Cosmological Bounce in Higher Dimensional Theories

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(Received October 6, 1984)

In higher dimensional Kaluza-Klein type of theories one loop quantum effective action drastically modifies near the Planck size the evolutionary equation for the Robertson-Walker scale factor. A mechanism of the cosmological bounce is found.

There is no observational support that the world we now know is built out of something more complicated than the usual four dimensional spacetime. Yet, the theoretical speculation that there exists at each 4-spacetime point an extra space of the size of the Planck length ($\sim 10^{-33}$cm), hence completely unobservable at the present level of experimentation, is very appealing, because the Einstein-Hilbert action in the high dimension leads, after averaging over the small extra space, to an effective theory of the gravity plus the Yang-Mills action in the four dimensions. The higher dimensional gravity, supplemented by a spontaneous compactification of the extra space, is an attractive idea to unify all known gauge interactions of elementary particles with gravity.

Needless to say, cosmology is crucial in understanding decoupling of the extra space. Some interesting works on this problem are in progress. We shall show in this note that the higher dimensional cosmology provides a mechanism of the bounce, thus a closed universe may repeat many cycles, recycling universe. It was long recognized that in the recycling universe accumulation of entropy may occur due to some dissipative processes, and that after many cycles the flatness and the horizon problems may be solved. The recycling universe based on the Kaluza-Klein cosmology thus gives an interesting alternative to the inflationary universe.

On the basis of an extension of our previous work, we shall first compute high temperature quantum action in a Robertson-Walker spacetime, and then proceed to construct a bounce model. A basic underlying assumption is that gravity is strong enough to maintain a thermal equilibrium, but the thermal equilibrium may be allowed to adiabatically change as the universe slowly evolves. The quantum 1-loop correction to the effective action in the dynamical spacetime profoundly modifies the basic equation of the cosmological evolution near the Planck size and finally gives a nonsingular bounce solution, as we shall see.

Computation of the quantum effective action is briefly summarized as follows. Consider the homogeneous, closed Robertson-Walker metric characterized by a scale factor $a(t)$ in the $(d+1)$-dimensional spacetime. One computes the 1-loop quantum effective action of graviton around this dynamical spacetime. As shown elsewhere, the graviton contribution for a large $d$ is essentially a spinless boson contribution times $(d+1)(d-2)/2$, the number of graviton degrees of freedom.

The adiabatic approximation allows one to expand the background metric in powers of the Euclidean time $\tau(-\beta/2 \leq \tau < \beta/2, \beta=1/T)$,

$$a(t-i\tau) = a(t) - i\tau \dot{a}(t) - \frac{\tau^2}{2} \ddot{a}(t) + \cdots. \quad (1)$$

The time $t$ and all the coefficients of this expansion, $a^n(t)$, should be regarded as given parameters in this calculation. Restriction to a particular periodic interval $[-\beta/2, \beta/2]$ is dictated by our insistence on that for a slowly varying $a(t)$ there should be dissipative term coming from an imaginary action. The adiabatic expansion (1) is valid when $|\beta \dot{a}/d| \leq 1$, which is usually satisfied if the expansion rate is much smaller than a microscopic reaction rate.

As is well known, one loop action is given by a Gaussian integral, hence is written as a trace log of a Laplacian operator. By inserting the known
eigenvalues of \( \lambda_i = \ell(\ell + d - 1) \) for the spatial Laplacian on the unit sphere, one reduces this operator to
\[
-(d/d\tau)^2 + \lambda_\alpha a^2(t - i\tau) + A[a(t - i\tau)].
\]
(2)
The functional \( A \) contains at least two powers of time derivatives, whose precise form does not concern us. We then introduce the generalized \( \zeta \)-function given by \( \zeta(s) = \text{tr}(\text{operator})^{-s} \), to regularize the effective action. To two powers of time derivatives one finds\(^9\) from (1) and (2) that
\[
\zeta(s) \sim G^{(s)} \int_0^\infty du u^{s-1} \sum_{n=0}^{\infty} \sum_{l=0}^{n} \xi_l
\]
\[
\times \left[ -u[(2\pi)^22^{n+1}n+1+\lambda_n a^2+M_{n,l}] \right],
\]
(3)
where the effective action is given by \(-\zeta'(0)/2\) and \( \xi_l \) is the degeneracy of the eigenvalue \( \lambda_l \). We have not written down a term of the zero mode, \( n=0 \), because this is not significant at high temperatures. As can explicitly be checked, dominant contribution for a large \( d \) is approximated by the first term of Eq. (4). We thus ignore the rest of the terms of (4) for the moment and mention effect of these ignored terms later.

It is then easy to see that the integrand of (3) is factorized after taking the mode sums over \( n \) and \( l \). The mode sum over \( l \) yields a function \( f(t/\alpha^2) \) with \( \alpha^2 = a^{-2} + \beta^2 \alpha^{-4}(a\bar{a} - 3\bar{a}^2)/12 \), where
\[
f(x) = \sum_{l=0}^{\infty} \xi_l \exp(-\lambda_l x).\]

At high temperatures of \( T \gg a^{-2} \), one only needs this function at a small argument \( x \). Namely,\(^9\) \( f(x) \sim G(d/2) G(d' 1)x^{-d/2} \). After this replacement one can perform the \( t \)-integration in (3) to obtain the usual \( \zeta \) function. One thus finds for the 1-loop effective action \( G^{(1)} \),
\[
-\Gamma^{(1)} \beta^{-1} = 2a^d T^{d+1} \xi (d+1)
\]
\[
\times [1 + \frac{1}{12}(a T)^{-2}(a\bar{a} - 3\bar{a}^2)]^{-d/2}.
\]
(5)
We may take the right-hand side of this equation as an effective local (in time) action induced by 1-loop effect. When expanded in powers of \( (a T)^{-2}(a\bar{a} - 3\bar{a}^2) \), the first term represents the well-known free energy density multiplied by the volume of the curved space \( S^d \). The rest of the expansion terms is a correction to the gravity at high temperatures.

The evolutionary equation for the scale factor \( a(t) \) is obtained by taking the functional derivative with respect to \( a(t) \) for the sum of the usual local action plus the effective action (5). This yields the space-space component of the Einstein equation, and, when supplemented by the conservation equation, \( T^{\mu \nu} = 0 \), describes a complete dynamical system. More conveniently, one may write the time-time component of the Einstein equation by taking the functional derivative with respect to the metric component \( g_{\mu \nu} \) around \( g_{\mu \nu} = 1 \). Since introduction of \( g_{\mu \nu}(\tau) \) is compensated by a new variable \( \hat{t}' = \sqrt{g_{\mu \nu}} d\tau \), it can be shown\(^9\) that for the effective action \( \Gamma \)
\[
\Gamma_{g_{\mu \nu}} = \frac{1}{2} \beta \hat{t}' \hat{t}' a^{-2} - \frac{1}{2} \alpha \hat{t}' \hat{t}' a^{-2} a^{-4}(a\bar{a} - 3\bar{a}^2)
\]
\[
\times \sum_{l=0}^{\infty} \xi_l \exp(-\lambda_l x).
\]
(4)
Thus, the time-time Einstein equation is found,\(^9\)
\[
\frac{1}{2} a^{-2} = \frac{1}{2} \alpha \hat{t}' \hat{t}' a^{-2} a^{-4}(a\bar{a} - 3\bar{a}^2)\]
\[
= 8\pi G\rho [1 + \frac{1}{12}(a T)^{-2}(a\bar{a} - 3\bar{a}^2)]^{-d/2}.
\]
(6)
with \( \rho = d a^{-1} (d+1) T^{d+1} \xi (d+1) T^{d+1} \), the energy density of \( d \)-dimensional ideal gas. On the other hand, the conservation equation is given by
\[
\rho \hat{t}' + \alpha a^{-3} \rho \hat{t}' = 0,
\]
(7)
where the energy-momentum tensor components, \( \rho \) and \( \rho' \), are modified from the usual ones of the ideal gas by the last factor of Eq. (5).

As the first approximation to Eqs. (6) and (7), it is sufficient to take the ideal gas limit of the conservation equation (7), thus \( (Ta)^d = \Sigma (\text{const}) \). Correction to this relation is estimated,
\[
(Ta)^d = [1 - \frac{1}{12}N^{-2d}
\]
\[
\times \int_0^t dt (a\bar{a}' - \alpha a^{-4} a\bar{a}' - \alpha a^{-2} a\bar{a}')],
\]
(8)
where at \( t_0 \) the correction is assumed negligible.

\(^{*}\) In deriving the 1-loop quantum action we ignored a term of the static induced gravity which has a form of \( GT^{d-1} a^{-2} \) if substituted in Eq. (6). This term is not significant in our discussion.
Effect of this correction will be discussed later, and we shall ignore it for the moment. The evolution equation (6) is then written in a dimensionless form,

$$[(\dot{\xi}^2 + 1)^{d-1} = 1 + k_0 (\xi^2 - 3 \xi \dot{\xi})]^{-4/2}$$

where $a = a_0 \xi$, $\dot{\xi} = d\xi / du$, $i = a_0 u$, $k_i = \Sigma^{-2i(d-1)}/12$ and the scale is given by

$$a_0 = [16\pi^{-(d-1)/2}(d-1)^{-1} \Gamma\left(d+1\right) / 2]
\times \xi^d (d+1) G^{1/2(d-1)} \Sigma^{(d+1)(d-1)}.$$

The constant $\Sigma$ measures the total entropy per comoving volume, and is assumed very large. We shall examine the solution of Eq. (9) around $u = 0$ at which $\dot{\xi}$ becomes an extremum.

Since $k_1$ is very small, one can first ignore this term, and obtain the usual Friedmann solution, $\xi \propto u^{2(d+1)}$ for a small positive $u$. This, however, cannot describe a solution all the way down to $u = 0$, because the $k_1$-term of (9) increasingly becomes important as one approaches the origin. In fact, one can easily establish that with the $k_1$-term included no solution behaving like $u^a (a > 0)$ exists. The only allowed solution near the origin is of the form, $c_0 + c_1 u + c_2 u^2 + O(u^3)$. A singular bounce solution, $c_0 + c_1 u^a (0 < a < 1)$, is thus excluded. The expansion coefficients of regular solutions with $c_1 = 0$ are related, so that the curvature $c_2 > 0$ when $0 < c_0 < 1$ and $c_0 < 0$ when $c_0 > 1$. The first solution corresponds to a smooth bounce near the "singularity", and the second to a solution near the maximum expansion (the $k_1$-term is hardly important in this case, $c_0 \approx 1$). With $c_1 = 0$, all odd terms of $u^{2n+1}$ do not appear in the expansion.

One must admit that with increasing importance of the correction term ($\propto k_0$) the expansion parameter $d/(aT)$ of (1) becomes larger, and the adiabatic expansion gets worse. It is however conceivable that around the bounce region the correction term does not overwhelm the original energy density and remains comparable to it: $d/(aT) \approx 1$. In this case, presence of our correction term is crucial in obtaining the bounce, but numerically the correction may not invalidate our approximation. A detailed analysis of this point, including a numerical analysis, is beyond the scope of this paper and is left to a future work.

The minimum size of scale for the regular bounce is roughly estimated as follows. A typical Friedmann solution of a unit amplitude, $u^{2(d+1)}$, is valid until the two terms of the right-hand side (9) start to compete around $\xi_0$,

$$\xi_0 \sim u_0^{2(d+1)} \sim k_1^{1/2(d-1)} \Sigma^{-2i(d-1)}.$$

Beyond this point the solution quickly approaches the bounce, and $\xi_0$ gives an order of magnitude of the minimum size. Translated to the actual size with (10), this leads to

$$a_{\text{min}} \sim G^{1/2(d-1)} \Sigma^{(d-1)/2(d-1)}, T_{\text{max}} \sim G^{-1/2(d-1)}. $$

We find it reasonable to have a maximum temperature independent of the entropy. The size at the maximum expansion, on the other hand, goes with the entropy like

$$a_{\text{max}} \sim G^{1/2(d-1)} \Sigma^{d(d-1)/d(d-1)}.$$ 

If the decoupling of the extra space occurs in a more realistic Kaluza-Klein cosmology, an effective dimension $d$ at the maximum size may actually be 3 and much smaller than $d$, and it can happen that $a_{\text{min}} < a_{\text{max}}$.

The arguments presented so far seem to indicate that the bounce always occurs. But for a small $d$ ignored terms come into play in a subtle way so that a singular solution cannot be avoided. Inclusion of the remaining terms in (4) gives additional contribution to the right-hand side of (9), and in the limit of slow variation the right-hand side should be multiplied by

$$1 + k_2 \xi^2 + k_3 [2(d-2)^{-1} \xi \ddot{\xi} + \dddot{\xi}]$$

with $k_2$ and $k_3$ known constants of $\sim \Sigma^{-2d}$. It is then possible to have a singular solution of the form, $c_0 u$, if, $1 + c_0 (k_2 + k_3) = 0$. Both of $k_2$ and $k_3$ are positive, but $k_2$ is negative and

$$(k_2 + k_3) \Sigma^{2d} = -\frac{1}{24} d
\quad + \frac{1}{8} (d^2 - d - 40)(d-1)^{-1} \xi(d-1)\dot{\xi}(d+1)^{-1}.$$ 

This number is negative only for an integer $d \leq 8$, at which the singular solution exists.

It is not clear whether in all cases $d \leq 8$ a Friedmann solution smoothly joins to this singular solution. The terms proportional to $\dot{\xi} \ddot{\xi}$ in (9) and (13) might dominate over those terms of $\dot{\xi}^2$ so as to block the approach to the singular solution. We would also like to estimate effect of the correction to the adiabatic relation (8). As a plausible check we may substitute the Friedmann behavior of $\xi \sim u^{2(d+1)}$ to the right-
The second term, representative of deviation from the Friedmann model, is negative only when $d \leq 5$, and in this case the Friedmann behavior can join to the singular solution. What happens at an intermediate dimension with $6 \leq d \leq 8$ presumably depends on details put in as the initial condition to the differential equation. The dimension 6 agrees with the one that can be obtained when one naively truncates the effective action to two powers of time derivatives and identifies a temperature dependent antigravity with a form of $(G^{-1} - cT^{d-1})a^2a^{-2}$.

In summary, gravity at a high dimension ($\geq 6$) is profoundly modified by high temperature quantum effects. In a closed universe model contraction may be turned into expansion, thus the bounce seems possible in a higher dimensional Kaluza-Klein cosmology.

   A. Salam and J. Strathdee, Ann. of Phys. 141 (1982), 316.
6) M. Yoshimura, to be published.

*) The last term in the square bracket gives correction caused by departure from the adiabatic condition (8). This term is always subdominant compared to the other terms.