiation-dominated stage just after it, respectively. Solving this junction condition, we find the following relation between $(A, B)$ and $(C, D)$:

\begin{equation}
AU_c(\zeta_*) = \frac{9}{10} K(\zeta_*) CU_+(\zeta_*), \quad (3.11a)
\end{equation}

\begin{equation}
BU_d(\zeta_*) = (1 - \frac{9}{10} K(\zeta_*)) CU_+(\zeta_*) + DU_-(\zeta_*), \quad (3.11b)
\end{equation}

where

\begin{equation}
K(z) = (z-1)^{-3} \left( z^2 - \frac{25}{9} z^2 + \frac{5}{3} z - \frac{5}{3} + \frac{16}{9} z^{1/2} \right). \quad (3.12)
\end{equation}

Here note that

\begin{equation}
K(z) \to \begin{cases} 10/9; & z \to 1, \\ 1; & z \to \infty. \end{cases} \quad (3.13)
\end{equation}

Now assuming that $t_H \gg t_*$, $a$ is given by

\begin{equation}
a = z_* / 3 f_*^2 = z_* / 3 \zeta_H^2, \quad (3.14)
\end{equation}

since $f$ is constant in the purely radiation-dominated stage and coincides with $\zeta_H$ at $t = t_H$. Hence in the limit $\zeta_1 \ll \zeta_2 \ll 1$, $\Delta(t_H)$ is estimated as

\begin{equation}
\Delta(t_H) \simeq -\frac{1}{10} \Gamma \ln \frac{\zeta_2}{\zeta_1^2}. \quad (3.15)
\end{equation}

Thus again $\Delta(t_H)$ is of order $\Gamma$ and reconfirm the result in paper I obtained by the iterative argument (see also Ref. 6)). On the other hand in the limit $\zeta_1 \ll 1 \ll \zeta_2$ we obtain

\begin{equation}
\Delta(t_H) \simeq -\frac{1}{10} \Gamma \left[ -\ln \zeta + \ln \frac{4}{3} - \frac{4}{3} \frac{1}{\zeta_2} \right]. \quad (3.16)
\end{equation}

Hence the amplitude is exactly the same as that in the case the universe remains radiation-dust dominated throughout. This result may look strange considering the fact that the growth of density perturbations is slower in the dust-dominated stage than in the radiation-dominated stage if expressed in terms of the cosmic scale factor: $\Delta \propto \zeta^2$ in the radiation-dominated stage and $\Delta \propto \zeta$ in the dust-dominated stage for the growing mode. This seeming contradiction is resolved if we note that the Hubble horizon size $H^{-1}$
increases more rapidly in the radiation-dominated stage than in the dust-dominated one if expressed in terms of the cosmic scale factor. The density perturbations come within the horizon earlier in the case the universe evolves into the radiation-dominated stage than in the case it remains in the radiation-dust stage.

(ii) $\Gamma = \gamma (z - 1)$

Note that $w' = \frac{\delta p}{\rho}$ in the velocity-orthogonal gauge stays nearly constant for $z \gg 1$ in this case. Now $A$ is given by

$$A = -\frac{2}{5} \frac{2}{\gamma} \ln \frac{3z_2 + 1}{3z_1 + 1}. \quad (3.17)$$

Since the contribution of the decaying mode is negligibly small as shown in (i), we do not consider it from now on. Further since $\Delta(t_H)$ does not depend on whether the universe undergoes the transition to the purely radiation-dominated stage or not as shown in case (i), we only consider the case in which the universe remains radiation-dust dominated throughout.

In the limit $\xi_1 < \xi_2 \ll 1$, $\Delta(t_H)$ is approximately given by

$$\Delta(t_H) \approx -\frac{4}{15} \gamma (z_2 - z_1) \approx -\frac{4}{15} \Gamma(z_2). \quad (3.18)$$

Hence the amplitude of the generated density perturbation is of the same order as that in case (i) except for the absence of the logarithmic factor. On the other hand in the limit $\xi_1 \ll \xi_2 \ll 1$, $\Delta(t_H)$ is given by

$$\Delta(t_H) \approx -\frac{2}{15} \gamma \ln \left( \frac{3}{4} \xi_2 \right) = -\frac{2}{15} \Gamma(z_2) \frac{1}{\xi_2} \ln \left( \frac{3}{4} \xi_2 \right). \quad (3.19)$$

Now the resultant amplitude is depressed by the factor $1/\xi_2$ compared with case (i) except for the unimportant logarithmic enhancement factor. This is easily understood by noting that $\Gamma$ is ineffective in the radiation-dominated stage due to the form of $\Gamma(\propto \xi)$ and that the effect of $\Gamma$ in the dust-dominated stage is depressed by $w$ as explained previously.

(iii) $\Gamma(z - 1)^n (n \geq 2)$

In this case $A$ is estimated as

$$A \approx \begin{cases} \frac{3}{10} \gamma [(z_2 - 1)^n - (z_1 - 1)^n] \approx -\frac{3}{10n} \Gamma(z_2) & \text{for } \xi_1 < \xi_2 \ll 1, \\ -\frac{2}{5(n-1)} \gamma z_2^{n-1} & \text{for } \xi_1 \ll \xi_2 \ll 1. \end{cases} \quad (3.20)$$

The result in the limit $\xi_1 < \xi_2 \ll 1$ is the same as in (i) and (ii). In the limit $\xi_1 \ll 1 \ll \xi_2$, $\Delta(t_H)$ is approximately given by

$$\Delta(t_H) \approx -\frac{2}{5(n-1)} \Gamma(z_2) \frac{1}{\xi_2}. \quad (3.21)$$

Hence the result in this limit is also the same as that for the corresponding limit in case (ii).

To summarize, as in the case of a small change in B-EOS studied in paper I, the amplitude of the density perturbations generated from isotropic stress perturbations is,
when they come within the horizon, at most of the same order as the strength of the original stress perturbations in the radiation-dust universe even if the stress perturbations continue to work while the universe changes from radiation-dominated to dust-dominated. In addition the density perturbations generated after the universe becomes dust-dominated are negligibly smaller than those generated at the radiation-dominated stage, unless the stress perturbations are rather enhanced in the dust-dominated stage.

So far we have considered only the generation from isotropic stress perturbations. The effect of anisotropic stress perturbations can be estimated in the same way. Of the source terms arising from the anisotropic stress perturbation in Eq. (2.5), the term $(2/3) w f^{-2} \Pi$ acts exactly in the same way as the one from the isotropic stress perturbation $-w f^{-2} \Gamma$. Hence the difference of the effect of the anisotropic stress perturbation comes from the part

$$S' = 2\left[3(w^2 + c_s^2) - 2w\right] \frac{\Pi}{\xi^2} - 2w \frac{d\Pi}{d\xi} = 2\left(\frac{1}{3z^2} + \frac{4}{1+3z} - \frac{2}{3z}\right) \frac{\Pi}{(z-1)^2} - 2\frac{1}{3z(z-1)} \frac{d\Pi}{dz}. \quad (3.22)$$

Replacing the right-hand side of Eq. (2.24a) by $S'$, we find that the amplitude of the generated growing mode is given by

$$A = -2 \int_{z_1}^{z_2} dz \frac{U_0}{W} \left(\frac{1}{3z^2} + \frac{4}{1+3z} - \frac{2}{3z}\right) \frac{\Pi}{(z-1)^2} + \frac{2}{3} \int_{z_1}^{z_2} dz \frac{U_0}{Wz(z-1)} \frac{d\Pi}{dz}. \quad (3.23)$$

By partial integration we find that the right-hand side of Eq. (3.23) exactly vanishes. Thus no density perturbation is generated from the source term $S'$ in the radiation-dust universe irrespective of the stage it works. This result and that in paper I, if combined, strongly suggest that isotropic and anisotropic stress perturbations act in the same way in generating density perturbations even in cases the B-EOS undergoes a more general change when the stress perturbations are working.

§4. Concluding remarks

In this paper the generation of density perturbations from stress perturbations in the radiation-dust universe has been investigated. It has been shown that the generation predominantly occurs while the universe is radiation-dominated unless the amplitude of stress perturbations increases in proportion to the cosmic scale factor or faster in the dust-dominated stage. The amplitude of the generated density perturbations when they come within the Hubble horizon is of the order of the amplitude of the original stress perturbations. These results hold irrespective of the type of the stress perturbation, isotropic or anisotropic.

Summarizing the results obtained in paper I and the present paper, it is concluded that density perturbations generated from stress perturbations associated with the ordinary transient phenomena in the early universe have too small amplitudes to be related with the present large scale structure of the universe (cf. Ref. 7). Hence there must have occurred
some exotic transition phenomena, such as inflation associated with the GUT phase transition,\textsuperscript{8)\textendash 11)} if the seed density fluctuations for the present structure of the universe were generated in the course of the cosmic evolution. The behavior of density perturbations in the inflationary universe is extensively studied in Ref. 3) (see also Ref. 12)).

Acknowledgements

The author thanks Misao Sasaki for valuable discussion and critical comments. This work has been supported in part by the Grant in Aid for Science Research Fund of the Ministry of Education, Science and Culture No. 59740126.

References