Weak Field Approximation of New General Relativity

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In the weak field approximation, gravitational field equations of new general relativity with arbitrary parameters are examined. Assuming a conservation law \( \partial^\nu T_{\mu\nu} = 0 \) of the energy-momentum tensor \( T_{\mu\nu} \) for matter fields in addition to the usual one \( \partial^\nu T_{\mu\nu} = 0 \), we show that the linearized gravitational field equations are decomposed into equations for a Lorentz scalar field and symmetric and antisymmetric Lorentz tensor fields.

§ 1. Introduction

New general relativity\(^1\) is a theory of gravitation based on the Weitzenböck spacetime endowed with absolute parallelism, which contains torsion and identically vanishing curvature. The gravitational field equations, which are derived from the sum of matter action and gravitational action quadratic in the torsion with arbitrary weights \( c_1, c_2 \) and \( c_3 \), can naturally contain spinor fields as source matter.

In Ref. 1), Hayashi and Shirafuji discussed the weak field approximation of new general relativity in the case with \( c_1 = c_2 = 0 \). They showed that there exists a characteristic antisymmetric field besides a graviton, which is originated from intrinsic spins of spinor fields. Then the field equations for the graviton and the antisymmetric field are completely decoupled, which is not in general realized in the case with \( c_1 \neq 0 \neq c_2 \).

In a previous paper,\(^2\) the authors determined linear field equations for massless tensor fields in the Minkowski spacetime. Then they required the assumptions, keeping in mind that massless fields propagate as transverse waves:

(a) The field equation is linear and contains only second order derivatives of the field with the metric \( \eta_{\mu\nu} \) and \( \eta^{\mu\nu} \).
(b) The field equation has the same symmetry as the field for tensor indices.
(c) The field equation allows a gauge transformation which preserves the symmetry of the field and is written by first order derivatives of other tensor fields whose ranks are lower than that of the original field.
(d) The field can be made traceless and divergence free by means of the gauge transformation.

In the case with a symmetric tensor field \( \chi_{\mu\nu} \) and an antisymmetric tensor field \( A_{\mu\nu} \), the free field equations and their gauge transformations are respectively given by:

\[
\Box \chi_{\mu\nu} - (\partial_\mu \partial^\nu \chi_{\rho\nu} + \partial_\nu \partial^\mu \chi_{\rho\mu}) + p \eta_{\mu\nu} \partial^\alpha \partial^\sigma \chi_{\rho\sigma} + (-p + q - 3pq) \eta_{\mu\nu} \Box \chi
+ (1 + 2q) \partial_\mu \partial_\nu \chi = 0,
\]

\[
\chi_{\mu\nu} = \chi_{\mu\nu} + (\partial_\mu \xi_\nu + \partial_\nu \xi_\mu) - \frac{2q}{1 + 4q} \eta_{\mu\nu} \partial^\alpha \xi_\alpha, \tag{1.1a}
\]

\[
\chi_{\mu\nu} = \chi_{\mu\nu} + (\partial_\mu \xi_\nu + \partial_\nu \xi_\mu) - \frac{2q}{1 + 4q} \eta_{\mu\nu} \partial^\alpha \xi_\alpha, \tag{1.1b}
\]
\[ \square A_{\mu \nu} - (\partial_{\mu} \partial^{\rho} A_{\rho \nu} - \partial_{\nu} \partial^{\rho} A_{\rho \mu}) = 0, \quad (1.2a) \]
\[ A_{\mu \nu} = A_{\mu \nu} + (\partial_{\mu} \xi_{\nu} - \partial_{\nu} \xi_{\mu}), \quad (1.2b) \]

with \( p \) and \( q \) being free parameters excluding \( q = -1/4 \). Here \( \xi_{\mu} \) and \( \zeta_{\mu} \) are arbitrary vector fields representing gauge transformations. Owing to the gauge transformations, there remain two independent components for \( \chi_{\mu \nu} \) and only one independent component for \( A_{\mu \nu} \), propagating as transverse waves.

In this paper, we examine the weak field approximation of new general relativity with free parameters, \( c_1, c_2 \) and \( c_3 \). By means of adequate definition of irreducible fields, the gravitational field equations can be decomposed into three parts in the case with \( c_1 + c_2 \neq 0 \), provided symmetric and antisymmetric parts of the energy-momentum tensor for matter fields individually satisfy conservation laws. Consequently, field contents of new general relativity are a massless Lorentz scalar field, a massless Lorentz tensor field with helicity 2 and parity +1, and a massless Lorentz tensor field with helicity 0 and parity +1, propagating as transverse waves.

In §2, new general relativity is briefly summarized. The linearized gravitational field equations are examined in §3. The last section is devoted to conclusions.

§ 2. New general relativity

In new general relativity, fundamental entity is parallel vector fields \( b^k = \{ b^k_\mu \} \) with their inverse \( b_k = \{ b_k^\mu \} \), where the Latin and Greek indices run from 0 to 3 representing an internal Lorentz frame and a general coordinate frame, respectively. The metric tensor is defined by

\[ g_{\mu \nu} = \eta_{km} b^k_\mu b^m_\nu \quad (2.1) \]

with \( \eta_{km} \) denoting the Lorentz metric \( \eta_{km} = \text{diag}(-1, 1, 1, 1) \).

Spacetime of new general relativity is the Weitzenböck spacetime endowed with absolute parallelism:

\[ D_\nu b^k = 0, \quad (2.2a) \]

or equivalently,

\[ D_\nu b^k_\mu = \partial_\nu b^k_\mu - \Gamma^k_{\lambda \mu} b^\lambda = 0. \quad (2.2b) \]

The affine connection \( \Gamma^k_{\lambda \mu} \) is solved as

\[ \Gamma^k_{\lambda \mu} = b^k_\lambda \partial_\nu b^\mu. \quad (2.3) \]

Curvature tensor \( R^k_{\sigma \nu \mu}(\Gamma) \) and torsion tensor \( T^k_{\nu \mu}(\Gamma) \) are respectively defined by

\[ R^k_{\sigma \nu \mu}(\Gamma) = \partial_\mu \Gamma^k_{\sigma \nu} - \partial_\nu \Gamma^k_{\sigma \mu} + \Gamma^k_{\lambda \mu} \Gamma^\lambda_{\sigma \nu} - \Gamma^k_{\lambda \nu} \Gamma^\lambda_{\sigma \mu} = 0, \quad (2.4) \]

\[ T^k_{\nu \mu}(\Gamma) = \Gamma^k_{\nu \mu} - \Gamma^k_{\nu \mu}. \quad (2.5) \]

The curvature tensor is identically vanishing because of the form of the affine connection (2.3).

Gravitational Lagrangian \( L_G \) is quadratic in the torsion and invariant under general coordinate transformations, internal global Lorentz transformations and parity.
transformation. It can be rewritten in the form,\(^1\)

\[
L_0 = \frac{1}{2\chi} \left[ R(\{ \}) + 2c_1(t^{\lambda\nu}t_{\lambda\nu}) + 2c_2(v^\nu v_\nu) + 2c_3(a^\nu a_\nu) \right] + \frac{1}{\sqrt{-g}} \times \text{(total derivative)},
\]

where \(\chi\) is the Einstein constant and the parameters \(c_1, c_2\) and \(c_3\) should be determined by experiments. The symbol \(R(\{ \})\) represents the Riemann-Christoffel scalar curvature made of the Christoffel symbols. Here the torsion tensor is divided into its irreducible parts as:

\[t_{\lambda\mu\nu} \equiv \frac{1}{2} (T_{\lambda\mu\nu} + T_{\mu\lambda\nu}) + \frac{1}{6} (g_{\lambda\nu} v_\mu + g_{\mu\nu} v_\lambda - 2g_{\lambda\mu} v_\nu),\]  

\[v_\mu \equiv T^\lambda_{\lambda\mu},\]  

\[a_\mu \equiv \frac{1}{6} \varepsilon_{\mu\nu\rho\sigma} T^{\nu\rho\sigma},\]

with \(\varepsilon_{\mu\nu\rho\sigma}\) being a totally antisymmetric tensor defined by \(\varepsilon_{0123} = -\sqrt{-g}\).

Total action \(I\) is constructed from the gravitational Lagrangian \(L_0\) and matter Lagrangian \(L_M\),

\[
I = \int d^4x \sqrt{-g} (L_0 + L_M).
\]

Gravitational field equations are obtained by varying the total action with respect to the parallel vector fields as

\[
G_{\mu\nu}(\{ \}) + 2D^4 F_{\mu\rho\lambda\nu} + 2v^\nu F_{\mu\rho\lambda\nu} + 2H_{\mu\nu} - g_{\mu\nu} L' = \chi T_{\mu\nu},
\]

where

\[
F_{\mu\rho\lambda\nu} = c_1 (t_{\mu\rho\lambda\nu} - t_{\mu\lambda\rho\nu}) + c_2 (g_{\mu\nu} v_\lambda - g_{\mu\lambda} v_\nu) - \frac{1}{3} c_3 \varepsilon_{\mu\nu\lambda\rho\sigma} a^\sigma
\]

\[= - F_{\mu\lambda\nu},\]  

\[H_{\mu\nu} = T_{\rho\sigma\nu} F^{\rho\sigma}_{\mu} - \frac{1}{2} T_{\nu\rho\sigma} F^{\rho\sigma}_{\mu} = H_{\nu\mu},\]  

\[L' = c_1 (t^{\lambda\nu} t_{\lambda\mu}) + c_2 (v^\nu v_\mu) + c_3 (a^\nu a_\mu),\]  

\[T_{\nu\mu} = -(1/\sqrt{-g}) \eta_{k\lambda} b^{\nu}_{\lambda} \delta (\sqrt{-g} L_M)/\delta b^\mu_{\nu} .\]

Here the tensor \(G_{\mu\nu}(\{ \})\) is the Einstein tensor made of the Christoffel symbols. The tensor \(T_{\mu\nu}\) denotes the energy-momentum tensor of the matter fields including not only gauge fields but also spinor fields.

\section*{§ 3. Weak field approximation}

In the weak field approximation, the parallel vector fields \(b^\mu_{\nu}\) slightly differ from those in the flat spacetime, \(b^\mu_{\nu} = \delta^\mu_{\nu}\), as
In the lowest order of the small field \( a^\mu \), the Latin indices are not distinguished from the Greek indices which we shall use hereafter. The Lorentz tensor \( a_{\mu \nu} = \delta_{\kappa \lambda} a^\kappa a^\lambda \) is divided into its symmetric and antisymmetric parts,

\[
a_{\mu \nu} = \frac{1}{2} h_{\mu \nu} + A_{\mu \nu} \tag{3.2}
\]

with \( h_{\mu \nu} = h_{\nu \mu} \) and \( A_{\mu \nu} = - A_{\nu \mu} \).

The linearized gravitational field equations of (2.9) in the weak field approximation are written in terms of these two fields as

\[
-\frac{1}{2} (1 - 3c_1) \Box h_{\mu \nu} + \frac{1}{2} (1 - 2c_1 + c_2) (\partial_\mu \partial^\sigma h_{\nu \rho \sigma} + \partial_\nu \partial^\sigma h_{\rho \mu \sigma})
\]

\[
-\frac{1}{2} (1 - c_1 + 2c_2) \eta_{\mu \nu} \partial^\rho \partial^\sigma h_{\rho \sigma} - \frac{1}{2} (c_1 + c_2) (\eta_{\mu \nu} \Box h - \partial_\mu \partial_\nu h)
\]

\[
+ (c_1 + c_2) (\partial_\mu \partial^\rho A_{\rho \nu} + \partial_\nu \partial^\rho A_{\rho \mu}) = \chi T_{(\mu \nu)},
\]

\[
(c_1 - \frac{4}{9} c_3) \Box A_{\mu \nu} + (c_2 + \frac{4}{9} c_3) (\partial_\mu \partial^\rho A_{\rho \nu} - \partial_\nu \partial^\rho A_{\rho \mu})
\]

\[
+ \frac{1}{2} (c_1 + c_2) (\partial_\mu \partial^\rho h_{\nu \rho} - \partial_\nu \partial^\rho h_{\mu \rho}) = \chi T_{(\mu \nu)}
\]

with \( h_{\mu \nu} \equiv h_{\mu \nu} - \frac{1}{2} \eta_{\mu \nu} h \) and \( h = - h = - \eta^{\rho \sigma} h_{\rho \sigma} \). The tensors \( T_{(\mu \nu)} \) and \( T_{(\mu \nu)} \) represent symmetric and antisymmetric parts of the energy-momentum tensor for matter fields. The combinations of parameters, \( 1 - 3c_1 \) and \( c_1 - \frac{4}{9} c_3 \), are assumed not to be vanishing. In the case with \( c_1 + c_2 = 0 \), the field \( h_{\mu \nu} \) and \( A_{\mu \nu} \) are decoupled, but it is not in general. We now examine the general case with \( c_1 + c_2 \neq 0 \).

By using the usual energy-momentum conservation law,

\[
\partial^\sigma T_{\sigma \nu} = 0,
\]

divergences of the gravitational field equations (3.3) and (3.4) are reduced to

\[
(c_1 + c_2) (\Box \phi_\nu - \partial_\nu \partial^\rho \phi_\rho) = \chi \partial^\nu T_{\rho \nu}.
\]

Here we define

\[
\phi_\nu \equiv \partial^\rho h_{\rho \nu} + \frac{1}{2} \partial_\nu h + 2 \partial^\rho A_{\rho \nu}.
\]

In terms of this field \( \phi_\nu \), the gravitational field equations (3.3) and (3.4) are rewritten as

\[
-\frac{1}{2} (1 - 3c_1) [\Box h_{\mu \nu} - (\partial_\mu \partial^\rho h_{\rho \nu} + \partial_\nu \partial^\rho h_{\rho \mu}) + \eta_{\mu \nu} \partial^\rho \partial^\sigma h_{\rho \sigma}]
\]

\[
-\frac{1}{2} (c_1 + c_2) [2 \eta_{\mu \nu} \partial^\rho \phi_\rho - (\partial_\mu \phi_\nu + \partial_\nu \phi_\mu)] = \chi T_{(\mu \nu)},
\]

\[
(c_1 - \frac{4}{9} c_3) [\Box A_{\mu \nu} - (\partial_\mu \partial^\rho A_{\rho \nu} - \partial_\nu \partial^\rho A_{\rho \mu})] + \frac{1}{2} (c_1 + c_2) (\partial_\mu \phi_\nu - \partial_\nu \phi_\mu) = \chi T_{(\mu \nu)}.
\]
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In order to decouple these field equations, we hereafter assume the following conservation law in addition to the usual one in (3.5):

\[ \partial \nu T_{\mu \nu} = 0 . \]  
(3.10)

Therefore, the symmetric and antisymmetric parts of the energy-momentum tensor are individually conserved,

\[ \partial \nu T_{(\mu \nu)} = 0 , \quad \partial \nu T_{[\mu \nu]} = 0 . \]  
(3.11)

From assumption (3.10), the field \( \phi_\mu \) in the wave equation (3.6) has nowhere a source field,

\[ \Box \phi_\mu - \partial_\mu \partial ^\rho \phi_\rho \equiv 0 . \]  
(3.12)

Hence the field \( \phi_\mu \) can be written as

\[ \phi_\mu = \partial_\mu \phi , \]  
(3.13)

where \( \phi \) is a scalar function. We get a relation taking divergence of (3.13) with the use of definition (3.7),

\[ \Box \phi = \partial ^\rho \phi_\rho = \partial ^\rho \partial ^\sigma \tilde{h}_{\rho \sigma} + \frac{1}{2} \Box \tilde{h} . \]  
(3.14)

Trace of the gravitational field equation (3.3) is given by

\[ -(1+3c_2)(\partial ^\rho \partial ^\sigma \tilde{h}_{\rho \sigma} + \frac{1}{2} \Box \tilde{h}) = x T \]  
(3.15)

with \( T \equiv \eta ^{\rho \sigma} T_{\rho \sigma} \). Here we assume \( 1+3c_2 \neq 0 \). From these relations, we find a field equation for the scalar field \( \phi \),

\[ -(1+3c_2)\Box \phi = x T . \]  
(3.16)

The gravitational field equations (3.8) and (3.9) are rewritten in terms of (3.13) as

\[ \frac{1}{2} (1-3c_1)[\Box \tilde{h}_{\mu \nu} - (\partial_\mu \partial ^\rho \tilde{h}_{\rho \nu} + \partial_\nu \partial ^\rho \tilde{h}_{\rho \mu}) + \eta _{\mu \nu} \partial ^\rho \partial ^\sigma \tilde{h}_{\rho \sigma}] \]

\[ -(c_1 + c_2)(\eta_{\mu \nu} \Box \phi - \partial_\mu \partial_\nu \phi ) = x T_{(\mu \nu)} , \]  
(3.17)

\[ (c_1 - \frac{4}{9}c_3)[\Box A_{\mu \nu} - (\partial_\mu \partial ^\rho A_{\rho \nu} - \partial_\nu \partial ^\rho A_{\rho \mu})] = x T_{[\mu \nu]} . \]  
(3.18)

If we define

\[ \chi_{\mu \nu} \equiv \tilde{h}_{\mu \nu} + \frac{c_1 + c_2}{1-3c_1} \eta_{\mu \nu} \phi , \]  
(3.19)

Eq. (3.17) is finally reduced to

\[ \frac{1}{2} (1-3c_1)[\Box \chi_{\mu \nu} - (\partial_\mu \partial ^\rho \chi_{\rho \nu} + \partial_\nu \partial ^\rho \chi_{\rho \mu}) + \eta _{\mu \nu} \partial ^\rho \partial ^\sigma \chi_{\rho \sigma}] = x T_{(\mu \nu)} . \]  
(3.20)

The gravitational field equation (3.18) in vacuum agrees with Eq. (1.2a), and (3.20) in vacuum is in accordance with Eq. (1.1a) if we choose \( p = 1 \) and \( q = -1/2 \).
Consequently, in the case with $1 - 3c_1 \neq 0$, $1 + 3c_2 \neq 0$, $c_1 - (4c_3/9) \neq 0$ and $c_1 + c_2 \neq 0$, we obtain these three equations (3·16), (3·18) and (3·20) rewriting the gravitational field equations (3·3) and (3·4) with assumption (3·10). The field $\phi$ is a massless Lorentz scalar field, $\chi_{\nu\mu}$ represents a massless Lorentz tensor field with helicity 2 and parity $+1$, and $A_{\mu\nu}$ propagates as a massless Lorentz tensor field with helicity 0 and parity $+1$.\(^1\)

§ 4. Conclusion

We have shown that the gravitational field equations in the weak field approximation of new general relativity with conditions of parameters, $1 - 3c_1 \neq 0$, $1 + 3c_2 \neq 0$, $c_1 - 4c_3/9 \neq 0$ and $c_1 + c_2 \neq 0$, are decoupled if a conservation law $\partial^\nu T_{\mu\nu} = 0$ of the matter field is required in addition to the usual one $\partial^\nu T_{\mu\nu} = 0$. This means that the symmetric and antisymmetric parts of the energy-momentum tensor for matter fields are individually conserved. This assumption is also required in the case with $c_1 + c_2 = 0$.\(^1\) It is noted that the symmetric and antisymmetric parts of the energy-momentum tensor for the Dirac field are individually conserved besides trivial cases of matter fields with symmetric energy-momentum tensors.\(^1\)

The parameters $c_1$ and $c_2$ are restricted to be much smaller than unity from experiments in the solar system.\(^1\) Therefore, the conditions $1 - 3c_1 \neq 0$ and $1 + 3c_2 \neq 0$ are experimentally satisfied. On the other hand, we regret that the parameter $c_3$ is not sufficiently restricted from experiments.\(^1,4\) However, in the framework of new general relativity, the condition $c_1 - 4c_3/9 \neq 0$ is necessary if there exists an antisymmetric part of the energy-momentum tensor of matter fields.

The field $\phi$, the symmetric field $\chi_{\nu\mu}$ and the antisymmetric field $A_{\mu\nu}$ propagate as a massless Lorentz scalar field, a massless Lorentz tensor field with helicity 2 and parity $+1$, and a Lorentz tensor field with helicity 0 and parity $+1$.\(^1\) It is noted that the field $A_{\mu\nu}$ is created by intrinsic spins of spinor fields and that the field $\phi$ does not appear in the case with $c_1 + c_2 = 0$. Therefore, these are characteristic fields of new general relativity against general relativity.

It should be emphasized that if we require the form of the field equation (1·1a) and (1·2a) for the gravitational field equations, it is in general impossible without assumption (3·10). Although the divergence of the left-hand side of the field equation (1·2a) is identically vanishing, the divergence of the antisymmetric part of the gravitational field equations is reduced to Eq. (3·12) whose left-hand side is not in general vanishing without assumption (3·10).

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References