Lattice Fermion and $\sigma$-Model

Where Could We Find the Remnant of Chiral Breaking Term in the Continuum Limit?

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In terms of $\sigma$-model constructed from a well-defined quantum theory with the aid of fermionic coherent states, we study the effect of chiral breaking term in the fermion action. We find the effect could not be wiped out, when gauge fields coming in, by any counter terms, then remains as the form of an anomaly. We also discuss the consequences of the anomaly in terms of $\pi^0\rightarrow 2\gamma$ decay and $\eta'$ mass being heavier than others ($U(1)$ problem).

§ 1. Introduction

Since the success of lattice gauge theory proving both the confinement$^1$ and the universal scaling with a sophisticated Monte Carlo simulation,$^2$ physicists try to understand the chiral spontaneous breakdown ($\chi$SB) within a lattice QCD. The no-go theorem,$^3$ on the other hand, prevents us from putting fermions on a lattice in chiral invariant way, so Karsten-Smit$^4$ proposed that a quark mass in the Wilson fermion action$^5$ should be subtracted to make the pion mass vanish in the weak coupling expansion. This definition of the quark mass was extended to the strong coupling by Kawamoto-Smit$^6$ and to the whole regime by Eguchi-Nakayama$^7$ who adopted the $1/N$ expansion in chiral Gross-Neveu model. Inspired by these works, one might study a $\chi$SB in the lattice QCD which is, however, far from our calculability, then one should take a rather tractable and phenomenological model or employ the Monte Carlo technique$^8$ with a larger lattice size than usually adopted ($>10^4$).

In this paper, we shall first discuss how to construct a lattice fermion action from a well-defined quantum theory, since various considerations$^9,10,11$ so far were mainly confined to avoid the species doubling, on a classical lattice action level, therefore their relationships to the quantum theory is still unclear. We attack this with the help of fermionic coherent states invented by Ohnuki and one of the present authors$^9$ and clarify that it is impossible to have a lattice fermion action which fulfills the conditions such as (1) causality, (2) space-time symmetry (STS), (3) locality, (4) hermiticity of Hamiltonian operator and (5) chiral invariance. Several points of these facts, of course, have already discussed in terms of a no-go theorem,$^3$ however, our approach is based on the operator formalism and can work even in finite degrees of freedom such as the case in the Monte Carlo simulation. Thus we believe our point of view provides the alternative way to lattice fermion actions and would be a key to solve the chiral fermion problem in the future. Armed with these, we propose a lattice $\sigma$-model action. Those are the contents of §2. The reason why we take a $\sigma$-model is that it is considered to be a most tractable and phenomenological model in studying $\chi$SB$^{12,13}$ so that one could make a systematic discussion on what are the remnants of chiral non-invariant term in the continuum.
However, more phenomenological be the model, more complicated it becomes, therefore one must study the one loop calculations to determine the counter terms, according to Karsten-Smit,\textsuperscript{4} which guarantee the chiral invariance in the continuum. We discuss these in §3. The next section 4 will be devoted to the examination of Ward-Takahashi identities, where we shall find the remnant of chiral non-invariant term in a fermion action when gauge fields come into the stage, and also discuss these consequences are responsible for \( \pi^0 \rightarrow 2\gamma \) and \( \eta' \) mass being heavier than pion's. (We can argue these \( \pi^0 \rightarrow 2\gamma \) and \( U(1) \) problems without any approximation such as \( 1/N \) or chiral perturbation,\textsuperscript{18,19} which is a merit of employing \( \sigma \)-model.) Some discussions will be made in §5 and a little bit longer appendixes are provided for estimating the lattice loop integral and the anomaly calculation. The needs for these are as follows: Although there have been several calculations on a chiral anomaly on lattice,\textsuperscript{4,13,20–22} those present only the final result or some explanation to derive it. We consider the anomaly itself is supposed as a topological quantity\textsuperscript{23} so any rough calculations lead one to the correct value,\textsuperscript{21} but which clearly brings some doubts about the anomaly.\textsuperscript{20} We thus there in the appendixes make a detailed calculation: We introduce a mass to control the infrared singularity, which is the only singularity on a lattice, and find the result is true to all orders of perturbation theory provided the whole gauge fields are external.

§ 2. Construction of a lattice \( \sigma \)-model

2.1. Fermion action\textsuperscript{*)}

The purpose of the present section is to elucidate how to get the lattice action starting from a well-defined quantum theory, which helps, we hope, the understanding of a lattice fermion problem. (The previous formulations\textsuperscript{11,13,10)} seem to be rather heuristic compared to ours.) We start with a well-defined quantum theory then adopt the fermion coherent states invented by Ohnuki and one of the present authors\textsuperscript{9) to obtain the path-integral formula: The coherent states of fermion operators are given by\textsuperscript{9})

\[
|\xi\rangle \equiv \exp(a^+ \cdot \xi)|0\rangle,
\quad \langle \xi| \equiv \langle 0| \delta(\xi_1 - a_1) \cdots \delta(\xi_K - a_K)
\]

\[
|\xi^*\rangle \equiv \delta(a_1^+ - \xi_1^*) \cdots \delta(a^K_1 - \xi^K_1)|0\rangle,
\quad \langle \xi^*| \equiv \langle 0| \exp(\xi^* \cdot a),
\]

where

\[
\{a_j, a_k^+\} = a_j a_k^+ + a_k^+ a_j = \delta_{k,j}, \quad (k, j = 1, \cdots, K), \text{ otherwise} = 0
\]

\[
(a^+ \cdot \xi) = \sum_{k=1}^K a_k^+ \xi_k
\]

with \( \xi_j(\xi_j^*) \) being Grassmann numbers satisfying

\[
\{\xi_j, \xi_k\} = \{\xi_j, \xi_k^*\} = \{\xi_j^*, \xi_k^*\} = 0,
\]

\[
\{\xi_j, a_k\} = \{\xi_j^*, a_k\} = \cdots = 0.
\]

\( |0\rangle \) is the vacuum, \( a_k|0\rangle = 0 \), and \( \delta(\xi_j - a_j) \) is the Grassmann delta function,

\textsuperscript{*) The discussion with Professor Y. Ohnuki was a great help to write out this section. The authors express sincere thanks to him.
\[ \delta(\xi_j - a_j) = \frac{1}{i}(\xi_j - a_j). \]  

(2.4)

The Grassmann integral is defined as

\[ \int d\xi = 0, \quad \int \xi d\xi = i = \sqrt{-1}. \]  

(2.5)

The following relations are easily verified with the aid of the above facts:

\[ a_k |\xi\rangle = \xi_k |\xi\rangle, \quad \langle \xi|a_k = \langle \xi|\xi_k, \]

\[ a_k^* |\xi^*\rangle = \xi_k^* |\xi^*\rangle, \quad \langle \xi^*|a_k^* = \langle \xi^*|\xi_k^*. \]  

(2.6)

and completeness relations,

\[ \int \int |\xi\rangle \langle \xi|(d\xi) = I, \quad \int (d\xi)^* |\xi^*\rangle \langle \xi^*| = I, \]  

(2.7)

where

\[ (d\xi) = d\xi_1 d\xi_{k-1} \cdots d\xi_1, \quad (d\xi)^* = d\xi_1^* d\xi_2^* \cdots d\xi_k^*. \]  

(2.8)

and \( I \) is the identity operator, and furthermore

\[ \langle \xi^*|\xi\rangle = \exp(\xi^* \cdot \xi), \quad \langle \xi|\xi^*\rangle = \exp(- (\xi^* \cdot \xi), \]

\[ \langle \xi^*|\xi\rangle = \delta(\xi_i - \xi_i'), \delta(\xi_k - \xi_k'), \langle \xi^*|\xi^*\rangle = \delta(\xi_i^* - \xi_i^*'), \delta(\xi_k^* - \xi_k^*'). \]  

(2.9)

Armed with these coherent states, we can built up the path-integral representation of the (imaginary) time evolution operator,

\[ U(T/2, -T/2) = \lim_{N \to \infty} (1 - \Delta tH_N)(1 - \Delta tH_{N-1}) \cdots (1 - \Delta tH_{N-k}), \]  

(2.10)

where \( \Delta t = T/(2N + 1) \), \( H_j \) is a hamiltonian,

\[ H_j = H(a^+, a) + (a^+ \cdot \eta_j) + (\eta_j^+ \cdot a), \]  

(2.11)

with Grassmann sources, \( \eta_j = \eta(t_j), \eta_j^+ = \eta^*(t_j) \) and all the operators are Schroedinger ones of course. Inserting the completeness (2.7) into (2.10) successively and then using (2.9), we find

\[ U(T/2, -T/2) = \lim_{N \to \infty} \int \int |\xi\rangle \exp\left[ - \sum_{j=0}^{N} \left( \xi_j^+ \cdot (\xi_j - \xi_{j-1}) \right) \right] \]

\[ \quad + \Delta t \left( h(\xi_j^{(a)}, \xi_j^{(a)}) + (\xi_j^+ \cdot \eta_j) + (\eta_j^+ \cdot \xi_j) \right) \right] <\xi_{-N-1}| \Pi_{j=-N}^{N} (d\xi_j)^* (d\xi_{-N-1}), \]  

(2.12)

where

\[ h(\xi^{(a)}, \xi) = \int \langle \xi + (1 + a) \xi^* | H(a^+, a)(\xi - (1 + a)\xi^*) \rangle \exp(- (\xi^* \cdot \xi))(d\xi)^* \]  

(2.13)

is Grassmann valued “classical” hamiltonian specified by the operator ordering parameter, \( \xi_j^{(a)} = (\frac{1}{2} - a)\xi_j + (\frac{1}{2} + a)\xi_{j-1} \), which says that \( a = 1/2 \), \( 0 \) and \( -1/2 \) correspond to the normal \((a^+ a)\), Weyl \((\frac{1}{2}(a^+ a - a a^+)\) and antinormal \((aa^+)\) respectively.\(^{25, a}\)

\(^{a}\) The operator hamiltonian is expressed as, by use of (2.13),

\[ H(a^+, a) = \int \int \xi + (\frac{1}{2} + a)\xi \langle \xi - (\frac{1}{2} - a)\xi \mid h(\xi^{(a)}, \xi) e^{-i a^+ \cdot \eta}(d\xi)^*(d\xi), \]

thus it is obvious that (2.12) is \( a \)-independent.
The generating function(al) could be obtained by sandwiching (2.15) between the true vacuum $|\Omega\rangle$, however, this is very difficult or impossible in most cases. We shall then utilize "Euclidean Technique". Let us introduce the generalized trace,

$$\text{Tr}^{(b)} U(T/2, -T/2) = \int \langle \xi | U(T/2, -T/2) e^{i\tau b} \xi \rangle (d\xi)$$

(2.14)

with $b$ being a real parameter. The ordinary trace corresponds to $b=1$ antiperiodic boundary condition (APBC), while $b=0$ to the periodic one (PBC). By assuming $\eta(t) = \eta^*(t) = 0$ for $|t| > T'/2$ with $T' < T$, (2.14) becomes

$$\sum_{m,n} \int \langle \xi | m \rangle \langle n | e^{i\tau b} \xi \rangle e^{-(T-T')\Delta t} \langle m | U(T'/2, -T'/2) \rangle \langle n | (d\xi),$$

where we have introduced the energy eigenstate $|m\rangle$ and used $\sum_m |m\rangle \langle m| = I$. After taking $T \to \infty$, it yields

$$\lim_{T \to \infty} (2.14) = \lim_{T \to \infty} \left( \int \langle \xi | \Omega \rangle \langle \Omega | e^{i\tau b} \xi \rangle e^{-(T-T')\Delta t} (d\xi) \right) \times \langle \Omega | U(T'/2, -T'/2) | \Omega \rangle.$$ 

(2.15)

The braces in (2.15) could be discarded as long as $\langle \xi | \Omega \rangle \neq 0$ and $\langle \Omega | e^{i\tau b} \xi \rangle \neq 0$. (This is guaranteed as far as the volume of the system $V$ is finite, therefore one should take $V \to \infty$ after all the calculations have been done.) Finally putting $\Delta t \to i\Delta t$, we have the desired generating function(al). These procedures are called "Euclidean Technique". Note that by "Euclidean Technique" one could get the path-integral formula independent of the representation space. (The expression of (2.12) depends strongly on the choice of the vacuum $|0\rangle$.)

Thus the path-integral formula of the generating functional is written as

$$\text{Tr}^{(b)} U(T/2, -T/2) = \lim_{N \to \infty} \int \exp \left[ - \sum_{j=-N}^{N} \left( \langle \xi_j^* \rho (\xi_j - \xi_{j-1}) \right) + \Delta t (b^{1/2}(\xi_j^* \xi_j)$$

$$+ (\eta_j^* \xi_j) + (\xi_j^* \eta_j)) \right] \times \prod_{j=-N}^{N} (d\xi_j) (d\xi_j^*)|_{\xi_{-N} = \xi_N}$$

(2.16)

where we have adopted the anti-normal ordering $a = -1/2$ for simplicity. The exponent without source terms in (2.16) will be called "action" and the first term be "kinetic term".

To get a fermi field action, one may tempt to use the Fock operator $a(k)$ defined through the Fourier transformation of the operator $\Psi(x)$, but this causes the non-local kinetic term when turning back to the $x$-representation since $a(k)$ itself is defined as the non-local expression; $a(k) \sim \int d^3x \tilde{u}(k) e^{ikx} \times (\partial/\partial t) \Psi(x, t)$. Hence we must work exclusively in the $x$-space.

To make a theory well-defined, we put the system in a cube with the length $L$ and discretize the space such as
with \(a\) being a lattice constant. Fermi field operators on a lattice should read as, by expressing \(n_j\) as \(n\),

\[
\hat{\Psi}_n = a^{3n} \hat{\Psi}(an) \tag{2.18}
\]

then the anti-commutation relations become

\[
\{ \hat{\Psi}_m, \hat{\Psi}_n^\dagger \} = \delta_{m,n}, \quad \text{otherwise}=0. \tag{2.19}
\]

We denote the operator quantity as with a caret in the following. We introduce the projection operators composed of \(\gamma\)-matrix,

\[
\Gamma'(+) + \Gamma'(-)=1, \quad \Gamma'(\pm)^2=\Gamma'(\pm), \quad \Gamma'(\pm)^* = \Gamma'(\mp). \tag{2.20}
\]

(Our convention of \(\gamma\)-matrix is given in (A.1).) With these projection operators, we can define annihilation operators

\[
\hat{\Psi}_n^{(+)} |0\rangle = \hat{\Psi}_n^{(-)} |0\rangle = 0,
\]

\[
\hat{\Psi}_n^{(+)} \equiv \Gamma'(+) \hat{\Psi}_n, \quad \hat{\Psi}_n^{(-)} \equiv \hat{\Psi}_n^\dagger \Gamma'(-), \tag{2.21}
\]

and creation operators

\[
\hat{\Psi}_n^{(-)} \equiv \Gamma'( - ) \hat{\Psi}_n^\dagger, \quad \hat{\Psi}_n^{(+)} \equiv \hat{\Psi}_n \Gamma'( + ). \tag{2.22}
\]

Those satisfy, with the aid of (2.19) and (2.20),

\[
\{ \hat{\Psi}_m^{(\pm)}, \hat{\Psi}_n^{(\mp)} \} = \delta_{m,n}, \quad \text{otherwise}=0. \tag{2.23}
\]

We can now have the path-integral representation of fermi field by noting the following correspondence:

\[
a_{k} \rightarrow (\hat{\Psi}_n^{(+)}, \hat{\Psi}_n^{(-)}), \quad a_{k}^\dagger \rightarrow (\hat{\Psi}_n^{(+)} \hat{\Psi}_n^{(-)}, \hat{\Psi}_n^{(*)}),
\]

\[
\xi_{k} \rightarrow (\Psi_n^{(*)}, \Psi_n^{(-)}), \quad \xi_{k}^\dagger \rightarrow (\Psi_n^{(*)} \Psi_n^{(-)}). \tag{2.24}
\]

Accordingly, the kinetic term is

\[
- \sum_{j=-N}^{N} \sum_{n=-N}^{N} \left\{ \Psi_{n,j}^{(*)}(\Psi_{n,j+1}^{(*)} - \Psi_{n,j}^{(*)}) + \Psi_{n,j}^{(-)}(\Psi_{n,j}^{(*)} - \Psi_{n,j+1}^{(*)}) \right\}. \tag{2.25}
\]

There could be several kinds of fermion actions corresponding to the choice of \(\Gamma'(\pm);\)

(i) \[\Gamma'(\pm) \equiv \frac{1 \pm \gamma_4 e^{i\theta}}{2} \tag{2.25}\]

with \(\theta\) being a real parameter \(0 \leq \theta \leq 2\pi,

\[
(2.24) = - \sum_{j=-N}^{N} \sum_{n=-N}^{N} \left\{ \frac{1}{2} \left( \Psi_{n,j}^{(*)} \Psi_{n,j+1} - \Psi_{n,j}^{(*)} \Psi_{n,j+1} \right) - \frac{1}{2} \left( \Psi_{n,j}^{(*)} \gamma_4 e^{i\theta} \Psi_{n,j+1} + \Psi_{n,j+1}^{(*)} \gamma_4 e^{i\theta} \Psi_{n,j} - 2 \Psi_{n,j}^{(*)} \gamma_4 e^{i\theta} \Psi_{n,j} \right) \right\}, \tag{2.26}
\]
where we have dropped out the surface term arising through the replacement such as
\[ \sum_j \Psi_j^* \Psi_{j+1} = \sum_j \Psi_j^* \Psi_{j+1} \Psi_j - (\Psi_{j+1}^* \Psi_n - \Psi_{j-1}^* \Psi_n) \] by employing APBC or PBC. Introducing a four dimensional notation
\[ n = (n_1, n_2, n_3, n_4 = j) \]
we get
\[ (2.26) = -\sum_n \left\{ \frac{1}{2} \left( \bar{\Psi}_n \gamma_5 \Psi_{n+1} - \bar{\Psi}_{n+1} \gamma_5 \Psi_n \right) - \frac{1}{2} \left( \bar{\Psi}_n \gamma_5 \Psi_{n+1} + \bar{\Psi}_{n+1} \gamma_5 \Psi_n - 2 \bar{\Psi}_n e^{i\theta \gamma_5} \Psi_n \right) \right\}, \]
where
\[ \bar{\Psi}_n = \Psi_n^* \gamma_4. \]
So far we have not specified the form of the Hamiltonian, but now we shall think about it. If one adopts the principle that the action should be respected the space-time symmetry (STS)\(^{22}\) even on a lattice, the Hamiltonian can be selected out,
\[ \tilde{H} = \frac{1}{a} \sum_n \left[ \sum_{\mu=1}^3 \left\{ \frac{1}{2} \left( \bar{\Psi}_n \gamma_\mu \Psi_{n+\hat{\mu}} - \bar{\Psi}_{n+\hat{\mu}} \gamma_\mu \Psi_n \right) \right. \right. \right.
\[ - \frac{1}{2} \left. \left. \left( \bar{\Psi}_n e^{i\theta \gamma_5} \Psi_{n+\hat{\mu}} + \bar{\Psi}_{n+\hat{\mu}} e^{i\theta \gamma_5} \Psi_n - 2 \bar{\Psi}_n e^{i\theta \gamma_5} \Psi_n \right) \right\} + \bar{\Psi}_n m \Psi_n \right], \]
where \(m\) is a lattice dimensionless mass,
\[ m = a \tilde{m}, \]
with \(\tilde{m}\) being a continuum mass. (In the following, all the quantities should be regarded as dimensionless. We always specify the continuum quantity by a tilde.) Thus, in the case where \(\Gamma' (\pm)\) is given by (2.25), the fermion action becomes
\[ I_{\text{fermion}} = -\sum_n \left[ \sum_{\mu=1}^3 \left\{ \frac{1}{2} \left( \bar{\Psi}_n \gamma_\mu \Psi_{n+\hat{\mu}} - \bar{\Psi}_{n+\hat{\mu}} \gamma_\mu \Psi_n \right) \right. \right. \]
\[ - \frac{1}{2} \left. \left. \left( \bar{\Psi}_n e^{i\theta \gamma_5} \Psi_{n+\hat{\mu}} + \bar{\Psi}_{n+\hat{\mu}} e^{i\theta \gamma_5} \Psi_n - 2 \bar{\Psi}_n e^{i\theta \gamma_5} \Psi_n \right) \right\} + \bar{\Psi}_n m \Psi_n \right], \]
where we have put \(a = \Delta t \ (L = T)\) according to STS. Three terms in (2.31) are named "naive Dirac (nD)\(^{1,1}\), "Wilson (W)\(^{3}\) and "Mass" term in order. \(I_{\text{fermion}}\) is nothing but a \(\theta\)-action found rather heuristically by Seiler and Stamatescu\(^{13}\) which becomes Wilson\(^{5}\) and Osterwalder-Seiler\(^{12}\) action when putting \(\theta = 0, \pi\) and \(\theta = \pi / 2, 3\pi / 2\), respectively.
There are other choices:
(ii) \(\Gamma' (+) = 1, \Gamma' (-) = 0\). (2.32)
The kinetic term becomes
\[ -\sum_n \bar{\Psi}_n \gamma_4 (\Psi_n - \Psi_{n-\hat{\mu}}), \]
then, a Hamiltonian must be, according to STS,
\[ \hat{H} = \sum_{n,k} \bar{\Psi}_n \gamma_k (\bar{\Psi}_n - \bar{\Psi}_{n-k}) + \text{mass term} \]

which is not a hermitian operator however.

(iii) \[ \Gamma(\pm) = \frac{1 \pm \gamma_5}{2}. \] (2.33)

The kinetic term is given by

\[-\sum_n \left[ \frac{1}{2} \bar{\Psi}_n \gamma_4 (\bar{\Psi}_{n+i} - \bar{\Psi}_{n-i}) - \frac{1}{2} \bar{\Psi}_n \gamma_4 \gamma_5 (\bar{\Psi}_{n+i} + \bar{\Psi}_{n-i} - 2 \bar{\Psi}_n) \right], \]

and a hamiltonian is from STS

\[ \hat{H} = \frac{1}{2} \sum_n \left[ \bar{\Psi}_n \gamma_k (\bar{\Psi}_{n+k} - \bar{\Psi}_{n-k}) - \frac{1}{2} \bar{\Psi}_n \gamma_k \gamma_5 (\bar{\Psi}_{n+k} + \bar{\Psi}_{n-k} - 2 \bar{\Psi}_n) \right] + \text{mass term} \]

whose second term is also non-hermitian. Note that a naive Dirac action could be obtained by setting \( \Gamma(+) = \Gamma(-) = 1/2 \) but in this case the vacuum defined by (2.21) reads as \( \bar{\Psi}_n \ket{0} = \bar{\Psi}_n \ket{\Psi} \) which means \( \ket{0} = \ket{0} \). Thus one cannot get a naive Dirac action from our formalism.

To summarize the result: (1) Causality which is the trivial consequence that our action is the faithful expression of the time evolution operator (2.10). (2) STS which may be considered as the requirement of euclidean (Lorentz) covariance on a lattice. (3) Localarity, that is, only finite differences should be allowed in the action (which is the inevitable consequence of (1) and (2)), (4) Hermiticity of hamiltonian select a \( \theta \)-action almost uniquely. Needless to say nD action cannot satisfy the causality. Moreover, chiral symmetry must always be broken under the conditions 1)~4), which seems to restate a no-go theorem in a different situation: The system is put in a box whose volume has been kept finite, so the topological argument employed in Ref. 3) cannot be applied so directly.

If we remove the restriction that our starting point is the time evolution operator (2.10), there could be the other parameter in front of \( \mathcal{W}^{\theta} \)

\[ \frac{r}{2} (\bar{\Psi}_n e^{i\theta \gamma_5} \Psi_{n+\hat{\mu}} + \bar{\Psi}_{n+\hat{\mu}} e^{i\theta \gamma_5} \Psi_n - 2 \bar{\Psi}_n e^{i\theta \gamma_5} \Psi_n). \] (2.34)

This action satisfies conditions 1)~4) as far as \( r \neq 0 \), however, the correspondence to the operator formalism is very complicated or might be lost in the case \( r \neq 1 \), therefore we shall set \( r = 1 \) in the following. \( \theta \neq 0 \) case itself is very interesting, but the \( CP \)-violating nature causes many cumbersome problems, hence we shall put \( \theta = 0 \) in the subsequent discussions.

2.2. \( \sigma \)-model action

The path-integral representation of bose fields can be obtained without any problems, therefore we may construct the lattice bose action by simply discretizing the continuum one. We then propose the \( \sigma \)-model action,

\[ I_0 = I_{\sigma \pi} + I_{\text{fermion}}, \] (2.35)

\[ I_{\sigma \pi} = -\sum_{n} \left[ \sum_{\mu} \left( \frac{1}{2} (\mathcal{P}_{\mu}(- \sigma_n \kappa)) + \frac{1}{2} (\mathcal{P}_{\mu}(- \pi_n \kappa)) \right) \right] + \frac{\chi^2}{2} (\sigma_n \kappa + (\pi_n \kappa))^2 \]
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\[ + \frac{A}{4} \left( (\sigma^a_n)^2 + (\pi^a_n)^2 \right) + G \overline{\Psi}_n (\sigma^a_n + i \gamma_5 \pi^a_n) \frac{\gamma^a}{2} \Psi_n + \epsilon \sigma^0_n \]

\[ = - \sum_n \left[ \text{Tr} (\mathcal{L}_\mu^{(-)} M_n \mathcal{L}_\mu^{(-)} M_n^+) + \lambda (\text{Tr} M_n M_n^+) \right] \]

\[ + G (\overline{\Psi}_{n,R} M_n \Psi_{n,L} + \overline{\Psi}_{n,L} M_n^+ \Psi_{n,R}) + \text{Tr} \frac{\gamma^0}{2} \epsilon (M_n + M_n^+) \],

(2.36)

where

\[ M_n = \sum_{\alpha=0}^8 \frac{f^\alpha}{2} (\sigma^a_\alpha + i \pi^a_\alpha), \quad \Psi_{n,R} \equiv \left( \frac{1 + \gamma_5}{2} \right) \Psi_n, \]

(2.37)

\[ \mathcal{L}_\mu^{(-)} f_n \equiv f_n - f_{n-\mu}, \quad \mathcal{L}_\mu^{(+)} f_n \equiv f_{n+\mu} - f_n, \]

(2.38)

and $I_{\text{fermion}}$ is (2.31) with $\theta = 0, \ m = 0$. Here we have considered the $U(3) \times U(3)$ model with $t^a/2 \ (a = 0 \sim 8)$ being $U(3)$ generators.

Chiral symmetry of our action (2.35) is explicitly broken by $W_t$ in $I_{\text{fermion}}$, thus the inclusion of fermion loops leads us to chiral breaking terms. Thus following the spirit of Karsten-Smit\(^9\) we should subtract all the chiral breaking terms to ensure the meson masses (almost) being massless within a weak coupling or a loop expansion (which differs from the ordinary methods of subtracting only divergent terms as $a \to 0$).

Accordingly our $\sigma$-model action reads

\[ I = I_0 + \Delta I_{\text{inv}} + \Delta I_{\text{br}}, \]

(2.39)

where $\Delta I_{\text{inv}} (\Delta I_{\text{br}})$ consists of chiral invariant (breaking) counter terms, whose explicit forms could be determined through the order by order perturbation theory.

We shall introduce, in the following, the color $SU(N_c)$ gauge fields coupled to quarks and the abelian one coupled to both quarks and mesons, to search for the effect of $W_t$. This introduction of color gauge $SU(N_c)$ does not mean we make double counting, since we shall not consider the bound state or we might be in a position where chiral symmetry is spontaneously broken but deconfining phase transition has not yet occurred.\(^{27,28}\) The “minimal substitution” on a lattice reads as

\[ \overline{\Psi}_n (\cdot) \Psi_{n+\mu} \to \overline{\Psi}_n (\cdot) U_{n,\mu} \Psi_{n+\mu}, \]

\[ \mathcal{L}_\mu^{(+)} M_n \to D_\mu^{(+)} M_n \equiv U_{n,\mu}^0 M_n + U_{n,\mu}^{+0} - M_n. \]

(2.40)

Here we write the gauge fields as

\[ U_{n,\mu} = \exp i \sum_{\alpha=0}^{N_c^2-1} \frac{T^a}{2} A^a_{\mu}(n) \quad \text{for quarks} \]

(2.41)

with $T^a/2$ being the charge operator, $T^0/2 = Q = e(t^3/2 + i^3/2\sqrt{3})$ and $T^a (a = 1 \sim N_c^2 - 1)$ color generators and

\^9\ The quartic interaction, $\text{Tr} M M^* M M^*$, could be possible under $U(3) \times U(3)$ which corresponds to $(\sigma^a \pi^a)^2$. We omit this possibility only for a technical simplicity. Without $\text{Tr} M M^* M M^*$, there is a redundant symmetry $O(18)$ in a meson potential which causes 17-folded degeneracies when the spontaneous breakdown occurs. Accordingly, from a phenomenological point of view we should take this term into account, but in the following our main interest is in the effect of $W_t$ in $a \to 0$, therefore we neglect this.
\[ U_{n\mu}^0 = \exp iQA_{\mu}^0(n) \quad \text{for mesons.} \quad (2.42) \]

In §4, we shall consider the three types of expectation value:

(i) \[ \langle \mathcal{O} \rangle = \int [d\sigma][d\Psi d\overline{\Psi}] \mathcal{O} e^I Z, \]
\[ Z = \int [d\sigma][d\Psi d\overline{\Psi}] e^I, \quad (2.43) \]

where \( I \) is given by (2.39).

(ii) \[ \langle \mathcal{O} \rangle_A = \int [d\sigma][d\Psi d\overline{\Psi}] \mathcal{O} e^{I_A} Z_A, \]
\[ Z_A = \int [d\sigma][d\Psi d\overline{\Psi}] e^{I_A}, \quad (2.44) \]

where \( I_A \) is given by (2.39) with the minimal substitution (2.40).

(iii) \[ \langle \mathcal{O} \rangle = \int [dA_{\mu}][d\sigma][d\Psi d\overline{\Psi}] \mathcal{O} e^{I_{\text{exchange}}} Z_A, \]
\[ Z_A = \int [dA_{\mu}][d\sigma][d\Psi d\overline{\Psi}] e^{I_{\text{exchange}}}. \quad (2.45) \]

The relation of (iii) to (ii) reads
\[ \langle \mathcal{O} \rangle = \int [dA_{\mu}] \langle \mathcal{O} \rangle_A Z_A e^{I_{\text{exchange}}} / \int [dA_{\mu}] Z_A e^{I_{\text{exchange}}}. \quad (2.46) \]

### §3. One loop calculations

In this section, we study a one-loop radiative correction to fix up the counter terms and then subtract all the chiral breaking terms according to the spirit of Karsten-Smit,\(^{4}\) that ensures chiral invariance in the continuum. For these purposes, the system size \( L \) and the time interval \( T \) are put to the infinity from scratch.\(^{4}\)

Let us first examine the effective potential of our model. Following ordinary procedures, we obtain
\[ V_{\text{eff}} = V_0 + V_{\text{1-loop}} + \Delta V_{\text{inv}} + \Delta V_{\text{br}}, \quad (3.1) \]

where \( V_0 \) is the \( \sigma, \pi \) potential part of (2.36), \( \Delta V_{\text{inv}} (\Delta V_{\text{br}}) \) is the contribution from \( \Delta I_{\text{inv}} (\Delta I_{\text{br}}) \) in (2.39), and
\[ V_{\text{1-loop}} = -\frac{1}{2} \text{Tr} F(G\sigma, G\pi) + \frac{1}{2} \sum_{\mu=1}^{18} B(\mu \sigma^2). \quad (3.2) \]

Here the first term is the fermionic contribution while the second is the bosonic, expressed as

\[ F_{n^r} = e^{\kappa n} F, \quad (b=0 \text{ for PBC, } b=1 \text{ for APBC}) \]
\[ F_{n^r} = \frac{1}{\sqrt{2N+1}} \exp \left\{ \frac{(2\tau+b)}{2N+1} n \right\}. \]

\(^{4}\) This makes the whole calculation rather simple. If not so, we should use the discrete eigenfunction \( F_{n^r} \):
\[ \sum_{n=1}^{N} F_{n^r} F_{n^r} = \delta_{r=r}, \quad \sum_{n=1}^{N} F_{n^r} F_n = \delta_{m=n} \]
\[ F_{n^r} = e^{\kappa n} F_{n^r}, \quad (b=0 \text{ for PBC, } b=1 \text{ for APBC}) \]
\[ F_{n^r} = \frac{1}{\sqrt{2N+1}} \exp \left\{ \frac{(2\tau+b)}{2N+1} n \right\}. \]
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\[ F(G\sigma, G\pi) = \frac{1}{2} \int_p \ln \{ \Delta^{-1}(p) \Delta^{-1}(p)[G\sigma, G\pi] \Delta(p)[G\sigma, G\pi] \}, \quad (3.3) \]

where

\[ \Delta^{-1}(p) = p^2 + (C(p) + G\sigma)^2 + (G\pi)^2 \quad (3.4) \]

with

\[ p^2 = \sum_{\mu} \sin^2 p_{\mu}, \quad C(p) = \sum_{\mu} (1 - \cos p_{\mu}), \quad (3.5) \]

\[ \sigma(\pi) = \sum_{a=0}^{8} \frac{i}{2} \sigma^a(\pi^a) \quad (3.6) \]

$[G\sigma, G\pi]$ is the commutator and the trace should be taken w.r.t. the flavor, and

\[ B(\mu_\beta^2) = \int_p \ln (p^2 + \mu_\beta^2) \quad (3.7) \]

with

\[ p^2 = \sum_{\mu} (2 \sin \frac{p_{\mu}}{2})^2 \quad (3.8) \]

and $\mu_\beta^2$ being eigenvalues of

\[ K_{a\beta} = \left( \begin{array}{cc} \frac{\partial^2 V_0}{\partial \sigma^a \partial \sigma^\beta} & \frac{\partial^2 V_0}{\partial \sigma^a \partial \pi^\beta} \\ \frac{\partial^2 V_0}{\partial \pi^a \partial \sigma^\beta} & \frac{\partial^2 V_0}{\partial \pi^a \partial \pi^\beta} \end{array} \right) \]

\[ = \left( \begin{array}{cc} \mu^2 \delta_{ab} + 2\lambda \sigma^a \sigma^b & 2\lambda \sigma^a \pi^b \\ 2\lambda \pi^a \sigma^b & \mu^2 \delta_{ab} + 2\lambda \pi^a \pi^b \end{array} \right) \quad (3.9) \]

with

\[ \mu^2 = \chi^2 + \sum_{a=0}^{8} \lambda \{(\sigma^a)^2 + (\pi^a)^2\}, \quad (3.10) \]

whose explicit values will be found in the following (3.23).

Chiral noninvariant pieces induced by $W_t$ (now written as $C(p)$ (3.5)) are obtained from (3.3)

\[ F(G\sigma, G\pi) = F_{\text{inv}}(G\sigma, G\pi) + F_{\text{br}}(G\sigma, G\pi), \quad (3.11) \]

where

\[ F_{\text{inv}}(G\sigma, G\pi) = \frac{1}{2} \int_p \ln \{ \Delta^{-2}(p) + \Delta^{-1}(p)[G\sigma, G\pi] \Delta(p)[G\sigma, G\pi] \} \quad (3.12) \]

with

\[ \Delta^{-1}(p) = p^2 + C^2(p) + (G\sigma)^2 + (G\pi)^2 \quad (3.13) \]

and

\[ F_{\text{br}}(G\sigma, G\pi) = A_{1,8} G(M + M^+) + A_{2,8} G^2(M + M^+)^2 \]
\[ + A_{3,0} G^3(M + M^+)^3 + A_{4,0} G^4(M + M^+)^4 \]
\[ + A_{1,1} G^3(M + M^+)[M, M^+] + A_{2,1} G^4(M + M^+)^2 [M, M^+] \]
\[ + \text{irrelevant terms}, \quad (3.14) \]

which could be obtained by expanding (3.4) around (3.13). (In (3.14), we have used the notation in terms of \( M, M^+ \) for simplicity. \( [M, M^+] \) is the anticommutator.) Thus the coefficients of (3.14),
\[ A_{1,0} = 4 \int \frac{C(p)}{(p_f^2 + C^2(p))^2}, \quad A_{2,0} = -2 \int \frac{C^2(p)}{(p_f^2 + C^2(p))^3}, \]
\[ A_{3,0} = \frac{16}{3} \int \frac{C^3(p)}{(p_f^2 + C^2(p))^3}, \quad A_{4,0} = - \int \frac{C^4(p)}{(p_f^2 + C^2(p))^4}, \]
\[ A_{1,1} = -4 \int \frac{C(p)}{(p_f^2 + C^2(p))^2}, \quad A_{2,1} = 4 \int \frac{C^2(p)}{(p_f^2 + C^2(p))^3}, \quad (3.15) \]

are finite while “irrelevant terms” vanish as \( a \to 0 \) owing to the power counting rules (A.19). Putting
\[ \Delta V_{br} = \frac{1}{2} \text{Tr} F_{br}(M, M^+), \quad (3.16) \]

we could have the effective potential free from effect of \( W_t \).

We shall next calculate \( F_{\text{inv}}, (3.12), \) and \( B(\mu^2), (3.7) \), to determine the form of \( \Delta V_{\text{inv}} \). To this end, we rewrite (3.12) as
\[ F_{\text{inv}}(G\sigma, G\pi) = F_{\text{inv}}(M, M^+) \]
\[ = \int \ln \left\{ p_f^2 + C^2(p) + \mu^2 + \left( \frac{G^2}{2} [M, M^+] - \mu^2 \right) \right\} \]
\[ + \frac{1}{2} \int \ln \left[ 1 + \left( p_f^2 + C^2(p) + \mu^2 + \left( \frac{G^2}{2} [M, M^+] - \mu^2 \right) \right)^{-1} \right. \]
\[ \times \left. \frac{G^2}{2i} [M, M^+] \left( p_f^2 + C^2(p) + \mu^2 + \left( \frac{G^2}{2} [M, M^+] - \mu^2 \right) \right)^{-1} \frac{G^2}{2i} [M, M^+] \right\}, \quad (3.17) \]

where we have introduced the parameter \( \mu^2 \) to ensure that there is no infrared singularity when expanding w.r.t. \( M, M^+ \) or \( G\sigma, G\pi \) whose result is
\[ F_{\text{inv}}(M, M^+) = F_1 \frac{G^2}{2} [M, M^+] + F_2 \frac{G^4}{2} (MM^+ MM^+ + M^+ MM^+ M) + O(\mu^6) + \text{const}, \quad (3.18) \]

where
\[ F_1 = \sum_{l=1}^{\infty} (\mu^2)^{l-1} \int \frac{1}{(p_f^2 + C^2(p) + \mu^2)^{l}}, \quad (3.19) \]
\[ F_2 = -\sum_{l=1}^{\infty} l(\mu^2)^{l-1} \int \frac{1}{(p_f^2 + C^2(p) + \mu^2)^{l+1}}, \quad (3.20) \]

which come from the first and the second term of (3.17) respectively, and “const” depends
only on $\mu^2$. Here we again use the power counting rules, (A·19), to find
\[
\lim_{\alpha \to 0} \frac{1}{a^2} F_1 = \lim_{\alpha \to 0} \left( \frac{1}{a^2} \int \frac{1}{p F_2 + C_\beta(p) + \mu^2 a^2} + \tilde{u}^2 \int p \left( p F_2 + C_\beta(p) + \mu^2 a^2 \right)^2 + \text{finite terms} \right) = \lim_{\alpha \to 0} \left( \frac{1}{a^2} \int \frac{1}{p F_2 + C_\beta(p)} \right) + \text{finite terms},
\]
(3·21)

\[
\lim_{\alpha \to 0} F_2 = \lim_{\alpha \to 0} \left( -\int p \left( p F_2 + C_\beta(p) + \mu^2 a^2 \right)^2 \right) + \text{finite terms} = \lim_{\alpha \to 0} \left( -\frac{1}{16 \pi^2} \ln \frac{1}{\mu^2 a^2} \right) + \text{finite terms},
\]
(3·22)

where we have put $\mu^2 = \mu^2 a^2$ and use has been made of the results in Appendix A. One should bear in mind that $U(3) \times U(3)$ invariant $\text{Tr} MM^+ MM^+$ term arises through the fermion loop effect. We, however, subtract them by adding a counter term to ensure that our tree lagrangian (2·36) becomes a renormalized one with a suitable renormalization as $a \to 0$.

The boson loop contribution (3·7) could be estimated similarly: The eigenvalues $\mu^2_{\beta}$ of (3·9) with (3·10) yield
\[
\mu^2_{\beta} = x^2 + 2\lambda \rho_\beta \text{Tr} MM^+ = x^2 + \lambda \rho_\beta \sum_{\beta=0}^{8} \left( (\sigma^\alpha)^2 + (\pi^\alpha)^2 \right),
\]
(3·23)

where $\rho_\beta = 1$ for $\beta = 1 \sim 17$ and $\rho_\beta = 3$ for $\beta = 18$, and then (3·7) becomes
\[
B(\mu^2_{\beta}) = \int p \ln \left( p F_2 + \mu^2 + \left( 2\lambda \rho_\beta \text{Tr} MM^+ + x^2 - \mu^2 \right) \right).
\]
(3·24)

By expanding w.r.t. ($\text{Tr} MM^+$), the result is
\[
(3·24) = B_1(2\lambda \rho_\beta \text{Tr} MM^+ + B_2(2\lambda \rho_\beta \text{Tr} MM^+)^2 + O(M^6) + \text{const},
\]
(3·25)

where
\[
B_1 = \sum_{i=1}^{\infty} \left( (\mu^2 - x^2)^{i-1} \left( \frac{1}{(p F_2 + \mu^2)^i} \right) \right),
\]
(3·26)

\[
B_2 = \sum_{i=1}^{\infty} \left( \left( \frac{1}{2} \right) (\mu^2 - x^2)^{i-1} \left( \frac{1}{(p F_2 + \mu^2)^{i+1}} \right) \right),
\]
(3·27)

and “const” is the term independent of $MM^+$ as above. Here we have also introduced $\mu^2$ as before, but, in this case, which guarantees (3·24) to have no tachionic poles when $x^2 < 0$. With the help of the power counting rules, (A·19), (3·26) and (3·27) become
\[
\lim_{\alpha \to 0} \frac{1}{a^2} B_1 = \lim_{\alpha \to 0} \left( \frac{1}{a^2} \int \frac{1}{p F_2 + \mu^2 a^2} \ln \frac{1}{\mu^2 a^2} \right) + \text{finite terms},
\]
(3·28)

\[
\lim_{\alpha \to 0} B_2 = \lim_{\alpha \to 0} \left( \frac{1}{16 \pi^2} \ln \frac{1}{\mu^2 a^2} \right) + \text{finite terms},
\]
(3·29)

where $x^2 = \tilde{x}^2 a^2$ and we have used (A·34) and (A·37).

We thus find the form of $\Delta V_{\text{inv}}$ in view of (3·2), (3·18) and (3·25),
\[ \Delta V_{\text{inv}} \equiv \left( -\Delta F_1 \frac{G_r^2}{2} + \Delta B_1 \sum_{\lambda=1}^{18} \lambda \rho_\lambda \right) \text{Tr} MM^+ \]
\[ + \Delta B_2 \sum_{\lambda=1}^{18} 2 \rho_\lambda \lambda^2 (\text{Tr} MM^+)^2 + F_2 \frac{G_r^4}{4} \text{Tr} MM^+ MM^+ , \]  \( (3.30) \)

where coefficients of counter terms, \( \Delta F_1, \Delta B_1 \) and \( \Delta B_2 \), should be chosen to cancel corresponding divergent terms, \( \Delta F_1 \to (3.21), \Delta B_1 \to (3.28) \) and \( \Delta B_2 \to (3.29) \), and \( F_2 \) term given by \( (3.20) \) wipes out the whole \( \text{Tr} MM^+ MM^+ \) term.

So much for the effective potential, now let us consider the remaining renormalization, but the calculations are straightforward so we only mention their structures: Meson self-energy; \( W_t \) causes the finite difference between the momentum dependent part so we must prepare the counter terms \( \Delta C_1 \bar{\nu}_\mu \bar{o} \nu_\mu o + \Delta C_2 \bar{\nu}_\mu \bar{\nu}_\mu \pi \) with \( \Delta C_1 \neq \Delta C_2 \). Consequently we have \( Z_\sigma \neq Z_\pi \). Quark self-energy and quark-meson vertex contain no \( W_t \) effect because of chiral symmetry. However, we need either \( \Delta D_1 \bar{\nu}_\sigma \nu_\sigma \) or \( \Delta D_2 \bar{\nu}_\pi \nu_\sigma \) to ensure chiral invariance because \( Z_\sigma \neq Z_\pi \).

Finally we shall consider \( \chi_{\text{SB}} \) of our model. Since we have subtracted all the chiral breaking terms, the situation is exactly the same as the continuum; if we assume \( \chi^2 < 0 \) then \( \chi_{\text{SB}} \) occurs yielding to the non-vanishing VEV \( \langle \sigma^0 \rangle \). Accordingly, all the quark masses are degenerate (in a renormalized form)

\[ m_u = m_d = m_s \equiv m_q \equiv \frac{G_r}{\sqrt{6}} \langle \sigma_R^0 \rangle . \] \( (3.31) \)

(If one wishes to have correct quark masses, one must add \( \varepsilon^2 \sigma^2 + \varepsilon^8 \sigma^8 \) to our model.)

Meson masses,

\[ \mu_\pi^2 = -\frac{\varepsilon_R}{\langle \sigma_R^0 \rangle} \] \( (3.32) \)

are 17-folded degenerate (\( \mu_\pi^2 = \mu_\sigma^2 \) except one \( \sigma \)-meson whose mass is given by \( 2|x_\pi|^2 - 3\varepsilon_R/\langle \sigma_R^0 \rangle \)) because of the \( O(18) \) symmetry. (The inclusion of \( \text{Tr} MM^+ MM^+ \) clears up this fault to give heavy masses to all \( \sigma \)-meson and to split \( \eta, \eta' \) and \( \pi \) simultaneously.)

§ 4. Examination of \( W_t \) effects in the Ward-Takahashi identities

In this section, we study the Ward-Takahashi (WT) identities under \( U(3) \times U(3) \) to see the effect of \( W_t \) in the continuum. The WT identity\(^{19,21}\) for some quantity \( \mathcal{O}_\mu \) reads as

\[ \langle \delta_\mu \mathcal{O}_\mu \rangle + \langle \mathcal{O}_\mu \delta_\mu \rangle = 0 , \] \( (4.1) \)

where the expectation value is defined by \( (2.43) \), \( I \) is given by \( (2.39) \) and \( \delta_\mu \) means a local variation.

The vector variation brings us the (fermionic part of) vector current,

\[ V_\mu = R_\mu^{(1)} + C_\mu^{(1)} , \] \( (4.2) \)

where

\[ R_\mu^{(1)} = \frac{i}{2} \left( \overline{\nu} \gamma_\mu \frac{1}{2} \nu \overline{\psi} \gamma_\mu \frac{1}{g} \psi \right) , \]
\[
C^{n(1)}_{\mu(n)} = -\frac{i}{2} \left( \overline{\psi}_n \frac{t^a}{2} \psi_{n+\mu} - \overline{\psi}_{n+\mu} \frac{t^a}{2} \psi_n \right). \tag{4.3}
\]

(The mesonic part has the naive form. Note that there is no contribution from \(\Delta I_{\text{inv}}\) and \(\Delta I_{\text{br}}\) (3.16), (3.30) since those locally invariant under the vector rotation.) Taking the expectation value of (4.2), the second term of the r.h.s. vanishes, according to (A.19), as \(a \to 0\). Thus as far as the 1-point function, \(\langle V_{\mu(a)}^{\alpha(1)} \rangle\), is concerned, we find no effect of \(W_t\) appearing as \(C^{n(1)}_{\mu(n)}\). The situation is not changed when studying the many-point function, but contrary to the continuum, the current \(V_{\mu(a)}^{\alpha(1)}(4.3)\) is not locally chiral invariant so one must take these effects into account. Two-point function, for instance, is given by

\[
\Pi^{\alpha(3)}_{\mu}(m, n) = \langle V_{\mu(a)}^{\alpha(1)} V_{\nu(n)}^{\alpha(1)} \delta_{\nu \mu} \delta_{m, n} V_{\mu(m)}^{\alpha(3)} \rangle, \tag{4.4}
\]

where

\[
V_{\mu(a)}^{\alpha(3)} = R_{\mu(a)}^{\alpha(3)} + C_{\mu(a)}^{\alpha(3)}
\]

with

\[
R_{\mu(a)}^{\alpha(3)} = -\frac{1}{2} \left( \overline{\psi}_n \gamma_{\mu} \frac{\{t^a/2, t^b/2\}}{2} \psi_{n+\mu} - \overline{\psi}_{n+\mu} \gamma_{\mu} \frac{\{t^a/2, t^b/2\}}{2} \psi_n \right),
\]

\[
C_{\mu(a)}^{\alpha(3)} = \frac{1}{2} \left( \overline{\psi}_n \frac{\{t^a/2, t^b/2\}}{2} \psi_{n+\mu} - \frac{\{t^a/2, t^b/2\}}{2} \psi_n \right), \tag{4.5}
\]

whose WT identity is expressed as

\[
V_{\mu(m)}^{(1)} \Pi_{\nu(m)}^{(1)}(m, n) + \frac{1}{2} (\delta_{m, n} + \delta_{m+\mu, n}) f^{\alpha(1)} \langle V_{\mu(m)}^{\gamma(1)} \rangle = 0. \tag{4.6}
\]

(The second term on the r.h.s. of (4.4) is due to the non-invariant nature of the current.) To find out the effect of \(W_t\) in (4.4) then we should estimate, for instance, \(\langle R_{\mu(a)}^{\alpha(1)} R_{\nu(a)}^{\beta(1)} \rangle\) and \(\langle R_{\mu(a)}^{\alpha(1)} C_{\nu(a)}^{\beta(1)} \rangle\). With the aid of Feynman rules, (A.5), we get (Fig. 1)

\[
\langle R_{\mu(a)}^{(1)} R_{\nu(a)}^{(1)} \rangle = \int_q \langle q + \frac{k}{2} \rangle \gamma_{\mu(a)}^{(1)}(q) S(q - \frac{k}{2}) \gamma_{\nu(a)}^{(1)}(q), \tag{4.7}
\]

\[
\langle R_{\mu(a)}^{(1)} C_{\nu(a)}^{(1)} \rangle = \int_q \langle q + \frac{k}{2} \rangle \gamma_{\mu(a)}^{(1)}(q) S(q - \frac{k}{2}) C_{\nu(a)}^{(1)}(q), \tag{4.8}
\]

where we have dropped, for notational simplicity, the flavor generator \(t^a/2\). Though one could perform a thorough calculation of (4.4),\(^{20}\) we make here the order estimation. After taking a trace (putting \(k = 0\)), we have

\[
(4.7) \sim \int_q A^2(q) \cos q_{\mu} \cos q_{\nu} (2 \sin q_{\mu} \sin q_{\nu} - \delta_{\mu \nu} q_\rho^2 + O(q^4)),
\]

\[
(4.8) \sim \int_q A^2(q) \cos q_{\mu} \sin q_{\nu} (2 \sin q_{\mu} \cos q_{\nu} + O(q^4)),
\]

whose infrared behavior reads (4.7) \(\sim \int q^2 / q^4\) and (4.8) \(\sim \int q^4 / q^4\) respectively. Thus (4.7) is more singular, that is, more dominant than (4.8).

Next let us make a (local) chiral transformation. Setting \(O_m = 1\) in (4.1), we get the

\(^{20}\) The role of the second term (Figs. 1(d) and (e)) is to cancel out the zero momentum part of the first term (Figs. 1(a)~(c)).
WT identity

\[ \langle \mathcal{V}_\mu^{(-)} \mathcal{V}^a_{\mu}(n) \rangle = \varepsilon \sqrt{\frac{2}{3}} \langle \pi^a(n) \rangle + \langle Y^a(n) \rangle, \quad (4.9) \]

where

\[ V^a_{\mu}(n) = i \left( \bar{\psi}_n \gamma_\mu \gamma_5 \frac{t^a}{2} \psi_{n+\tilde{\mu}} + \bar{\psi}_{n+\tilde{\mu}} \gamma_\mu \gamma_5 \frac{t^a}{2} \psi_n \right) \]

\[ + i \text{Tr} \left( \left[ \frac{t^a}{2}, M_{n+\tilde{\mu}} \right] \mathcal{V}_\mu^{(+) M_n} - \left[ \frac{t^a}{2}, M^{+}_{n+\tilde{\mu}} \right] \mathcal{V}_\mu^{(+)} M_n \right), \quad (4.10) \]

\[ Y^a(n) = X^a(n) + \delta_\alpha \Delta V_{br}(M, M^+) \quad (4.11) \]

with

\[ X^a(n) = \frac{i}{2} \left[ 4 \bar{\psi}_n \gamma_5 \frac{t^a}{2} \psi_n - \bar{\psi}_n \gamma_5 \frac{t^a}{2} \psi_{n+\tilde{\mu}} - \bar{\psi}_{n+\tilde{\mu}} \gamma_5 \frac{t^a}{2} \psi_n \right. \]

\[ - \left. \bar{\psi}_{n-\tilde{\mu}} \gamma_5 \frac{t^a}{2} \psi_n - \bar{\psi}_n \gamma_5 \frac{t^a}{2} \psi_{n-\tilde{\mu}} \right] \quad (4.12) \]

and \( \delta_\alpha \Delta V_{br}(M, M^+) \) being a chiral transformation of \((3.16)\). In this case, contrary to the vector case, one cannot express \( X^a \), which is the consequence of WT, as the difference of some quantity. One may then expect that there is the remnant of WT, however \( Y^a \) vanishes as \( a \to 0 \). The reason is simple: \( \Delta V_{br} \) is determined to cancel out the contribution
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from Wt, while $X^\sigma$ is a chiral transformation of Wt so by definition (4·11) itself tends to zero. Consequently we see no effects of Wt within pure $\sigma$-model.

When gauge fields take part in the model, however, the situation could be changed. To see this, we make the minimal substitution (2·40) with (2·41) and (2·42), regard gauge fields as external then adopt the notation (2·44). By differentiating twice w.r.t. gauge fields, we get

$$\lim_{\delta a \to 0} \frac{\delta^2}{\delta A_\mu^b(n_1)\delta A_\nu^c(n_2)} \langle Y^\sigma(m) \rangle_A$$

$$= \lim_{\delta a \to 0} \left[ \langle Y^\sigma(m) V_\mu^b(n_1) V_\nu^c(n_2) \rangle_A + \left\{ \langle Y^\sigma(m) \frac{\partial V_\mu^b(n_1)}{\partial A_\nu^c(n_2)} \rangle_A + (\mu \leftrightarrow \nu, n_1 \leftrightarrow n_2, b \leftrightarrow c) \right\} + \left\{ \frac{\partial X_\mu^b(n_1)}{\partial A_\nu^c(n_2)} V_\nu^c(n_2) \right\}_A \right. \right.$$  

$$+ (\mu \leftrightarrow \nu, n_1 \leftrightarrow n_2, b \leftrightarrow c) \left. \right] + \left\{ \frac{\partial^2 X_\mu^b(n_1)}{\partial A_\nu^c(n_1)\partial A_\nu^c(n_2)} \right\}_A, \quad (4·13)$$

where

$$V_{\mu}^{\nu}(n) = -\frac{1}{2} \bar{\psi}_n(\gamma_{\mu} - 1) \frac{\partial U_{n_{\nu}^{\mu}}(n)}{\partial A_{\mu}^{b}(n)} \psi_{n_{\mu}^{\nu}} + \frac{1}{2} \bar{\psi}_n(\gamma_{\nu} + 1) \frac{\partial U_{n_{\nu}^{\mu}}(n)}{\partial A_{\mu}^{b}(n)} \psi_n$$

$$- i\delta_{\beta,0} \text{Tr}[\left\{ \frac{T^0}{2}, M_{n_{\mu}^{\nu}} \right\} D_\mu^{(+)} M_{n_{\nu}^{\mu}} + \left\{ \frac{T^0}{2}, M_{n_{\nu}^{\mu}} \right\} D_\nu^{(+)} M_{n_{\mu}^{\nu}}], \quad (4·14)$$

and we have used $\delta Y^\sigma/\delta A_{\mu}^{b} = \delta X^\sigma/\delta A_{\mu}^{b}$, because of $\delta a \Delta V_{br}$ being gauge invariant. Up to the one-loop approximation, the right-hand side of (4·13) is expressed as Fig. 2 (the first term corresponds to (a), the second braces to (b), the third braces to (c) and the final term to (d)). Note that $\Delta V_{br}$ was defined as the result of the one-loop calculation so the meson loop effect becomes a higher order. According to the Feynman rules in Appendix A, the only surviving contribution is from (a):

$$\lim_{\delta a \to 0} \langle Y^\sigma(m) V_{\mu}^{b}(n_1) V_{\nu}^{c}(n_2) \rangle_A$$

$$= \lim_{\delta a \to 0} \left\{ \frac{\partial}{\partial \text{color}} \int_{p}^{\infty} \frac{T^0}{2} \frac{T^c}{2} \int_{k}^{\infty} \frac{T^b}{2} \frac{T^c}{2} e^{i p_{(m-n_1-(\bar{\nu}+\bar{\bar{\nu}}))/2+i \Lambda(m-n_2-\bar{\nu}))} I_{\mu\nu}(\tilde{p}, k) \right\}, \quad (4·15)$$

where

$$I_{\mu\nu}(\tilde{p}, k) = \frac{i}{2} \int_{q}^{\infty} \text{Tr} \left[ \gamma_{\mu} \left( C(q + \frac{\tilde{p} + k}{2}) + C(q - \frac{\tilde{p} + k}{2}) \right) \right] S(q + \frac{\tilde{p} + k}{2})$$

$$\times V_{\nu}^{(1)}(q + \frac{k}{2}) S(q - \frac{\tilde{q} + k}{2}) V_{\nu}^{(1)}(q - \frac{\tilde{p}}{2}) S(q - \frac{\tilde{p} + k}{2}) + (\mu \leftrightarrow \nu, \nu \leftrightarrow k). \quad (4·16)$$

Due to the power counting rules (A·19), the continuum quantity is defined as

$$\tilde{I}_{\mu\nu} (\tilde{p}, k) = \lim_{\delta a \to 0} \frac{\partial}{\partial \tilde{a}} I_{\mu\nu} (\tilde{p}, k) \quad (4·17)$$

and then using the result of Appendix B (B·20), we get
which is the anomaly in the continuum. (For detailed calculations see Appendix B.)

From the above argument, the continuum limit of $Y^a(n)$ is written as

$$\lim_{a\to 0} \langle Y^a(n) \rangle_A = -a^4 \frac{i}{16\pi^2} \varepsilon_{\mu
u\lambda\rho} \text{Tr}_{\text{color}} \left[ \frac{f_a}{2} F_{\mu\nu}(x) F_{\lambda\rho}(x) \right], \quad (4.19)$$

where

$$F_{\mu\nu} = \sum_{a=0}^{N_c-1} T^a_{\mu\nu}. \quad (4.20)$$

(In (4.19), $x_\mu = an_\mu$ and the factor $a^4$ turns into the integration measure when combined with the summation.) Consequently, we have the same PCAC relation as the continuum,

$$\langle \partial_\mu V^a_{5\mu}(x) \rangle_A = \varepsilon \sqrt{\frac{2}{3}} \langle \pi^a(x) \rangle_A - \frac{i}{16\pi^2} \varepsilon_{\mu
u\lambda\rho} \text{Tr} \left[ \frac{f_a}{2} F_{\mu\nu} F_{\lambda\rho}(x) \right], \quad (4.21)$$

which shows $\pi^a \to 2\gamma$ decay correctly occurs in our $\sigma$-model.\footnote{This study tells us \textit{when gauge fields are switched on the effect of $W_t$ emerges}. Making use of this fact, let us attack $U(1)$ problem.}

Putting $O_m = Y^a(m)$ in (4.1), we get

$$\langle Y^a(m) | \mathcal{S} \rangle = \frac{1}{2} \eta'(n) - Y^0(n) \rangle + \delta_{m,0} \langle Y^0(n) \rangle = 0, \quad (4.21)$$

where $Z^a$ is the chiral transformation of $Y^a$.\footnote{This study tells us \textit{when gauge fields are switched on the effect of $W_t$ emerges}. Making use of this fact, let us attack $U(1)$ problem.}
Lattice Fermion and σ-Model

\[
Z^o(n) = -\frac{1}{6} \left[ 4 \bar{\psi}_n \psi_n - \bar{\psi}_n U_{n,\mu} \psi_{n+\mu} - \bar{\psi}_{n+\mu} U_{n,\mu} \psi_n \right. \\
\left. - \bar{\psi}_{n-\mu} U_{n,\mu} \psi_n - \bar{\psi}_n U_{n-\mu,\mu} \psi_{n-\mu} \right] + (\delta_0)^2 \Delta V_{br}
\]  

(4.22)

and we write \( \pi^o = \eta' \). In this case, we regard gauge fields as dynamical variables and therefore the expectation value in (4.21) reads as (2.45), and also regard the anomaly as a topological quantity.\(^{18,19,20,30}\) We next substitute (4.21) into the another WT identity,

\[
\sum_n \langle \eta'(m) Y^o(n) \rangle = -\sqrt{\frac{2}{3}} \left\{ \sum_n \langle \eta'(m) \eta'(n) \rangle + \langle \sigma^o(m) \rangle \right\}
\]

\[
= -\sqrt{\frac{2}{3}} \left\{ \left( \frac{\mu_{\eta'}}{\mu_{\eta}} \right) + \langle \sigma^o(m) \rangle \right\},
\]  

(4.23)

where \( \mu_{\eta'} \) is the physical mass of \( \eta' \). ((4.23) comes from (4.1) by \( O_m = \eta'(m) \) and by performing a summation w.r.t. site \( n \).) The result is

\[
\frac{1}{\mu_{\eta'}^2} = \frac{1}{\epsilon} \sum_n \frac{1}{V_4} \sum_{m} \langle \sigma^o(m) \rangle + \frac{3}{2\epsilon} \sum_m \langle Y^o(m) Y^o(n) \rangle - \sqrt{\frac{3}{2}} \frac{1}{\epsilon} \sum_n \langle Z^o(n) \rangle,
\]  

(4.24)

where we have introduced a 4-dimensional volume \( V_4 \) which, however, could be eliminated by using a translational invariance, \( \langle Y^o(m) Y^o(n) \rangle = \langle Y^o(m + l) Y^o(n + l) \rangle \), which might be guaranteed by shifting all fields \( \phi_n \rightarrow \phi_{n+l} \) in the path-integral. (One should then take PBC or APBC.) Thus (4.24) turns out to be

\[
\frac{1}{\mu_{\eta'}^2} = -\frac{\langle \sigma^o \rangle}{\epsilon} + \frac{3}{2\epsilon} \sum_m \langle Y^o(m) Y^o(0) \rangle - \sqrt{\frac{3}{2}} \frac{1}{\epsilon} \langle Z^o(0) \rangle.
\]  

(4.25)

Although the perturbative effects are all canceled between the second and the third term, in view of the anomaly term (4.19) we might expect the non-perturbative contribution from \( \sum_n \langle Y^o(m) Y^o(0) \rangle \) corresponding to the continuum quantity \( \sim -\int d^4x \langle K(x) K(0) \rangle \) with

\[
K(x) = \frac{g^2}{32\pi^2} \sum_{\mu,\nu,\lambda} \frac{N^2}{4} \langle F_{\mu\nu}(x) F_{\lambda\rho}(x) \rangle.
\]

If this is the case, the \( U(1) \) problem could be solved

\[
\frac{1}{\mu_{\eta'}^2} = \frac{1}{\mu_\pi^2} - \frac{3}{2\bar{f}_\pi^2 \mu_{\eta'}} \langle d^4x \langle K(x) K(0) \rangle \rangle + \sqrt{\frac{3}{2}} \frac{1}{\bar{f}_\pi \mu_{\eta'}} \langle Z^o(0) \rangle,
\]  

(4.26)

where we have used (3.32) and \( \langle \sigma^o \rangle = \bar{f}_\pi \). Although our model is far from a real world (it has \( O(18) \) symmetry instead of \( U(3) \times U(3) \) in the meson sector), we would find the identical relations of \( U(1) \) meson as (4.26) when taking \( Tr M^+M^+ \) into account in our primary lagrangian (2.36). (It is necessary to estimate third term together with \( \langle Z^o \rangle \) in (4.26) by the help of a Monte Carlo simulation for instance.\(^{31,32}\))

§ 5. Discussion

In the foregoing sections, we have discussed the effects of \( W_t \). We learned that one could wipe out the whole memory of \( W_t \) according to the spirit of Karsten-Smit\(^4\) provided
there are no gauge fields, but that the situation is changed when gauge fields are switched on, thus one must have the remnant of Wt in the chiral WT identity. (One should note the introduction of gauge fields brings no effects to the vector WT identity, (4·6) for instance, since in the vector identities Wt is in the definition of the current (4·2) or (4·4), contrary to the axial case where $X^a$ cannot be expressed as the divergence of some quantity, so the contributions of gauge fields are all absorbed into the redefinition of $R_\mu$ and $C_\mu$.) Accordingly, this Wt is responsible for $\pi^0\to 2\gamma$ and $U(1)$ meson mass.

We comment on the action with $\theta$-parameters. From our analysis, it is expected that we might have a non-trivial $\theta$-dependence in the continuum limit as far as quark mass exists. This is indeed the case as was discussed by Seiler and Stamatescu.\(^{13}\) If quark mass vanishes, $\theta$-parameter also does through a chiral rotation like the discussions using the $\theta$-vacuum.\(^{33}\) It may then be interesting to investigate the effect of $\theta$ to our world, which for example is the study of CP-violation under the $\theta$-action, although several calculations have been done.\(^{34}\)

Finally, we mention about another fermion action defined by (2·32) or (2·33). Their action produces no anomaly. While the determinant obtained by integrating fermions may have the complex eigenvalue. The authors do not understand these consequences but it is worth while studying.

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**Appendix A**

--- **Feynman Rules, Power Counting and the Evaluation of Loop Integrals** ---

$\gamma$-matrix convention;

\[
\{\gamma_\mu, \gamma_\nu\} = 2\delta_{\mu\nu}, \quad \gamma_5 = \gamma_1 \gamma_2 \gamma_3 \gamma_4 = \gamma_5^+, \quad (A\cdot1)
\]

Feynman rules:

**Propagators:**

- **quarks**
  \[
  S(p)_{ij} \equiv (i\not{p} + C(p) + m_\psi)_{ij}^1
  \]

- **$\sigma$-mesons**
  \[
  D_{\sigma}^{ab}(p) \equiv \frac{\delta_{ab}}{p^2 + (\mu_\sigma^a)^2}
  \]

- **$\pi$-mesons**
  \[
  D_{\pi}^{ab}(p) \equiv \frac{\delta_{ab}}{p^2 + (\mu_\pi^a)^2}
  \]

- **gauge bosons**

where $(m_\psi)_{ij}$ is the quark mass matrix, $\mu_\sigma^a$ and $\mu_\pi^a$ are the meson masses which are all degenerate in a symmetric phase, and

\[
C(p) \equiv \sum_\mu (1 - \cos p_\mu), \quad (A\cdot2)
\]
Lattice Fermion and $\sigma$-Model

\[
\begin{align*}
\beta_F &= \sum_{\mu} \gamma_\mu \sin p_\mu, \quad \beta_F^2 &= \sum_{\mu} \sin^2 p_\mu, \\
\beta_F^2 &= \sum_{\mu} \left(2 \sin \frac{p_\mu}{2}\right)^2.
\end{align*}
\tag{A.3}
\]

Vertices:

quark-gauge boson

\[
\begin{align*}
\frac{T^a_1}{2} V^{(1)}_{\mu} \left(\frac{\dot{p} + q}{2}\right),
\end{align*}
\tag{A.4}
\]

where

\[
\begin{align*}
V^{(1)}_{\mu}(p) &= i \gamma_\mu \cos p_\mu + \sin p_\mu = r^{(1)}_{\mu}(p) + C^{(1)}_{\mu}(p), \\
V^{(n)}_{\mu_1 \cdots \mu_n}(p) &= \delta_{\mu_1 \mu_2} \cdots \delta_{\mu_n} \left(\frac{\partial}{\partial p_{\mu_1}}\right)^{n-1} V^{(1)}_{\mu_1}(p),
\end{align*}
\tag{A.5}
\]

$X^a$ vertices (4.12) in chiral WT identity:

\[
\begin{align*}
\frac{\gamma_5}{2} \frac{t^a_1}{2} (C(p) + C(q)),
\end{align*}
\tag{A.6}
\]

where

\[
\begin{align*}
C^{(n)}_{\mu_1 \cdots \mu_n}(p) &= \delta_{\mu_1 \mu_2} \cdots \delta_{\mu_n} \left(\frac{\partial}{\partial p_{\mu_1}}\right)^n C(p).
\end{align*}
\tag{A.8}
\]

Loop integral:

\[
\int_\pi \equiv \int_{-\pi}^{\pi} \frac{d^4p}{(2\pi)^4}.
\tag{A.7}
\]

Charge conjugation properties:

\[
\begin{align*}
C^{-1} S(p) C &= S^T(-p), \\
C^{-1} V^{(n)}_{\mu_1 \cdots \mu_n}(p) C &= (-)^n V^{(n)}_{\mu_1 \cdots \mu_n}(-p),
\end{align*}
\tag{A.8}
\]

where $C$ is a charge conjugation matrix, $C^{-1} \gamma_\mu C = - \gamma^T_\mu$.

Power counting: Although new interactions might be appeared on a lattice as is the case of the lattice gauge theory, we always have the same power counting rules as the continuum: Let us begin with a boson self-energy part of the effective action
\[ \sum_{m,n} \phi_m \phi_n \int e^{i\phi(m-n)\pi(p)} \]  
where \( \phi_m \) is the generic boson field on a lattice which by our definition is dimensionless and is related to the continuum quantity such as
\[ \phi_n = a \tilde{\phi}(x), \quad x_\mu = a n_\mu. \]  

Now we transform (A·9) into a continuum one given
\[ (A·9) = (a^{-4})^2(a^2)(a^4) \int d^4x d^4y \phi(x) \phi(y) \int_{\pi/\alpha}^{\pi/\alpha} \int_{\pi/\alpha}^{\pi/\alpha} d^4\tilde{p} d^4\tilde{p} e^{i\phi(x-y)\pi(a\tilde{p})}, \]
where the first factor \((a^{-4})^2\) comes from
\[ a^4 \sum_m \to \int d^4x, \]  
the second \(a^2\) from (A·10) and the last \(a^4\) from
\[ \phi_\mu = a \tilde{\phi}_\mu. \]  

Note that we always define the corresponding continuum quantity with a tilde. Thus the degree of divergence of a boson self-energy reads
\[ D = -8 + 2 + 4 = -2, \]
then yielding
\[ \tilde{\pi}(\tilde{p}) = \frac{1}{a^2} \pi(a\tilde{p}). \]  

Similarly a fermion self-energy part given as
\[ \sum_{m,n} \tilde{\Psi}_m \int e^{i\phi(m-n)\Sigma(p)} \Psi_n \]
becomes
\[ (A·18) = (a^4)^{-2}(a^{32})^2 a^4 \int d^4x d^4y \tilde{\Psi}(x) \tilde{\Psi}(y) \int_{\pi/\alpha}^{\pi/\alpha} \int_{\pi/\alpha}^{\pi/\alpha} d^4\tilde{p} d^4\tilde{p} e^{i\tilde{\phi}(x-y)\Sigma(a\tilde{p})} \tilde{\Psi}(y), \]
where \( \Psi_m(\tilde{\Psi}_m) = a^{32} \tilde{\Psi}(x)(\tilde{\Psi}(x)) \) in view of (2·18) with \( x \) being (A·10). Thus as above
\[ D = -8 + 3 + 4 = -1, \quad \Sigma(\tilde{p}) = \frac{1}{a} \Sigma(a\tilde{p}). \]

According to these examples, we could present the power counting theorem on a lattice: Assume there are \( E_b \) boson fields and \( E_f \) fermion fields in a certain graph \( \Gamma(p) \). Thus the superficial degree of divergence is
\[ D = E_b + \frac{3}{2} E_f - 4. \]  

Consequently, the corresponding quantity is
\[ \tilde{\Gamma}(\tilde{p}) = a^{E_b + (3/2)E_f - 4} \Gamma(a\tilde{p}). \]
When there is the interaction with a dimensional coupling constant, (A·18) should be modified as

\[ D = E_0 + \frac{3}{2} E_F - 4 + \sum_i n_i \delta_i, \]  
\[ (A·20) \]

where \( \delta_i \) is the power of the lattice constant \( a \) to make the dimensionless coupling constant and \( n_i \) is the number of these vertices containing in the graph.

Evaluation of lattice loop integrals:

Let us illustrate how to evaluate a loop integral. First, expand \( \tilde{P}(a\tilde{p}) \) of (A·19) in a Taylor series to the divergent order determined by (A·18) or (A·20), then put external momenta to zero. We usually meet the following three cases.

Case I. \( D > 0 \) The example is a 6-point function of scalar field induced by fermion loop \( (D=2) \),

\[ I_t = a^2 \int \frac{1}{(p_F^2 + (C(p) + m)^2)^3} \]  
\[ (A·21) \]

whose integrand behaves as \( p^{-6} \) when \( m \to 0 \), therefore we have \( p^{-2} \) singularity around \( p \to 0 \).

Case II. \( D = 0 \) In this case, the loop integrals are logarithmically divergent

\[ I_{I(1)} = \int \frac{1}{(p_F^2 + (C(p) + m)^2)^2} \]  
\[ (A·22) \]

or finite

\[ I_{I(2)} = \int \frac{C(p)}{(p_F^2 + (C(p) + m)^2)^3}. \]  
\[ (A·23) \]

The example of (A·22) is 4-point function of \( \pi \)-meson and of (A·23) is \( A_{1,1} \) in (3·15). If we have \( I_{I(1)} \) in a certain order of the differentiation, the next order differentiation leads us to Case I, on the other hand if we meet \( I_{I(2)} \) the remaining orders all vanish as \( a \to 0 \). To see this, suppose we have \( I_{I(2)} \), then differentiating twice more, we get

\[ \sim a^2 \int \frac{C(p)}{(p_F^2 + (C(p) + m)^2)^3}, \]

whose integrand is logarithmically divergent, therefore \( a^2 \ln a \to 0 \).

Case III. \( D < 0 \) In this case, loop integrals are all free from infrared divergences. An example is a \( \pi \)-meson self-energy induced by fermion loop,

\[ I_{Iu} = \frac{1}{a^2} \int \frac{1}{p_F^2 + (C(p) + m)^2}. \]  
\[ (A·24) \]

If one performs more differentiation, one always obtains Case II and Case III integral.

The calculation of \( I_t(A·21) \) could be done as follows:

\[ I_t = a^2 \int \frac{1}{(p^2 + m^2)^3} + a^2 \int \left( \frac{p^2 + m^2}{(p_F^2 + (C(p) + m)^2)^3} - \frac{(p_F^2 + (C(p) + m)^2)^3}{(p_F^2 + (C(p) + m)^2)^3} \right), \]  
\[ (A·25) \]

where we have introduced \( p^2 = \sum \mu p_\mu^2 \). To estimate the first term of (A·25), we separate the range of integration, \( [-\pi, \pi]^4 \) as \( [-\pi, \pi]^4 = R_1 + R_2 \), where
\[ R_1 = \{(p_\mu)_{\mu=1,\ldots,4} | 0 \leq p^2 \leq \pi\} \quad (A.26) \]

and \( R_2 \) is the remainder to find
\[ a^2 \int_{R_1} \frac{d^4p}{(2\pi)^4 (p^2 + m^2)^3} + a^2 \int_{R_2} \frac{d^4p}{(2\pi)^4 (p^2 + m^2)^3} . \quad (A.27) \]

An elementary manipulation to the first term leads us to
\[ a^2 \int_{R_1} \frac{d^4p}{(2\pi)^4 (p^2 + m^2)^3} = a^2 \left( \frac{1}{32\pi^2 m^2} + \text{const} \right) , \quad (A.28) \]

where "const" is independent of \( m \). Introducing
\[ m = \tilde{m} a , \quad (A.29) \]

we get
\[ (A.28) = \frac{1}{32\pi^2 \tilde{m}^2} + \text{irrelevant terms} . \quad (A.30) \]

The second term in (A.27) vanishes as \( a \to 0 \), since there are no singularities because of the integration range. The infrared behavior of the second term in (A.25) could be understood by noting that the denominator behaves as \( p^{12} \) while the numerator as \( p^8 \), \( p^6 \tilde{m}^2 a^2 \), \( \ldots \), therefore it diverges at most logarithmically. Thus the second term vanishes as \( a^2 \ln a \). The result is
\[ \lim_{a \to 0} I = \frac{1}{32\pi^2 \tilde{m}^2} , \quad (A.31) \]

which is the same value as in the continuum.

We shall next attack \( I_{n(1)} \). As above, we rewrite it as
\[ I_{n(1)} = \int_{R_1} \frac{d^4p}{(2\pi)^4 (p^2 + \tilde{m}^2 a^2)^2} + \left( \int_{R_2} \frac{d^4p}{(2\pi)^4 (p^2 + \tilde{m}^2 a^2)^2} \right) \]
\[ + \int_{R_2} \left( \frac{(p^2 + \tilde{m}^2 a^2)^2 - (p^2 + (C(p) + \tilde{m}a)^2)^2}{(p^2 + (C(p) + \tilde{m}a)^2)^2} \right) , \quad (A.32) \]

whose braces are finite and calculable numerically. The divergent part is obtained by a direct integration of the first term to give
\[ \text{the first term of (A.32)} = \frac{1}{16\pi^2} \ln \frac{1}{\tilde{m}^2 a^2} + \text{const} . \quad (A.33) \]

Thus
\[ \lim_{a \to 0} I_{n(1)} = \frac{1}{16\pi^2} \ln \frac{1}{\tilde{m}^2 a^2} + \text{finite} . \quad (A.34) \]

\( I_{n(2)} \) (A.23) is finite thus calculable by numerical methods. Finally we must evaluate (A.24). To this end, let us use the identity,
\[ \frac{1}{p^2 + (C(p) + \tilde{m}a)^2} = \frac{1}{p^2 + C(p)} \left( \frac{2C(p)\tilde{m}a + \tilde{m}^2 a^2}{(p^2 + C(p))(p^2 + (C(p) + \tilde{m}a)^2)} \right) , \quad (A.35) \]

then \( I_m \) turns into
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\[I_m = \frac{1}{a^3} \int \frac{1}{p^2 + C^2(p)} - \frac{2\bar{m}}{a} \int \frac{C(p)}{p^2 + (C(p) + \bar{m}a)^2} \]

\[-\frac{\bar{m}^2}{a} \int \frac{1}{p^2 + C^2(p)} \left( p^2 + (C(p) + \bar{m}a)^2 \right) \cdot \]  \hspace{1cm} (A.36)

The integrals of the first and the second terms are (infrared) finite then could be numerically calculated, but the third term has a logarithmic singularity. Thus with the help of the previous result (A.33), we get

\[\lim_{\bar{m} \to 0} I_m = \lim_{\bar{m} \to 0} \left\{ \frac{1}{a^2} \int \frac{1}{p^2 + C^2(p)} - \frac{2\bar{m}}{a} \int \frac{C(p)}{p^2 + C^2(p)} \right\} - \frac{\bar{m}^2}{16\pi^2} \ln \frac{1}{\bar{m}a^2} + \text{finite} \} \hspace{1cm} (A.37)\]

The infrared singularity with which we have dealt might be distinguished from the ordinary one which arises because $\bar{m} = 0$. In those cases, we must introduce a fictitious mass, $\mu$, to control the infrared divergence by keeping $\bar{\mu}$ fixed then putting $\mu = \bar{\mu}a$ to zero.

Appendix B

--- Anomaly Calculations ---

In this appendix, we shall calculate (4.19). We introduce the flavor singlet quark mass term to control an infrared singularity. We adopt a $\theta$-action with a most general $W_t$ (2.34), since by which we can reproduce the result of other method such as by using Wilson action or Osterwalder-Seiler action.\(^{(5,12)}\) The propagator and the vertices are generalized by changing $C(p)$ (A.2) to $re^{i\theta_7}C(p)$ as follows:

\[S(p) \to (i\slashed{p} + re^{i\theta_7}C(p) + m_q)^{-1},\]

\[V_\mu^{(1)}(p) \to r^{(1)}(p) + re^{i\theta_7}C_\mu^{(1)}(p),\]  \hspace{1cm} (B.1)

then $X^a$ vertices change as

\[i\gamma_\mu^{(a)}(C(p) + C(q)) \to i\gamma_\mu re^{i\theta_7}r^{(a)}(C(p) + C(q)).\]  \hspace{1cm} (B.2)

$X^a(n)$ (4.12) could be rewritten as

\[X^a(n) = \sum_m \left[ \overline{\Psi}_m i\gamma_5 e^{i\theta_7}r^{(a)}(C(n, m) \Psi_m + \overline{\Psi}_m i\gamma_5 e^{i\theta_7}r^{(a)}(C(m, n) \Psi_n) \right] \hspace{1cm} (B.3)\]

with

\[C(n, m) = \delta_{n,m} - \frac{1}{2}(\delta_{n+m,1}U_{n,m} + \delta_{n,m+1}U_{n,m}),\]  \hspace{1cm} (B.4)

and the quark propagator under external gauge fields is defined with the help of (2.44) by

\[S_A(m, n) = \langle \Psi_m \overline{\Psi}_n \rangle_A.\]  \hspace{1cm} (B.5)

Now according to these notations, we write

\[\langle X^a(n) \rangle_A = -r \sum_m Tr i\gamma_\mu e^{i\theta_7} r^{(a)}(C(n, m) S_A(m, n) + S_A(n, m) C(m, n)) \]

\[\hspace{1cm} (B.6)\]

then put $A^\mu_a = 0$ after taking the second derivative w.r.t. $A^\mu_a$ to find the quadratic dependence of the gauge fields such as
\[
\frac{\delta^2}{\delta A_\mu^a(m_1)\delta A_\nu^b(m_2)}\langle X^n(n) \rangle_{\Lambda} \bigg|_{A=0} \\
= -\text{Tr} \left( \frac{T^a}{2} \frac{T^b}{2} \right) \int_{p,k} e^{i\phi(n-m_1-(\bar{p}+\bar{k}))} e^{i\phi(n-m_2-(\bar{\nu}+\bar{k}))} I_{\mu\nu}(p,k), \tag{B.7}
\]

where

\[
I_{\mu\nu}(p,k) = r \int \text{Tr} i\gamma_5 e^{i\theta} \left[ \left( C(q+\frac{p+k}{2}) + C(q-\frac{p+k}{2}) \right) \times S\left( q+\frac{p+k}{2} \right) V_{\mu}^{(\alpha)}(q+\frac{k}{2}) S\left( q-\frac{p-k}{2} \right) V_{\nu}^{(\beta)}(q-\frac{p}{2}) \times S\left( q-\frac{p+k}{2} \right) \right] (\mu \leftrightarrow \nu, p \leftrightarrow k) \left( q+\frac{p+k}{2} \right) \delta_{\mu\nu} V_{\mu}^{(\alpha)}(q) S\left( q-\frac{p+k}{2} \right) + C\left( q-\frac{p+k}{2} \right) S\left( q+\frac{p+k}{2} \right) \delta_{\mu\nu} V_{\mu}^{(\alpha)}(q) S\left( q-\frac{p+k}{2} \right) - \left[ \left( C_{\mu}^{(1)}(q+\frac{p+k}{2}) + C_{\mu}^{(1)}(q-\frac{p+k}{2}) \right) S\left( q+\frac{k}{2} \right) V_{\nu}^{(\beta)}(q) S\left( q-\frac{p-k}{2} \right) \right] (\mu \leftrightarrow \nu, p \leftrightarrow k) + \delta_{\mu\nu} \left( C_{\mu}^{(2)}(q+\frac{p+k}{2}) + C_{\mu}^{(2)}(q-\frac{p+k}{2}) \right) S(q). \tag{B.8}
\]

Here the four terms are the contributions from Figs. (2. a) \sim (2. d) in this order. (If \( \theta = 0 \), the only first braces survives because of \( \gamma_5 \) trace property.) Owing to the gauge and the charge conjugation (A.8) invariance and the bose symmetry,

\[
\sum_{\mu} \sin \frac{p_\mu}{2} I_{\mu\nu}(p,k) = \sum_{\nu} \sin \frac{k_\nu}{2} I_{\mu\nu}(p,k) = 0,
\]

\[
I_{\mu\nu}(p,k) = I_{\mu\nu}(-p,-k) = I_{\mu\nu}(k,p), \tag{B.9}
\]

we find \( I_{\mu\nu}(0,0) = 0 \). Thus

\[
I_{\mu\nu}(p,k) \equiv I_{\mu\nu,\rho} k_\rho + O(p^2 k^2), \tag{B.10}
\]

where

\[
I_{\mu\nu,\rho} = \frac{\delta^2}{\partial p_\mu \partial k_\rho} I_{\mu\nu}(p,k) \bigg|_{p=k=0} \tag{B.11}
\]

whose explicit form can be found after a long and a tedious calculation as

\[
I_{\mu\nu,\rho} = (-16i) \epsilon_{\mu\nu,\rho} \int q \{ \cos q_\mu \cos q_\rho \cos q_\lambda \cos q_\rho (rC(q) + m_4 \cos \theta) \}
- (\cos q_\mu \cos q_3 \sin^2 q_\rho + \mu, \nu, \lambda, \rho \text{ cyclic}) rC(q) \Delta^3(q)
+ 4 m_4 \sin \theta (\delta_{\mu\nu} \delta_{3\rho} - \delta_{\mu\rho} \delta_{3\nu}) \int q \cos q_\mu \cos q_3 r^2 C(q) \Delta^3(q)
+ 16 m_4 \sin \theta \int \left[ 3 \left( G_{\mu\nu}(q) G_{3\rho}(q) - G_{\mu\rho}(q) G_{3\nu}(q) \right) \right]
- \left( G_{\mu\nu}(q) \frac{\partial Q_{\rho}(q)}{\partial q_\rho} + G_{3\rho}(q) \frac{\partial Q_{\nu}(q)}{\partial q_3} - G_{\mu\rho}(q) \frac{\partial Q_{\nu}(q)}{\partial q_\mu} - G_{3\nu}(q) \frac{\partial Q_{\rho}(q)}{\partial q_3} \right)
\]
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\[ -\left( \frac{\partial Q_\mu(q)}{\partial q_\nu} \frac{\partial Q_\lambda(q)}{\partial q_\rho} - \frac{\partial Q_\mu(q)}{\partial q_\rho} \frac{\partial Q_\nu(q)}{\partial q_\lambda} \right) r C(q) \Delta^3_s(q) \]

\[ -32 m_s \sin \theta \int \left[ G_{\mu\nu}(q) Q_\mu(q) Q_\nu(q) + G_{\lambda\rho}(q) Q_\lambda(q) Q_\rho(q) - G_{\mu\rho}(q) Q_\mu(q) Q_\rho(q) \right] + 2 \left( \frac{\partial Q_\mu(q)}{\partial q_\nu} Q_\lambda(q) Q_\rho(q) + \frac{\partial Q_{\lambda}(q)}{\partial q_\rho} Q_\mu(q) Q_\nu(q) \right) \]

\[ - \frac{\partial Q_\mu(q)}{\partial q_\rho} Q_\nu(q) Q_\lambda(q) - \frac{\partial Q_\nu(q)}{\partial q_\lambda} Q_\mu(q) Q_\rho(q) \right) r C(q) \Delta^3_s(q), \]  

(B.12)

where

\[ \Delta_s(q) \equiv \{ q^2 + (r C(q) + m_s \cos \theta)^2 + m_s^2 \sin^2 \theta \}^{-1}, \]

\[ G_{\mu\nu}(q) \equiv \delta_{\mu\nu} q \cos q_\mu \cos q_\nu + r^2 \sin q_\mu \sin q_\nu, \]

\[ Q_\mu(q) \equiv \sin q_\mu (\cos q_\mu + r C(q) + m_s \cos \theta). \]  

(B.13)

One may notice

\[ \frac{\partial \Delta_s^{-1}}{\partial \rho^\mu} = 2 Q_\mu(q). \]  

(B.14)

Note that the introduction of the quark mass term $m_q$ makes us possible to perform the integration by part freely, since there are neither infrared nor ultraviolet divergences. If $r \to 0$ with $m_q$ being fixed which means the calculations of the anomaly by means of naive Dirac action, we have the vanishing result as it should be.\(^3\)

Now let us consider the $a \to 0$ limit. According to the power counting rule (A.19), the continuum quantity of $I_{\mu\nu}$ is given by

\[ I_{\mu\nu}(p, k) \equiv \frac{1}{a^q} \tilde{I}_{\mu\nu}(\tilde{p}, \tilde{k}) \]  

(B.15)

then from (B.10)

\[ \tilde{I}_{\mu\nu}(\tilde{p}, \tilde{k}) = \sum_{\lambda\rho} I_{\mu\nu,\lambda\rho} \tilde{p}_\lambda \tilde{k}_\rho + O(a^2 \tilde{p}^4). \]  

(B.16)

By introducing the continuum mass such as $m_\sigma = m_q a$, the three terms with $m_\sigma \sin \theta$ coefficient in (B.12) vanish, since the singularity of their integrals is at most logarithmic (note that $C(q) \sim q^2/2$, $G_{\mu\nu} \sim 1$, $Q_\mu \sim q_\mu$ and $\Delta_s \sim q^{-2}$ around $q \sim 0$) therefore they behave $a \ln a$. While the order of the first term in (B.12) is $q^3/q^a$ thus it survives, yielding

\[ \lim_{a \to 0} I_{\mu\nu,\lambda\rho} = -16 i \epsilon_{\mu\nu,\lambda\rho} \int_q (\cos q_\mu \cos q_\nu \cos q_\lambda \cos q_\rho r^2 C(q) \]

\[ -4 r^2 \cos q_\mu \cos q_\nu \cos q_\lambda \sin^2 q_\rho) C(q) \Delta^3_s(q), \]  

(B.17)

where we have dropped $m_\sigma \cos \theta$ term also because $O(a \ln a)$ and use has been made of a symmetric property to the second term giving a factor 4 instead of $(\mu, \nu, \lambda, \rho)$ cyclic permutation. Equation (B.17) belongs to $I_{\mu(2)}$ type integral in (A.23) which is finite so numerically calculable, however, we can find the analytic answer by noting identity\(^4\)
\[ T. \text{ Kashiwa and H. So} \]

\[ \varepsilon_{\mu\nu\lambda\rho}(\cos q_{\mu}\cos q_{\nu}\cos q_{\lambda}\cos q_{\rho}r^2C(q) - 4\cos q_{\mu}\cos q_{\nu}\cos q_{\lambda}\sin^2 q_{\rho}r^2C(q))\Delta_8^3(q) \]

\[ = \varepsilon_{\mu\nu\lambda\rho}\frac{\partial}{\partial q_{\rho}}(\cos q_{\mu}\cos q_{\nu}\cos q_{\lambda}\sin q_{\rho}\Delta_8^3(q)) \]

\[ - \varepsilon_{\mu\nu\lambda\rho}(\cos q_{\mu}\cos q_{\nu}\cos q_{\lambda}\cos q_{\rho}(\sum_{k=1}^{4}\sin^2 q_{k} - 4\sin^2 q_{\rho}) \]

\[ + \cos q_{\mu}\cos q_{\nu}\cos q_{\lambda}\cos q_{\rho}2rC(q)m_{\omega}\cos \theta - 4\cos q_{\mu}\cos q_{\nu}\cos q_{\lambda}\sin^2 q_{\rho}r\cos \theta \]

\[ + m_{\omega}^2\cos q_{\mu}\cos q_{\nu}\cos q_{\lambda}\cos q_{\rho}\Delta_8^3(q). \quad (B\cdot18) \]

With the aid of (B\cdot18), (B\cdot17) becomes

\[ \lim_{a\to 0}I_{\mu\nu\lambda\rho} = +16i\varepsilon_{\mu\nu\lambda\rho}m_{\omega}^2\int_{q}\cos q_{\mu}\cos q_{\nu}\cos q_{\lambda}\cos q_{\rho}\Delta_8^3(q), \quad (B\cdot19) \]

where we have discarded all the terms the r.h.s. of (B\cdot18) except the last term: The first term is a total derivative, the second vanishes because of a symmetrical reason, the third and the fourth behave as \( O(\alpha m_{\omega}) \). Thus by identity (B\cdot18) \( I_{\mu\nu\lambda\rho} \) now belongs to Case I integral whose result (A\cdot31) leads us to

\[ \lim_{a\to 0}I_{\mu\nu\lambda\rho} = 16i\varepsilon_{\mu\nu\lambda\rho}\frac{1}{32\pi^2}\frac{a^2}{m_{\omega}^2}\frac{1}{a^2} \]

\[ = \frac{i}{2\pi^2}\varepsilon_{\mu\nu\lambda\rho}. \quad (B\cdot20) \]

One should note that (B\cdot20) is independent of \( \tilde{m}_{\omega} \) as well as \( \theta \) and \( r \). Going back to (B\cdot7) together with (B\cdot20), (B\cdot16) and (B\cdot15), we get

\[ \lim_{a\to 0}\langle X^a(n)\rangle_A = -\frac{a^4i}{16\pi^2}\varepsilon_{\mu\nu}\text{Tr}\left(\frac{T^a}{2}\frac{T^b}{2}\frac{T^c}{2}\right)(\partial_\mu A_\nu^a - \partial_\nu A_\mu^a)(\partial_\lambda A_\rho^b - \partial_\rho A_\lambda^b). \quad (B\cdot21) \]

The cubic and the quartic terms in \( a\neq0\neq b \) (non-abelian) case could be obtained with a similar manner whose coefficient is the same as (B\cdot20). Thus we conclude that

\[ \lim_{a\to 0}\langle X^a(n)\rangle_A = -\frac{a^4i}{16\pi^2}\varepsilon_{\mu\nu}\text{Tr}\left(\frac{T^a}{2}F_{\mu\nu}F_{\lambda\rho}\right), \quad (B\cdot22) \]

where \( F_{\mu\nu} = \partial_\rho A_\nu - \partial_\nu A_\rho + [A_\mu, A_\nu] \) with

\[ A_\mu = \sum_{a=0}^{N_{\lambda\mu}^a-1} A_\mu^a\frac{T^a}{2}. \quad (B\cdot23) \]

The remaining questions may be what is the contribution from higher order terms in (B\cdot10). The answer has already been obtained, since \( I_{\mu\nu\lambda\rho} \) was in the class \( I_{\mu(0)} \) so taking a higher derivative lead us to the vanishing result, as was explained in Appendix A (below (A\cdot23)). If one takes more derivatives w.r.t. gauge fields than the fourth, resulting integrals also vanish by the same mechanism. (A differential operation of gauge fields causes the same effects as that of momenta which is nothing but the content of Ward identity.) Thus we prove (B\cdot22) holds to all orders of \( A_\mu \).

References

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