Non-Abelian Anomaly and Vector Mesons as Dynamical Gauge Bosons of Hidden Local Symmetries

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We present general solutions to the Wess-Zumino anomaly equation which incorporate vector mesons as dynamical gauge bosons of the hidden local symmetry in the nonlinear chiral Lagrangian. In contrast to the previous attempts to introduce the vector mesons, our formalism enables one to treat consistently and systematically various processes associated with pseudoscalar mesons (Nambu-Goldstone bosons) and vector mesons (dynamical gauge bosons); it is automatic in this framework that the on-shell amplitudes of photons and pseudoscalar mesons such as \( \pi^0 \rightarrow 2\gamma \) and \( \gamma \rightarrow 3\pi \), which are fixed by the low energy theorems, are not affected no matter how the vector mesons participate in those processes. We further show that the "complete vector meson dominance" hypothesis of photon couplings is invalid in either \( \pi^0 \rightarrow 2\gamma \) or \( \gamma \rightarrow 3\pi \) process.

§ 1. Introduction

Recently much progress has been made on our (mathematical) understanding of non-Abelian anomaly and the effective chiral Lagrangian incorporating the anomaly.\(^{1,2}\) This revived our interest in the old Wess-Zumino\(^3\) method to construct effective action by imposing the anomaly equation. The method actually provides us with the easiest way to obtain the results of various low energy theorems concerning Nambu-Goldstone pseudoscalar mesons and photons.

In this context, the relations of vector mesons with the anomaly have also been considered by several authors,\(^{4-7}\) in the hope that the anomaly will determine or give important restrictions on the amplitudes of the processes including vector mesons. In particular, the "intrinsic parity" odd processes such as \( \omega \rightarrow \pi^0 \gamma, \omega \rightarrow 3\pi \) may be determined by the anomalous low energy theorems for such processes as \( \pi^0 \rightarrow 2\gamma, \gamma \rightarrow 3\pi \), if one assumes the "vector meson dominance" for the photon coupling to hadrons. [The "intrinsic parity" of a particle is defined to be even, if its parity equals \((-)^J\) (\(J\): spin), and odd otherwise; e.g., \( \rho, A_1 \) are odd while \( \gamma, \rho \) are even.]

Among these works, Kaymakcalan, Rajeev and Schechter\(^4\) proposed a systematic way to incorporate vector mesons (as well as axial vector mesons) into the chiral Lagrangian of pseudoscalar mesons in the presence of the Wess-Zumino term. Their results, however, have some unsatisfactory points as follows.

Their method is based on the old "massive Yang-Mills (YM) approach",\(^8\) in which the vector and axial-vector mesons are introduced as YM fields gauging the chiral \( U(3)_L \times U(3)_R \times U(3)_L \) symmetry with the mass terms put in by hand. There all the generators of \( U(3)_L \times U(3)_R \) are consumed as source charges of vector and axial-vector mesons. Therefore photon coupling can no longer be introduced in the usual manner gauging the
electromagnetic charge $Q = Q_L + Q_R = (2/3, 1/3)$ but are introduced instead by adding the interaction terms of photon-vector meson transition à la Sakurai\textsuperscript{90} and employing the vector meson dominance assumption. It is noted that they were obliged to introduce the axial vector mesons as gauge fields simultaneously with the vector mesons in order to preserve the global chiral $U(3)_L \times U(3)_R$ symmetry. But the gauging of the anomalous axial vector channel has a danger that the equations of motion may be mutually inconsistent, as was noticed by themselves. In addition, the introduction of axial vector mesons in this way forces them to choose Bardeen's form of the anomaly because the left-right symmetric form no longer keeps the vector current conservation and hence violates the low-energy theorems. The Bardeen form of the anomaly, however, explicitly breaks the chiral $U(3)_L \times U(3)_R$ symmetry. The problem is that this explicit breaking term survives even in the absence of external gauge fields in their formalism.

If one follows the usual procedure to introduce the photon field or other external gauge fields by gauging subgroup of $U(3)_L \times U(3)_R$ also for the system containing vector mesons, then the following questions will naturally arise: i) The Wess-Zumino original effective action including Nambu-Goldstone (NG) bosons and external gauge fields alone already saturates the anomaly, hence giving correct predictions for, e.g., $\pi^0 \rightarrow 2\gamma, \gamma \rightarrow 3\pi$. Are these predictions affected by the inclusion of vector meson interactions? ii) How is the vector meson dominance possible, e.g., in $\pi^0 \rightarrow 2\gamma$, despite the fact that photons directly couple to pseudoscalar mesons in the original Wess-Zumino action? iii) Why do such amplitudes depend on the parameters such as the $\rho \rightarrow \gamma, \omega \rightarrow \gamma$ transition coupling constants or the vector meson masses, being determined solely by the anomaly magnitude?

In this paper we shall consider these problems in a recently proposed framework\textsuperscript{10}\textsuperscript{12} in which the vector mesons are identified with dynamical gauge bosons of the hidden local symmetry $[U(3)_L]_{\text{local}}$ in the nonlinear chiral Lagrangian. This formalism in fact is found to satisfy the required consistency mentioned above. It is automatic in this framework that the amplitudes subject to the low energy theorems, like those of $\pi^0 \rightarrow 2\gamma$ and $\gamma \rightarrow 3\pi$, remain unaffected no matter how the vector mesons participate in those processes. In §5, we shall explain how this consistency is achieved in our formalism and give a complete answer to the above questions.

The other part of this paper is organized as follows. In §2 we consider the anomaly equation and the Wess-Zumino effective action. In §3 we briefly review the basic points of our framework in which the vector mesons are introduced as dynamical gauge bosons of the hidden local symmetry in the $U(3)_L \times U(3)_R$ chiral Lagrangian. The general solutions to the anomaly equation, which are responsible for the intrinsic parity violating processes, are given in §4 for the present system including vector mesons. In §6 we discuss the implications of vector meson dominance assumption in the context of this new approach. We argue that the “complete vector meson dominance” is not valid in either $\pi^0 \rightarrow 2\gamma$ or $\gamma \rightarrow 3\pi$ process (i.e., the direct coupling of photon to pions exist) and construct an effective action for the system of the pseudoscalar and vector mesons and external gauge bosons, which is consistent with all the low energy theorems and the present experimental data.
§ 2. The Wess-Zumino anomaly equation

Let us start with a nonlinear sigma model based on the manifold $G/H = U(3)_L \times U(3)_R/U(3)_V$ and briefly review the non-Abelian anomaly. The Lagrangian is given by

$$\mathcal{L} = \frac{f_\pi^2}{4} \text{Tr}(\partial^a U \partial_\mu U^\dagger),$$

(2.1)

where $U$ is written in terms of the Nambu-Goldstone (NG) pion fields as

$$U(x) = \exp(2i\pi(x)/f_\pi), \quad \pi = \pi^a T^a,$$

(2.2)

$T^a$ being the $U(3)$ generators normalized as $\text{Tr}(T^a T^b) = \frac{1}{2}\delta^{ab}$ and $f_\pi \approx 93\text{MeV}$ the pion decay constant. $U(x)$ transforms under chiral $U(3)_L \times U(3)_R$ as $U(x) \rightarrow e^{i\epsilon_L U \rightarrow e^{-i\epsilon_R}}$, where $\epsilon_L$ and $\epsilon_R$ are the group parameters of $U(3)_L$ and $U(3)_R$, respectively.

We now gauge these symmetries by introducing external gauge fields $A_{LM} = A_{L\mu}^a T^a$ and $A_{RM} = A_{R\mu}^a T^a$ which transforms as

$$\delta A_{LM} = \partial_\mu \epsilon_L + i[\epsilon_L, A_{LM}],$$

$$\delta A_{RM} = \partial_\mu \epsilon_R + i[\epsilon_R, A_{RM}].$$

(2.3)

The essential point of the Wess-Zumino idea is to notice that the anomaly at composite level should coincide with that at constituent (quark) level. This point of view is shared with 't Hooft in a different context. Therefore the effective action $\Gamma$ which describes low energy phenomena must satisfy the same anomalous Ward identity as that in QCD,

$$\delta \Gamma[U, A_L, A_R] = G[A_L, A_R],$$

(2.4)

to which we refer hereafter as the Wess-Zumino anomaly equation. Here $G[A_L, A_R]$ is the well-known non-Abelian anomaly, whose explicit form will be given shortly.

The so-called Wess-Zumino action $\Gamma_{wz}$ is a solution to Eq. (2.4). Since it is convenient to use the language of differential forms, we define 1-forms following the notation of Ref. 4:

$$\alpha = (\partial_\mu U) U^{-1} dx^\mu = (dU) U^{-1},$$

$$\beta = U^{-1} dU = U^{-1} aU ,$$

$$A_L = A_{L\mu} dx^\mu ,$$

$$A_R = A_{R\mu} dx^\mu .$$

(2.5)

Then the anomaly is

$$G[A_L, A_R] = -\frac{N_c}{24\pi^2} \int_M \text{Tr} \left[ \epsilon_L \left( (dA_L)^3 - \frac{i}{2} dA_L \right) - (l \leftrightarrow R) \right],$$

(2.6)

where $N_c = 3$ is the number of colors.

As Witten showed, the Wess-Zumino action is given concisely by gauge-covariantizing the integral.
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\[ C \int_{M^8} \text{Tr} \alpha^5, \quad C = \frac{-iN_c}{240\pi^2}. \]  \tag{2.7}

Here the integral is over a five-dimensional manifold \( M^8 \) whose boundary is ordinary Minkowski space \( M^4 \). The explicit form is:

\[
\Gamma_{\mu\nu}(U, A_L, A_R) = C \left[ \int_{M^8} \text{Tr} \alpha^5 \right]_{\text{covariantization}} \\
= C \int_{M^8} \text{Tr} \alpha^5 + 5C \int_{M^4} \text{Tr}(A_L \alpha^3 + A_R \beta^3) \\
- 5C \int_{M^4} \text{Tr}[(dA_L A_L + A_L dA_L)\alpha + (dA_R A_R + A_R dA_R)\beta] \\
+ 5C \int_{M^4} \text{Tr}(dA_L dU A_R U^{-1} - dA_R d(U^{-1}) A_L U) \\
+ 5C \int_{M^4} \text{Tr}(A_R U^{-1} A_L U^2 - A_L U A_R U^{-1} A^2) \\
+ \frac{5C}{2} \int_{M^4} \text{Tr}[(A_L \alpha)^2 - (A_R \beta)^2] + 5C \int_{M^4} \text{Tr}(A_L^2 \alpha + A_R^2 \beta) \\
+ 5C \int_{M^4} \text{Tr}[(dA_R A_R + A_R dA_R)U^{-1} A_L U - (dA_L A_L + A_L dA_L)UA_R U^{-1}] \\
+ 5C \int_{M^4} \text{Tr}(A_L UA_R U^{-1} A_L \alpha + A_R U^{-1} A_L UA_R \beta) \\
+ 5C \int_{M^4} \text{Tr}[A_R^3 U^{-1} A_L U - A_L^3 UA_R U^{-1} + \frac{1}{2} (UA_R U^{-1} A_L)^2]. \] \tag{2.8}

§ 3. Vector mesons as dynamical gauge bosons of hidden local symmetries

As is well known, the \( U(3)_L \times U(3)_R / U(3)_V \) nonlinear sigma model is a low energy effective Lagrangian of the massless 3-flavoured QCD. On the other hand, any nonlinear sigma model based on the manifold \( G/H \) is known to be gauge equivalent to a "linear" model with \( \text{G}_{\text{global}} \times \text{H}_{\text{local}} \) symmetry, and the gauge bosons corresponding to the hidden local symmetry, \( \text{H}_{\text{local}} \), are composite fields. Recently a framework has been proposed\(^{10-12}\) in which vector mesons are identified with the dynamical gauge bosons of hidden local \( U(3) \) symmetry. In that framework we can successfully explain the Kawarabayashi-Suzuki-Riazuddin-Fayyazuddin (KSRF) relation,\(^{13}\) the universality of \( \rho \)-coupling,\(^9\) and \( \rho \)-dominance of the photon coupling to pions.\(^9\)

Let us explain our framework briefly. We introduce new \( U(3) \) matrix-valued variables \( \xi_L(x) \) and \( \xi_R(x) \) such that

\[ U(x) = \xi_L(x) \xi_R(x), \] \tag{3.1}

and non-Abelian gauge field \( V_\mu(x) = V_\mu^a(x) T^a \). The transformation properties of these variables under \([U(3)_L \times U(3)_R]_{\text{global}} \times [U(3)_V]_{\text{local}}\) are

\[ \xi_L(x) \to e^{i\nu(x)} \xi_L(x) e^{-i\epsilon_L}. \]
\[\xi_R(x)e^{\imath\kappa(x)}\xi_R(x)e^{-\imath\kappa},\]
\[\delta_{\nu}(v)V_{\mu}(x) = \partial_{\nu}v(x) + ig[v, V_{\mu}], \tag{3.2}\]

where \(\nu(x)\) is the group parameter of the hidden local \(U(3)_{\nu}\) transformation.

If we gauge the global group \(U(3)_L \times U(3)_R\) with external gauge fields \(A_{\mu L}\) and \(A_{\mu R}\) which transform as (2.3), derivatives for \(\xi_{\mu R}\) fields are given by
\[D_{\mu}\xi_L = (\partial_{\mu} - igV_{\mu})\xi_L + i\xi_LA_{\mu L},\]
\[D_{\mu}\xi_R = (\partial_{\mu} - igV_{\mu})\xi_R + i\xi_RA_{\mu R}. \tag{3.3}\]

In particular, the electromagnetic field \(B_{\mu}\) couples to the charge \(Q = I_3^{(L)} + I_3^{(R)} + \frac{1}{2}(Y^{(L)} + Y^{(R)})\) with \(I_3^{(L,R)}\) and \(Y^{(L,R)}\) being the \(U(3)_L \times U(3)_R\) isospin and hypercharge, respectively. Thus, if only \(B_{\mu}\) exists as the external gauge field, then we should set
\[A_{L\mu} = A_{R\mu} = eB_{\mu}Q, \quad Q = \begin{pmatrix} 2/3 \\ -1/3 \\ -1/3 \end{pmatrix} \tag{3.4}\]
with \(e\) being the electromagnetic coupling constant.

The Lagrangian is given by
\[\mathcal{L} = \mathcal{L}_A + a\mathcal{L}_\nu + \mathcal{L}_{\text{gauge fields}} \tag{3.5}\]
with an arbitrary parameter \(a\), where
\[\mathcal{L}_A = -\frac{f^2}{4} \text{Tr}(D_{\mu}\xi_L \cdot \xi_L^\dagger \mp D_{\mu}\xi_R \cdot \xi_R^\dagger)^2, \tag{3.6}\]

which are the only invariants when the number of derivatives are less than three. \(\mathcal{L}_{\text{gauge fields}}\) stands for the kinetic terms of the external gauge fields \(A_{\mu L,R}\). We, however, include there also the kinetic term of the hidden local gauge field \(V_{\mu}\):
\[\mathcal{L}_{\text{gauge fields}} = -\frac{1}{2} \text{Tr}(F_{\mu\nu}^V)^2 - \frac{1}{2} \text{Tr}(F_{\mu\nu}^L)^2 - \frac{1}{2} \text{Tr}(F_{\mu\nu}^R)^2,\]
\[F_{\mu\nu}^V = \partial_{\nu}V_{\mu} - \partial_{\nu}V_{\mu} - ig[V_{\mu}, V_{\nu}]. \tag{3.7}\]

It is the central assumption of our approach that this kinetic term of \(V_{\mu}\) is developed by the quantum effects or by the underlying dynamics of QCD. As is easily shown, Lagrangian (3.5) reduces to the usual nonlinear Lagrangian (2.1) in the low energy limit when the external gauge fields \(A_{\mu L,R}\) are absent.

It would be instructive to give more explicit expression of Lagrangians (3.6), \(\mathcal{L}_A\) and \(\mathcal{L}_\nu\), for the case of the external gauge field being the electromagnetic field. After fixing the hidden local \(U(3)_{\nu}\) gauge by \(\xi_L = \xi_L^\dagger = e^{\imath\kappa_{\mu L}},\) we have
\[\mathcal{L}_A = -\frac{f^2}{4} \text{Tr} \partial_{\mu}U \partial^\mu U - 2ie \text{Tr} B^\rho Q[\pi, \partial_{\mu}\pi] + \cdots, \tag{3.8}\]
\[a\mathcal{L}_\nu = af^2 \text{Tr}\left(gV_{\mu} - eB_{\mu}Q - \frac{i}{2f^2}[\pi, \partial_{\mu}\pi] + \cdots\right)^2 \tag{3.9a}\]
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\[ = m_v^2 \text{Tr} V^\mu \cdot \text{Tr} V_\mu Q B^\mu + \left( \frac{eg_v}{m_v} \right)^2 \text{Tr} B_\mu Q B^\mu Q \]

\[- ig_v \cdot \text{Tr} V^\mu \pi, \varphi, \text{Tr} B^\mu \pi, \varphi \cdots \] (3.9b)

Here we read the following relations:

vector-photon mixing; \[ g_v = af_{\pi}^2 , \] (3.10a)

vector-meson mass; \[ m_v^2 = ag^2 f_{\pi}^2 , \] (3.10b)

vector-\pi \cdot \pi coupling; \[ g_{\pi \pi} = \frac{1}{2} ag . \] (3.10c)

We notice that the two relations

\[ \frac{1}{m_v^2} g_v = \frac{1}{g} \] (3.11)

and

\[ \frac{1}{m_v^2} g_{\pi \pi} = \frac{1}{2f_{\pi}^2 g} \] (3.12)

hold independently of the parameter \( a \). These equations will play important roles later in guaranteeing the consistency of this formalism in connection with anomaly as will be discussed in §5.

The ratio of (3.11) to (3.12) further yields

\[ \frac{g_v}{g_{\pi \pi}} = 2f_{\pi}^2 , \] (3.13)

which is independent of \( g \) also and agrees well with experiments. This relation, being known as KSRF relation, is characteristic to our framework and can be regarded as a "low energy theorem" of hidden local symmetry.\(^{11,12}\)

If we further take the parameter \( a \) as

\[ a = 2 , \] (3.14)

we have another form of the KSRF relation, \( m_v^2 = 2g_{\pi \pi} f_{\pi}^2 \), and the universality, \( g_{\pi \pi} = g \), together with the vector meson dominance of the electromagnetic form factor, i.e., the vanishing of \( B\pi \pi \) vertex.

§ 4. Solutions to the anomaly equation in the presence of vector mesons

In this section, we discuss general solutions to the Wess-Zumino anomaly equation for the present system including vector mesons. Of course we are only concerned with the part of the action having the smallest possible number of derivatives which is responsible for the intrinsic-parity violating processes.

The anomaly equation to be solved takes the same form as before:

\[ \delta \Gamma(\xi_L, \xi_R, V, A_L, A_R) = -10Ci \int_M \text{Tr} \left[ \varepsilon_L \left( (dA_L)^2 - \frac{i}{2} dA_L^3 \right) \right] - (L \leftrightarrow R) \] (4.1)
Here the gauge transformation $\delta$ may contain hidden local symmetry transformation $\delta_{\nu}(\nu)$, (3·2), since the action $\Gamma'$ must be invariant under it. On the other hand, since it is convenient to treat $\delta_{\nu}(\nu)$ on the same footing as the external gauge transformations $\delta_{L,R}(e_{L,R})$, we understand hereafter that $\delta$ contains $\delta_{\nu}(\nu)$ also, $\delta = \delta_{L}(e_{L}) + \delta_{\nu}(\nu) + \delta_{R}(e_{R})$:

$$\xi_{L,R} \rightarrow e^{i\alpha_{L,R}} \xi_{L,R} e^{-i\alpha_{L,R}},$$
$$\delta A_{L,R} = d e_{L,R} + i[e_{L,R}, A_{L,R}],$$
$$\delta V = d\nu + ig[\nu, V]. \quad (4·2)$$

The general solutions to inhomogeneous linear differential equation like (4·1) are given by a special solution of the inhomogeneous equation plus general solutions of the homogeneous equation. As a special solution we choose the original Wess-Zumino action (2·8):

$$\Gamma_{WZ}[U = \xi_{L}^{*}, \xi_{R}, A_{L}, A_{R}] = \left[ C \int_{M^{4}} \text{Tr} \ a^{2} \right]_{\text{covariantization}}. \quad (4·3)$$

The solutions of homogeneous equation (i.e., anomaly-free terms) are such ones as can be written explicitly as four-dimensional integrals over $M^{4}$ and they are made of gauge-covariant building blocks

$$\tilde{a}_{L(R)} = D_{L(R)} \cdot \xi_{L(R)}^{*} \xi_{L(R)} = a_{L(R)} - ig V + iA_{L(R)} ,$$
$$\tilde{a}_{L(R)} = d_{L(R)} \cdot \xi_{L(R)}^{*} \xi_{L(R)} , \quad \tilde{A}_{L(R)} = \xi_{L(R)} A_{L(R)} \xi_{L(R)} ,$$
$$F_{L(R)} = dV - ig V^{2} ,$$
$$\tilde{F}_{L(R)} = \xi_{L(R)}^{*} F_{L(R)} \cdot \xi_{L(R)} = \xi_{L(R)} (dA_{L(R)} - iA_{L(R)}^{2}) \xi_{L(R)} , \quad (4·4)$$

so that they are manifestly fully gauge-invariant (anomaly-free). We have six invariants in total that conserve parity but violate intrinsic parity:

$$\mathcal{L}_{1} = \text{Tr}(\tilde{a}_{L}^{3} \tilde{a}_{R} - \tilde{a}_{R}^{3} \tilde{a}_{L}) ,$$
$$\mathcal{L}_{2} = \text{Tr}(\tilde{a}_{L} \tilde{a}_{R} \tilde{a}_{L} \tilde{a}_{R}) ,$$
$$\mathcal{L}_{3} = i \text{Tr} \ F_{V}(\tilde{a}_{L}^{2} - \tilde{a}_{R}^{2}) ,$$
$$\mathcal{L}_{4} = i \text{Tr} \ F_{V}(\tilde{a}_{L} \tilde{a}_{R} - \tilde{a}_{R} \tilde{a}_{L}) ,$$
$$\mathcal{L}_{5} = i \text{Tr} \ F_{L}(\tilde{a}_{L}^{2} - \tilde{F}_{R} \tilde{a}_{L}^{2}) ,$$
$$\mathcal{L}_{6} = i \text{Tr} \ F_{L}(\tilde{a}_{L} \tilde{a}_{R} - \tilde{F}_{R} \tilde{a}_{R} \tilde{a}_{L}) . \quad (4·5)$$

Other possible terms such as $\text{Tr}(\tilde{a}_{L}^{2} \tilde{F}_{R} - \tilde{a}_{R}^{2} \tilde{F}_{L})$, $\text{Tr}(\tilde{a}_{L}^{2} D \tilde{a}_{R} - \tilde{a}_{R}^{2} D \tilde{a}_{L})$ are easily shown to be reducible to linear combinations of these six $\mathcal{L}$'s if we use the identities

$$D \tilde{a}_{L(R)} = -i F_{V} + i \tilde{F}_{L(R)} + \tilde{a}_{L(R)}^{2} ,$$
$$D \tilde{F}_{L(R)} = [\tilde{a}_{L(R)}, \tilde{F}_{L(R)}] , \quad DD \tilde{a}_{L(R)} = i [\tilde{a}_{L(R)}, F_{V}] . \quad (4·6)$$

Thus, as a part of the action responsible for the intrinsic parity violation processes, we have
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\[ \Gamma = \Gamma_{\text{WZ}}[\xi_L, A_L, A_R] + \frac{\delta}{\delta A} \int_M c_i L_i, \tag{4.7} \]

c_i being arbitrary constants which are to be fixed by experiments. Note that as a solution to the anomaly equation (4.1), the Wess-Zumino action \( \Gamma_{\text{WZ}}[\xi_L, A_L, A_R] \), i.e., the solution with \( c_i = 0 \), is by no means more "irreducible" or "minimal" than the others with \( c_i \neq 0 \). For instance, the action

\[ \Gamma_{\text{WZ}}[\xi_R, V, A_R] - \Gamma_{\text{WZ}}[\xi_L, V, A_L] = [C \int_M \text{Tr}(a_R^5 - a_L^5)]_{\text{covariantization}} \tag{4.8} \]

is also a solution to the anomaly equation (4.1) as is seen from the property of the Wess-Zumino function \( \Gamma_{\text{WZ}} \):

\[ \delta \Gamma_{\text{WZ}}[\xi_{LR}, V, A_{LR}] = -10Ci \int_M \text{Tr} \left[ \nu \left( (dV)^2 - \frac{i}{2} dV^3 \right) - \epsilon_{LR} \left( (dA_{LR})^2 - \frac{i}{2} dA_{LR} \right) \right]. \tag{4.9} \]

Solution (4.8) does have non-zero \( c_i \)'s but still looks "irreducible" as the original Wess-Zumino action (4.3).

§ 5. Consistency of vector meson contribution

with low energy theorems

Now we come to the point where we can completely answer the questions given in the Introduction.

The amplitudes for \( \pi^0 \rightarrow 2\gamma \) and \( \gamma \rightarrow 3\pi \) are given as low energy theorems: Those values at soft-momentum limit are determined solely by the anomaly, and do not depend on other parameters of the theory. This implies for our system (4.7) that any of the Lagrangians \( L_i (i = 1 \sim 6) \) should not contribute to \( \pi^0 \rightarrow 2\gamma \) and \( \gamma \rightarrow 3\pi \), since each \( L_i \) has no anomaly, being a solution of the homogeneous equation.

In order to see that the consistency is actually realized in our framework, let us examine Lagrangian \( L_6 \) as an example.

Setting the external gauge fields \( A_L \) and \( A_R \) to be the photon field \( B \), \( A_L = A_R = eB_{\mu}d\xi^\mu = eB^\xi \), and using (4.4), \( \alpha_{LR} = \mp if_{\pi^{-1}}d\pi + \cdots \), the relevant terms in \( L_6 \) become

\[ L_6 = i \text{Tr}(\tilde{F}_{\xi L}^\xi \tilde{a}_{LR} - \tilde{F}_{\xi R}^\xi \tilde{a}_{RL}) \]
\[ = ief_{\pi^{-1}}[2g \text{Tr}(Vd\xi^0) + 2g \text{Tr}(dB^0 Vd\pi) - 4e \text{Tr}(B^0 dB^0 d\pi)] \]
\[ + i(if_{\pi^{-1}})^3 4e \text{Tr} B^0(d\pi)^3 + \cdots. \tag{5.1} \]

The third term \( \text{Tr} B^0 dB^0 d\pi \) directly contributes to \( \pi^0 \rightarrow 2\gamma \) and the fourth term \( \text{Tr} B^0(d\pi)^3 \) to \( \gamma \rightarrow 3\pi \). They are, however, in fact cancelled by the contribution from the first and second terms representing \( \pi \cdot \gamma \cdot V \) (vector mesons) vertices.

The behavior of vector mesons is described by the following part of the Lagrangian in §3:

\[ \mathcal{L} = -af_{2}^2 \text{Tr}(\bar{a}_\mu - igV_\mu + i\bar{A}_\mu)^2 - \frac{1}{2} \text{Tr}(F_{\mu\nu}^V)^2 \]
\[ = 2\text{Tr} \left\{ \frac{1}{2} V^\mu [(\Box + m^2)g_{\mu\nu} - \partial_\mu \partial_\nu] V^\nu - eg V^\mu B_\mu^0 - ig_{\nu\pi} V^\mu [\pi, \partial_\nu] + \cdots \right\}, \tag{5.2} \]
where
\[ \tilde{a}_\mu = \frac{1}{2} (a_{L\mu} + a_{R\mu}) = [\partial_{\mu} \pi, \pi]/2f_\pi^2 + \cdots, \] (5.3a)
\[ \tilde{A}_\mu = \frac{1}{2} (A_{L\mu} + A_{R\mu}) \Rightarrow eB^\rho \mu, \] (5.3b)
and \( g_\nu, m_\nu^2 \) and \( g_{\nu\pi \pi} \) are given by (3.10) and satisfy (3.11) and (3.12). Equation (5.2) leads to the Feynman rules for the vector meson propagator and the \( V\cdot \gamma \) and \( V\cdot \pi\cdot \pi \) vertices, as shown in Fig. 1. Consider a subprocess in a Feynman diagram in which a vector \( V_\nu^a \) is created at a vertex, propagates and then changes into an on-shell photon with physical polarization \( B_\nu(\rho) \). From the rules (a) and (b) in Fig. 1, we have for that part of the diagram

\[ V_\nu^a \rightarrow \frac{e}{g} 2\text{Tr} T^a Q B_\nu(\rho). \] (5.4)

(a) \[ \frac{v_\mu^a}{p} \frac{v_\nu^b}{v} = -\delta^{ab} (g_{\mu\nu} - \frac{p_\mu p_\nu}{m_\nu^2})/(m_\nu^2 - p^2), \]
(b) \[ \frac{\mu^a}{a} \frac{v}{b} = -eg_\nu g_{\mu\nu} (2\text{Tr} T^a Q), \]
(c) \[ \frac{p^a}{a} \frac{q}{b} \frac{c}{k} = (-ig_{\nu\pi \pi}) i(k - q)_\mu (2\text{Tr} T^a[T^b, T^c]). \]

Fig. 1. Feynman rules for (a) vector propagator, (b) vector-photon vertex and (c) vector-pseudoscalar-pseudoscalar vertex.

Notice here that \( m_\nu^2 \) was cancelled by \( g_\nu = m_\nu^2/g \) due to relation (3.11). This implies that the vector meson \( V_\nu^a \) at a vertex, when connected via \( V\cdot \gamma \) vertex with an external physical photon line, is effectively replaceable by a photon field \( B_\mu \) (multiplied by a factor):

\[ V_\nu^a \rightarrow B_\mu \times \left( \frac{e}{g} \right) 2\text{Tr} T^a Q. \] (5.5)

If the \( V_\nu^a \)'s appear at the vertex in a matrix form as \( \text{Tr} MV_\mu, M \) being a certain matrix, then this replacement becomes

\[ g \text{Tr} MV_\mu = g \sum_{a=0}^8 (\text{Tr} M T^a) V_\nu^a \]
\[ \rightarrow e^a \sum_{p^2=0}^8 (\text{Tr} M T^a) 2(\text{Tr} T^a Q) B_\mu, \]
\[ = e\text{Tr}(MB_\mu^a), \] (5.6)
where use has been made of the completeness relation \( \sum_{a=0}^{3} (\text{Tr} A T^a)2(\text{Tr} T^a B) = \text{Tr} AB \); that is, the vector field (matrix) \( g V_\mu \) becomes equivalent to the physical photon field \( eB_\mu \). Similarly, if the vector line is connected via \( V\cdot \pi \cdot \pi \) vertex to two on-shell pseudoscalar particles (which are massless now in the chiral symmetry limit), the vector field \( V_\mu \) is replaceable by \([\pi, \partial_\mu \pi]\) times a constant:

\[
q
\frac{p}{V_\mu} - \frac{p_\mu p_\nu}{m_\pi^2 - p^2} (g_{\nu \pi}^{\pi} i(k - q)^\nu (2\text{Tr} q^a [\pi(q), \pi(k)])) \quad \text{times a constant.}
\]

\[
i.e., 
\frac{g}{2 f_\pi} M V_\mu 
\rightarrow 
\frac{i}{2 f_\pi} M V_\mu.
\]

Here notice that the "propagator" \( m_\pi^{-2} \) was again cancelled by \( g_{\nu \pi}^{\pi} = m_\pi^2/2g_\pi^2 \) (3·12).

Summarizing (5·6) and (5·8), we can make the replacement

\[
g V_\mu \rightarrow eB_\mu + i(2f_\pi^2)^{-1}[\pi, \partial_\mu \pi]
\]

effectively in the interaction Lagrangians when the \( V_\mu \) is converted into a photon or two pseudoscalars via \( V\cdot \gamma \) or \( V\cdot \pi \cdot \pi \) vertices. It should be kept in mind that replacement (5·9) becomes exact only when

\[
\begin{align*}
\text{the square of 4-momentum of } V_\mu \text{ propagator is zero,} \\
\gamma \text{ and } \pi \text{ are on the zero-mass shell,} \\
\gamma \text{ has physical polarization,}
\end{align*}
\]

as is clear from the above derivation.

Now with this result (5·9) we can easily show that the Lagrangian \( \mathcal{L}_5 \) does neither contribute to \( \pi^0 \rightarrow 2\gamma \) nor \( \gamma \rightarrow 3\pi \) as announced above. Actually, the first two terms of (5·1) containing the vector meson \( V_\mu \) contribute to \( \pi^0 \rightarrow 2\gamma \) and \( \gamma \rightarrow 3\pi \) effectively as

\[
2gef_\pi^{-1}\text{Tr}\{V, dB^0\}d\pi
\quad \rightarrow 
4e^2f_\pi^{-1}\text{Tr} B^0 dB^0 d\pi + ief_\pi^{-3}\text{Tr}\{[\pi, d\pi], dB^0\}d\pi.
\]

These first and second terms on the RHS come from the diagrams in Figs. 2(a) and (b), respectively, and just cancel the direct \( \pi^0 \rightarrow 2\gamma \) and \( \gamma \rightarrow 3\pi \) terms, the third and fourth terms of (5·1).

Similarly we can easily show that all the \( \mathcal{L}_i \) 's do not contribute to \( \pi^0 \rightarrow 2\gamma \) and \( \gamma \rightarrow 3\pi \) owing to the cancellations between direct terms and vector meson mediated terms, which is consistent with the fact that they are anomaly-free. Here we should recall that these cancellations are complete only at a special set of momentum values:

\[
i
q^2 = q_1^2 = q_2^2 = 0 \quad \text{for } \pi^0(\rho) \rightarrow \gamma(q_1) + \gamma(q_2),
\]

\[
ii
q^2 = q_1^2 = q_2^2 = q_3^2 = q_1 \cdot q_2 = q_2 \cdot q_3 = q_3 \cdot q_1 = 0
\quad \text{for } \gamma(\rho) \rightarrow \pi^+(q_1) + \pi^-(q_2) + \pi^0(q_3).
\]
These conditions come from (5\cdot10) and exactly coincide with the momentum values at which the low energy theorems determine the amplitudes for these processes.\textsuperscript{16,17}

Thus for photon processes $\pi^0 \rightarrow 2\gamma$ and $\gamma \rightarrow 3\pi$ we have shown that anomaly-free actions do not contribute to their amplitudes. This consistency of our framework is not restricted to the photon case but still holds for general $U(3) \times U(3)$ external gauge fields $A_L$ and $A_R$. This is shown as follows. The essential point in the above proof for the photon case was the replacement rule (5\cdot9). In contrast (5\cdot9) is nothing but an equation of motion of $V_\mu$ coming from Lagrangian (5\cdot2) if the kinetic term $-\frac{1}{2}\mathrm{Tr}(F_{\mu\nu}^2)$ is neglected:

\[
g V_\mu = \frac{1}{2} (\alpha_{L\mu} + \alpha_{R\mu}) + \frac{1}{2} (A_{L\mu} + A_{R\mu}) \tag{5\cdot13a}
\]

or

\[
\tilde{\alpha}_{L\mu} + \tilde{\alpha}_{R\mu} = 0. \tag{5\cdot13b}
\]

Indeed, when the external field is only the photon, $A_{L\mu} = A_{R\mu} = eB_\mu Q$, Eq. (5\cdot13a) reduces to (5\cdot9) up to higher-order terms owing to (5\cdot3a). Therefore for the general external gauge field case, the contributions of vector meson mediated diagrams are equivalently evaluated by eliminating $V_\mu$ by the use of the equation of motion (5\cdot13a), or equivalently (5\cdot13b). [The condition for the vector meson kinetic term to be neglected is met when $p^2 = 0$ with $p_\mu$ being the 4-momentum of vector meson line and the external particles are set on their physical (zero) mass-shells and physical polarizations.] Since $\tilde{\alpha} = DU \cdot U^\dagger$ and $U = \xi_L^L \xi_R$, we have $\tilde{\alpha} = \xi_L^L (\tilde{\alpha}_R - \tilde{\alpha}_L) \xi_L$. Hence the use of (5\cdot13b) implies the following replacements for $\tilde{\alpha}_R$ and $\tilde{\alpha}_L$:

\[
\tilde{\alpha}_L^R = \frac{1}{2}(\tilde{\alpha}_R + \tilde{\alpha}_L) \pm \frac{1}{2}(\tilde{\alpha}_R - \tilde{\alpha}_L) \rightarrow \pm \frac{1}{2} \xi_L \tilde{\alpha} \xi_L^L. \tag{5\cdot14}
\]

With (5\cdot14) alone the Lagrangians $\mathcal{L}_{1-6}$ in (4\cdot5) are shown not to contribute:

\[
\mathcal{L}_{1,2} \rightarrow \frac{1}{2} \mathrm{Tr} \tilde{\alpha}^4 = \mathrm{Tr} \tilde{\alpha}^3 \tilde{\alpha} = - \mathrm{Tr} \tilde{\alpha} \tilde{\alpha}^2 = 0, \tag{5\cdot15}
\]

\[
\mathcal{L}_{3,4} \rightarrow \frac{1}{2} \mathrm{Tr} \xi_L^L F_{\mu\nu} \xi_L (\tilde{\alpha}^2 - \tilde{\alpha}^2) = 0, \tag{5\cdot16}
\]

\[
\mathcal{L}_{3,4} \rightarrow \mathrm{Tr} (F_L - UF_R U^\dagger) \tilde{\alpha}^2 = i \mathrm{Tr} (D \tilde{\alpha} \cdot \tilde{\alpha}^2 - \tilde{\alpha}^4)
\]

\[
= i \mathrm{Tr} D \tilde{\alpha}^2 = id(\mathrm{Tr} \tilde{\alpha}^2) = \text{total derivative}, \tag{5\cdot17}
\]

where we have used in (5\cdot17) the identity $D \tilde{\alpha} - \tilde{\alpha}^2 = -i(F_L - UF_R U^\dagger)$. 

---

Fig. 2. Diagrams (a) and (b) producing the first and second terms of (5\cdot11) respectively.
§ 6. Vector meson dominance?

We have shown that inclusion of the Lagrangians $L_{1-6}$ does not change the amplitudes of the processes containing only pseudoscalar and $A_{L,R}$ particles in the initial and final states (at “zero-momentum” points like (5.12)). However it changes the Feynman diagram contents contributing to such processes. Varying the parameters $c_{1-6}$ in Lagrangian (4.7) changes the contribution rate of vector meson mediated diagrams to direct diagrams.

There is a particular choice of parameters in which an exact vector meson dominance is realized for a particular process in question. Let us discuss the vector meson dominance for the $\pi^0 \rightarrow 2\gamma$ process and for the $\gamma \rightarrow 3\pi$. The relevant terms are denoted by
\[
\gamma\pi^3 = (-2ie^2/f_\pi) Tr(A_dA_d\pi + dAA_d\pi),
\]
\[
V\gamma\pi^3 = (-2ige/f_\pi) Tr(VdA_d\pi + dAVd\pi),
\]
\[
VV\pi^3 = (-2ig^2/f_\pi) Tr(VdVd\pi + dVVd\pi),
\]
\[
\gamma\pi^3 = 4ie(i/f_\pi) Tr A(d\pi)^3,
\]
\[
V\pi^3 = 4ig(i/f_\pi) Tr V(d\pi)^3,
\]
and they are contained in each term of Lagrangian (4.7) as follows:
\[
(5C)^{-1} \Gamma_{W_2} = 3(\gamma\pi^3) + 0 + 0 + 4(\gamma\pi^3) + 0 + \ldots,
\]
\[
L_1 = 0 + 0 + 0 + 0 - 1(\gamma\pi^3) + 1(V\pi^3) + \ldots,
\]
\[
L_2 = 0 + 0 + 0 + 1(\gamma\pi^3) - 1(V\pi^3) + \ldots,
\]
\[
L_3 = 0 - 1(V\gamma\pi) + 0 + 0 + 0 + \ldots,
\]
\[
L_4 = 0 + 1(V\gamma\pi) - 1(VV\pi) + 0 + 1(V\pi^3) + \ldots,
\]
\[
L_5 = 0 - 1(V\gamma\pi) + 0 + 0 + 0 + \ldots,
\]
\[
L_6 = 1(\gamma\pi^3) + 0 + 1(\gamma\pi^3) + 0 + \ldots.
\]

Here the “photon” field $A$ may be the true photon $B_{\mu}dx^\mu Q$ or may be its $U(3)$ generalization, $A_L = A_R = eA$. The Wess-Zumino action $\Gamma_{W_2}$ of course contains direct $\pi^0 \rightarrow 2\gamma$ and $\gamma \rightarrow 3\pi$ terms only. From (6.2) we see that if we add $L_6$ with coefficient $c_6 = -3.5C$ to $\Gamma_{W_2}$ then the $\pi^0 \rightarrow 2\gamma$ process is converted to that occurring solely via $\pi \rightarrow V\gamma$ vertex, and by adding $L_4$ with $c_4 = -3.5C$ it is further converted to that only through $\pi \rightarrow VV$ vertex:
\[
\Gamma_{W_2} - 15C(L_4 + L_6) = 5C[3(VV\pi) + (\gamma\pi^3) - 3(V\pi^3)] + \ldots.
\]

At this stage, $\pi^0 \rightarrow 2\gamma$ is completely dominated by the vertex $\pi^0 \rightarrow \rho^0 + \omega^0$ followed by $\rho^0 \rightarrow \gamma$, $\omega^0 \rightarrow \gamma$, but $\gamma \rightarrow 3\pi$ process still has a contribution from direct $\gamma \rightarrow 3\pi$ vertex $(\gamma\pi^3)$. If we further add $5C(aL_1 + bL_2)$ with $a - b = 1$ to (6.3), the direct $(\gamma\pi^3)$ term is converted to $(V\pi^3)$ term, realizing vector dominance also for $\gamma \rightarrow 3\pi$ process accordingly:
\[
\Gamma_{W_2} - 15C(L_4 + L_6) + 5C(aL_1 + bL_2) = 5C[3(VV\pi) - 2(V\pi^3)] + \ldots.
\]
If this action were good, we would have a simple principle, say "complete vector meson dominance", to determine the arbitrary parameters \( c_{1}\ldots c_{6} \). Action (6.4), however, predicts the \( \omega \to 3\pi \) amplitude which does not agree so well with the experiment as will be seen shortly. The phenomenology seems prefer the existence of the direct \( (\gamma\pi^{3}) \) term to that of the \( (\omega\pi^{3}) \) one. Thus the action consistent with experiments is now taken as

\[
\Gamma_{WZ} - 15C(\mathcal{L}_{4} + \mathcal{L}_{5} + \bar{c}_{1}\mathcal{L}_{1} + \bar{c}_{2}\mathcal{L}_{2})|_{\bar{c}_{1}, \bar{c}_{2} = -1} = 5C[3(VV\pi) - 2(\gamma\pi^{3})] + \cdots, \tag{6.5}
\]

which we propose as a phenomenological action for the intrinsic parity violating processes of Ps, \( \gamma \) and vector mesons.

We now explicitly give some numerical predictions of the action (6.5). First of all, the prediction for \( \pi^{0} \to 2\gamma \) is of course identical with the current algebra result: \(^{16,17}\)

\[
\Gamma(\pi^{0} \to 2\gamma) = \frac{\alpha^{2}}{64\pi^{3}} \frac{m_{\pi}^{2}}{f_{\pi}^{2}} \approx 7.6 \text{ eV}, \tag{6.6}
\]

by the use of the values \( \alpha \approx 1/137, m_{\pi} \approx 135\text{MeV} \) and \( f_{\pi} \approx 93\text{MeV} \), which excellently agrees with the experimental value of \( 7.9\text{eV} \). Although (6.6) is identical with the current algebra result, \(^{16,17}\) this process now by construction occurs totally through vertex as is shown in Fig. 3. The coupling constant \( g_{\omega\rho\pi} \) which is defined by

\[
\mathcal{L}_{V\rho\pi} = 2g_{\omega\rho\pi} \epsilon_{\mu\nu\rho\sigma} \text{Tr}(\partial_{\mu} V_{\nu} \partial_{\rho} V_{\sigma\pi})
= g_{\omega\rho\pi} \epsilon_{\mu\nu\rho\sigma} \partial_{\mu} \omega_{\nu} \partial_{\rho} \sigma \cdot \pi + \cdots, \tag{6.7}
\]

is now fixed by the anomaly:

\[
g_{\omega\rho\pi} = -\frac{3g^{2}}{8\pi^{2}f_{\pi}}, \tag{6.8}
\]

where \( g \) is the \( V_{\mu} \)-gauge coupling constant and phenomenologically equals \( g_{\rho\pi} \) through (3.10c) and (3.13). Next, the decay \( \omega \to \pi^{0}\gamma \) is also completely dominated by the \( \omega\rho\pi \) vertex as shown in Fig. 4 and is predicted as

\[
\Gamma(\omega \to \pi^{0}\gamma) = \frac{3\alpha}{64\pi^{3}} \frac{g^{2}}{f_{\pi}^{2}} |q_{\pi}|^{2} \approx 0.84 \text{ MeV}, \tag{6.9}
\]

in a nice agreement with the observed value \( 0.86 \pm 0.05\text{MeV} \), where the value for \( g \approx g_{\rho\pi} \) is taken as \( g_{\rho\pi}^{2} / 4\pi \approx 3.0 \) so that

\[
\Gamma(\rho \to 2\pi) = \frac{2}{3m_{\rho}^{2}} \frac{g_{\rho\pi}^{2}}{4\pi} |q_{\pi}|^{2} \tag{6.10}
\]

reproduces the experimental width. Finally the decay \( \omega \to 3\pi \) is also dominated by the \( \omega\rho\pi \) vertex. Only the diagrams of the type shown in Fig. 5 contribute and give the width:

![Fig. 3. Feynman diagram for \( \pi^0 \to 2\gamma \).](image-url)
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\[
\Gamma(\omega \rightarrow 3\pi) = \frac{m_\omega}{192\pi^3} \left| \int dE_+ dE_- [q_+ q_+ - (q_+ q_-)^2] F^{\omega \rightarrow 3\pi} \right|^2,
\]

\[
F^{\omega \rightarrow 3\pi} = 2 g_{\omega\rho\pi} g_{\rho\pi\pi} \left[ \frac{1}{(q_+ + q_-)^2 - m_{\rho_0}^2} + \frac{1}{(q_+ + q_0)^2 - m_{\rho_+}^2} + \frac{1}{(q_- + q_0)^2 - m_{\rho_-}^2} \right],
\]

where \( E_+ \) and \( E_- \) are the energies of \( \pi^+ \) and \( \pi^- \) in the rest frame of \( \omega \). Integral (6.11) is evaluated numerically by taking \( m_{\pi_\nu}^2 \approx 1.91 \times 10^{-2} \text{GeV}^2 \), \( m_{\rho} = 0.769 \text{GeV} \), \( m_\omega = 0.783 \text{GeV} \) and the same values as the above for \( g \approx g_{\rho\pi\pi} \) and \( f_\pi \), yielding

\[
\Gamma(\omega \rightarrow 3\pi) \approx 9.1 \text{ MeV},
\]

which also nicely agrees with the experimental value 8.9\( \pm \)0.3 MeV.

The diagrams of Figs. 3–5 contributing to \( \pi^0 \rightarrow 2\gamma \), \( \omega \rightarrow \pi^0 \gamma \) and \( \omega \rightarrow 3\pi \), respectively, are exactly identical with those of Gell-Mann, Sharp and Wagner (GSW) model and therefore the above nice agreement with experimental data may be considered to be reconfirmation of the GSW model, aside from a point that the absolute value of \( \Gamma(\pi^0 \rightarrow 2\gamma) \) is now predicted by the anomaly.

As promised we here show how “bad” prediction the choice of action (6.4) corresponding to “complete vector meson dominance” assumption gives for the \( \omega \rightarrow 3\pi \) decay. Since action (6.4) contains \( (V\pi^3) \) contact term, the amplitude \( F^{\omega \rightarrow 3\pi} \) in (6.11) is now replaced by

\[
F^{\omega \rightarrow 3\pi} = \frac{3 g_\pi g_{\rho\pi\pi}}{4\pi^2 f_\pi} \left[ \frac{1}{m_{\rho_0}^2 - (q_+ + q_-)^2} + 2\text{-terms} \right] \frac{3 g}{8\pi^2 f_\pi^3}. \tag{6.13}
\]

Here the last term is the new contribution of the contact \( \omega \rightarrow 3\pi \) vertex shown in Fig. 6. Using the same values as above for the parameters, we obtain from (6.13)

\[
\Gamma(\omega \rightarrow 3\pi) \approx 6.1 \text{MeV} \text{ “complete vector meson dominance”} \tag{6.14}
\]

which is about 30% lower than the experimental value 8.9 MeV.

In view of the excellent agreements with data of the previous predictions (6.6), (6.9) and (6.12) based on action (6.5), and from the experiences that current-algebra predictions can usually be trusted to within 20%, we should rather judge the 30% disagreement of (6.14) being significant and hence the “complete vector meson dominance” assumption is not valid in either \( \gamma \rightarrow 3\pi \) or \( \pi^0 \rightarrow 2\gamma \) process. In our action (6.5) the direct photon-3\( \pi \) coupling term \( (\gamma\pi^3) \) is present since we have chosen the parameters \( c_i \) so that the vector meson remains dominant in \( \gamma \rightarrow 2\gamma \).

It should finally be remarked that this result implies that the naive GSW model does not apply to the \( \gamma \rightarrow 3\pi \) process. In a recent paper Freund and Zee have calculated \( \gamma \rightarrow 3\pi \) amplitude as well as \( \pi^0 \rightarrow 2\gamma \) based on the GSW model (=vector meson dominance)
and compared them with the anomaly predictions and obtained*)

\[ \frac{g_{\rho}}{g_{\rho\pi\pi}} = 3 f_\pi^2. \]  \hspace{1cm} (6.15)

This disagrees with the successful KSRF relation (3.13). If one correctly takes account of the direct \((\gamma\pi^3)\) term of (6.5), the factor 3 on the RHS of (6.15) becomes 2, hence agreeing with KSRF relation.

§ 7. Summary and discussion

We have presented general solutions to the Wess-Zumino anomaly equation which incorporate the vector mesons as dynamical gauge bosons of the hidden local symmetry in the nonlinear chiral Lagrangian. The vector mesons have been naturally introduced into the chiral Lagrangian with Wess-Zumino term so as not to disturb the well-established predictions of the low energy theorems associated with the anomaly such as the amplitudes of \(\pi^0 \to 2\gamma\) and \(\gamma \to 3\pi\). This is sharply contrasted with the previous attempts to incorporate the vector mesons based on the old “massive Yang-Mills” approach, in which the above consistency is not so straightforwardly satisfied if one keeps the chiral symmetry. We have further clarified the meaning of the “vector meson dominance” of photon couplings and constructed an effective action consistent with the low energy theorems and the present experimental data.

Finally we should make a comment on relations (3.11) and (3.12) among mass and coupling constants of vector mesons:

\[ \frac{g_{\nu}}{m_{\nu}^2} = \frac{1}{g}, \]

\[ \frac{g_{\nu\pi\pi}}{m_{\nu}^2} = \frac{1}{2f_\pi^2g}. \]  \hspace{1cm} (7.1)

By using these relations we have shown the consistency of our framework that anomaly free actions do not contribute to the processes such as \(\pi^0 \to 2\gamma\) and \(\gamma \to 3\pi\) which are subject to low energy theorems. However we can reverse this argument. If instead we demand that those amplitudes of \(\pi^0 \to 2\gamma\) and \(\gamma \to 3\pi\) have no contributions from any anomaly free actions allowed by the symmetry in our framework, then we can derive the above relations at “soft” momentum points:

\[ \frac{g_{\nu}(p^2=0)}{m_{\nu}^2(p^2=0)} = \frac{1}{g}, \]

\[ \frac{g_{\nu\pi}(p^2=0, q_1^2=0, q_2^2=0)}{m_{\nu}^2(p^2=0)} = \frac{1}{2f_\pi^2g}. \]  \hspace{1cm} (7.2)

*) See the Erratum of their paper.*)
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Notice that \( g_V, g_{\pi\pi} \) in (7·1) are the coupling constants given by the particular choice of Lagrangian (3·5), whereas \( g_V(p^2), g_{\pi\pi}(p^2, q_1^2, q_2^2) \) in (7·2) are the corresponding quantities given by a general Lagrangian.\(^\text{(12)}\) Equation (7·2) is derived irrespectively of particular form of the Lagrangian. In this sense the relations are the "low energy theorems" concerning vector mesons as gauge bosons of hidden local symmetry. Precisely the same type of argument was given in Ref. 12, where the same relation (7·2) was derived by appealing to the low energy theorems about the \( \pi-\pi \) scattering and on-shell photon coupling to matter.

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