Two-Loop Finite Temperature Effective Potential with Fermion

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(Received April 2, 1985)

We apply a new real time finite temperature formalism, Thermo Field Dynamics, to the theories involving fermion. A detailed calculation of two-loop finite temperature effective potential is given for the model with a fermion and a real scalar. As another application we provide a concise derivation of the imaginary part in a four-fermion theory by the use of a simple formula obtained for the thermo propagator. We observe a difference from the previous two-loop calculations based on the imaginary time formalism. A summary is given of all the essential techniques of perturbation in Thermo Field Dynamics with respect to the evaluation of both real and imaginary parts of finite temperature Feynman diagrams.

§ 1. Introduction

The present work is concerned with an application of a newly developed real time finite temperature formalism, Thermo Field Dynamics (=T.F.D.). We shall apply the technique of TFD to the basic theories which contains a fermion field and calculate two-loop finite temperature Feynman diagrams. Our main goal is to obtain the two-loop finite temperature effective potential (=F.T.E.P.) for the model with a fermion and a real scalar. We shall also provide a calculation of the imaginary part of a two-loop self-energy diagram in a two-dimensional four fermion theory.

We have had various motivations for undertaking the higher order calculation. Above all the effective potential is the only means available for investigating the nature of vacuum in the relativistic quantum field theoretic models. Most of the existing calculations at finite temperature (=T≠0) are limited to the one-loop level while at zero temperature (=T=0) higher order effects have been vigorously studied. We like to know, for example, how the critical temperature in the spontaneously broken theories is modified due to higher order corrections. Further we would expect that near the origin of F.T.E.P. where the effective mass is small the higher order effect may be significant, for instance, when one discusses the tunneling problem by the use of F.T.E.P. analytically continued with respect to its real part. Secondly T.F.D. provides us with a perturbative scheme convenient for higher order calculations. In T.F.D. calculation temperature (=T) dependent and independent parts are clearly separated and thus the divergence extraction and renormalization proceed in the same manner as in T=0 case. In particular it allows one a direct use of the dimensional regularization in contrast to the conventional imaginary time formalism (=I.T.F.). The third motivation is related to the renormalization scheme dependence. At two-loop level one encounters divergences, both T dependent and independent. They are to be cancelled by those in the counter term contributions. The detail of the counter terms depends on the renormalization conditions and regularization scheme. (In the present work the dimensional regularization is
employed. Accordingly, after cancellation of divergences, the residual finite part differs. This point does not appear to be well exposed in the two-loop calculations performed in I.T.F.\textsuperscript{,4,6} As the fourth motivations we may refer to another technical point. In evaluating the effective potential one must necessarily deal with integrals with massive particles. At two-loop level when the internal masses are different the finite part becomes rather difficult to evaluate and is rarely given in literature.\textsuperscript{**} We like to supplement these bits by providing a detailed calculation. (See the Appendix.) Furthermore, knowing effective potential of a given model is not only important to examine its behaviour at the critical temperature but also useful for finding the coefficients of the renormalization group equation as has been noted in Ref. 6. (For the renormalization group equation at $T \neq 0$ see Matsumoto et al.\textsuperscript{7}) And lastly we note that in studying the non-equilibrium phenomenon within the framework of linear response theory one must often go to two-loop to calculate the imaginary part and obtain the relaxation time.

The present work is the third of the series of higher order calculation in T.F.D. following Refs. 8 and 9. In the first of the series\textsuperscript{4} we dealt with single real scalar theory. In the second\textsuperscript{6} we considered the singular cases where T.F.D. perturbation becomes rather complicated due to the presence of many diagrams. In this and the subsequent works we calculate two-loop diagrams for basic theories. In this work we shall consider two theories involving fermions. We shall later study Wess-Zumino model,\textsuperscript{10} two-scalar theory, $O(N)$ model and scalar Q.E.D. (Numerical calculation will be given in a latter publication, where we will also investigate the validity of high temperature approximation often used at one-loop level.) Working out these basic theories should be enough to cope with more complicated theories such as unified theories.

For those who may be unfamiliar with the perturbative techniques of T.F.D. we like to give a brief survey of the essential aspects of T.F.D. in § 2. It contains prescriptions for calculating both real and imaginary parts of Feynman diagrams. Also we connect T.F.D. with the more general real time formalisms. In the previous works\textsuperscript{8,11} only the real part was obtained. The imaginary part is also of physical significance, for instance, in the area of linear response theory. (For a review and calculations in the relativistic case one may consult with works by Hosoya et al.\textsuperscript{12}) Hence we shall provide a model calculation in a four fermion theory in § 4. Our calculation makes use of a remarkable formula\textsuperscript{10} obtained recently and is completely different in appearance from I.T.F. calculation.

\section*{§ 2. Survey of T.F.D. perturbation}

In the beginning we like to make clear where T.F.D. stands as one of the real time formalisms. T.F.D. is a special case of the more general real time formalisms. The most general finite temperature formalisms are obtained by performing arbitrary analytic continuations in the complex time plane (see Fig. 1).\textsuperscript{9,13--16} And so far as the real time formalism\textsuperscript{13,14} is concerned, the general path is "uniquely" given as in Fig. 2. (In Fig. 2 the vertical segments of the continuation path are in the end pulled to infinity. Spacings between paths running horizontally are arbitrary.) The resulting so-called $2N$-component theory ($2N$: the number of horizontal paths in Fig. 2) consists of the ordinary

\textsuperscript{*} We shall come to this point at the end of § 3.

\textsuperscript{**} As a special case see Kang.\textsuperscript{8} In the case of degenerate internal mass see Collins\textsuperscript{9} and Ref. 8.)
field $\phi$, and $2N-1$ new fields, $\phi_i$ ($i = 1, 2, \ldots, 2N-1$). From a practical point of view the $2N$-component theory appears to be redundant in the sense that to extract the physical part one needs beside the ordinary field which is defined on the first horizontal path only one new field which is originally defined on one of the even numbered horizontal paths. The contribution from other fields is vanishing in calculating the physical quantity. To examplify the point one may take single real scalar theory and consider its 4-component formulation, and calculate two-loop self-energy diagrams (Fig. 3) using the 4-component thermo propagator,\(^{15, a)}

\[
\Delta(k) = \begin{vmatrix}
\langle \phi \phi \rangle, \langle \phi \phi_1 \rangle, \langle \phi \phi_2 \rangle, \langle \phi \phi_3 \rangle \\
\langle \phi_1 \phi \rangle, \langle \phi_1 \phi_1 \rangle, \langle \phi_1 \phi_2 \rangle, \langle \phi_1 \phi_3 \rangle \\
\langle \phi_2 \phi \rangle, \langle \phi_2 \phi_1 \rangle, \langle \phi_2 \phi_2 \rangle, \langle \phi_2 \phi_3 \rangle \\
\langle \phi_3 \phi \rangle, \langle \phi_3 \phi_1 \rangle, \langle \phi_3 \phi_2 \rangle, \langle \phi_3 \phi_3 \rangle \\
\end{vmatrix} = F_+(k) \frac{i}{k^2 - m^2 + i\varepsilon} F_-(k) \frac{i}{k^2 - m^2 - i\varepsilon}
\]

\[
F_+(k) = \begin{vmatrix}
e^{\omega}, e^{\rho_1 \omega}, e^{\sigma_1 \omega}, e^{\sigma_2 \omega} \\
e^{(\beta - \rho_2) \omega}, e^{\rho_2 \omega}, e^{(\rho_1 - \rho_2) \omega}, e^{(\sigma_2 - \rho_2) \omega} \\
e^{(\beta - \sigma_1) \omega}, e^{(\rho_2 - \sigma_1) \omega}, e^{(\sigma_2 - \sigma_1) \omega}, 1 \\
e^{(\beta - \sigma_2) \omega}, e^{(\rho_1 - \sigma_2) \omega}, e^{(\rho_1 - \sigma_2) \omega}, 1 \\
\end{vmatrix}
\]

\(^{a)}\beta, \rho, \sigma, \sigma_2 \text{ in } (2\cdot1) \text{ are given in Fig. 2.}\]
The vertices for the scalar self-coupling are
\[ \lambda \phi^4, \quad -\lambda \phi_1^4, \quad \lambda \phi_2^4, \quad -\lambda \phi_3^4. \]

Then one finds, among diagrams (I) \sim (IV) in Fig. 3, (I)+(II) (or (I)+(IV)) gives the correct result,
\[ -i \frac{\lambda^2}{4(4\pi)^3} \left( \frac{2}{\varepsilon + \gamma - 1 + \ln m^2 + 4} \int_1^\infty dx \frac{\ln^2 \left( x^2 - 1 \right)}{e^{\beta x} - 1} \right) \left( \frac{2}{\varepsilon + \gamma - \ln m^2 + 8} \int_1^\infty dx \frac{\ln^2 \left( x^2 - 1 \right)}{e^{\beta x} - 1} \right) \]
and (III)+(IV) (or (II)+(III)) cancel each other. Therefore for practical purposes it is sufficient to consider only 2-component theories. Formulation wise 2N-component theory may be useful. In fact a four-component theory has been applied to Hawking radiation. (See M. Horibe, A. Hosoya and N. Yamamoto, Osaka University Preprint (1984).)

Among 2-component theories T.F.D. whose second horizontal path runs at half way between \( t=0 \) and \( t=-i\beta \) (see (II) of Fig. 1) turns out to be unique in that it satisfies self-adjointness of Hamiltonian, according to Ref. 15.

T.F.D. perturbation has been formulated in Ref. 1). We shall not repeat the derivation of Lagrangian, propagator, etc. and only collect here what is needed to carry out perturbation.

1. **Scalar Thermo Propagator**

\[ \Delta(k) = \begin{vmatrix} \Delta_A(k) + \Delta_A(k) & \tilde{\Delta}_A(k) \\ \tilde{\Delta}_A(k) & -\Delta(k) + \Delta_A(k) \end{vmatrix} = \begin{vmatrix} \Delta_{11}(k), \Delta_{12}(k) \\ \Delta_{21}(k), \Delta_{22}(k) \end{vmatrix} = \langle \phi\phi \rangle, \langle \phi\bar{\phi} \rangle, \]

where
\[ \Delta_A(k) = \frac{i}{k^2 - m^2 + i\varepsilon}, \quad \tilde{\Delta}_A(k) = \frac{i}{k^2 - m^2 - i\varepsilon}, \]
\[ \Delta_\beta(k) = 2\pi \delta(k^2 - m^2) f_\beta(k), \quad \tilde{\Delta}_\beta(k) = 2\pi \delta(k^2 - m^2) g_\beta(k), \]
\[ f_\beta(k) = \frac{1}{e^{\beta k_0} + 1}, \quad g_\beta(k) = \frac{e^{\beta k_0}}{e^{\beta k_0} + 1}. \]

2. **Fermion Thermo Propagator**

\[ S_F(k) = (k + m) \times \begin{vmatrix} \Delta(k) - \Delta_\beta(k) & \tilde{\Delta}(k) \\ \tilde{\Delta}(k) & -\Delta(k) + \Delta_\beta(k) \end{vmatrix}, \]

where
\[ \Delta(k) = \frac{i}{k^2 - m^2 + i\varepsilon}, \quad \tilde{\Delta}(k) = \frac{i}{k^2 - m^2 - i\varepsilon}, \]
\[ \Delta_\beta(k) = 2\pi \delta(k^2 - m^2) f_\beta(k), \quad \tilde{\Delta}_\beta(k) = \epsilon(k_0) 2\pi \delta(k^2 - m^2) g_\beta(k), \]
\[ f_\beta(k) = \frac{1}{e^{\beta k_0} + 1}, \quad g_\beta(k) = \frac{e^{\beta k_0}}{e^{\beta k_0} + 1}. \]
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\[ \epsilon(k_0) = \begin{cases} +1, & (k_0 > 0) \\ -1, & (k_0 < 0) \end{cases} \] (2.6)

These propagators satisfy the following relations:

**Mass Derivative Formula:**

\[
\frac{1}{N!} \left( i \frac{\partial}{\partial m^2} \right)^N \Delta_\varphi(k) \tau = (\Delta_\varphi(k) \tau)^{N+1},
\]

\[
\frac{1}{N!} \left( i \frac{\partial}{\partial m} \right)^N S_\varphi(k) = (S_\varphi(k))^{N+1},
\]

where

\[
\tau = \begin{pmatrix} 1, & 0 \\ 0, & -1 \end{pmatrix}.
\] (2.7)

That the thermo propagators satisfy mass derivative formulae is the proof that \( \delta^N(k^2 - m^2) \) \((N \geq 2)\) type singularity is absent and the integrals become well-defined. In T.F.D. the new field is called tilde field and is often denoted as \( \tilde{\phi} \). Due to \( \tilde{\phi} \) the vertices are doubled in number.

**Vertex**

1. Scalar self-coupling

\[
g\phi^3, \quad \lambda \phi^4, \quad -g\tilde{\phi}^3, \quad -\lambda \tilde{\phi}^4.
\] (2.9)

2. Yukawa coupling

\[
f\tilde{\phi} \phi, \quad -f\tilde{\phi} \tilde{\phi} = + f(\tilde{\phi} \gamma_0) (\tilde{\phi}^+) \tilde{\phi}.
\] (2.10)

3. Four-fermion coupling

\[
g(\tilde{\phi} \phi)^2, \quad -g(\tilde{\phi} \phi)^2 = -g(\tilde{\phi} \gamma_0 \tilde{\phi}^+)^2.
\] (2.11)

Since we are usually concerned with physical part, let us state the rules\(^*\) for pulling it out.

(1) If the external legs of Feynman diagrams carry non-vanishing four momenta and then fix the legs to be ordinary fields, \((\phi \text{ or } \phi)\), and draw all diagrams consistent with the form of vertices.

(2) If there is no external leg (for instance, vacuum graphs) fix one of the vertices to be the one for ordinary field and draw all possible diagrams.

(3) If an external leg has vanishing four-momenta fix it to be ordinary field and draw diagrams and then fix the leg to be tilde field and draw diagrams. The physical real part is given the sum of both sets of diagrams.

In (3) we are considering the case where the external moment of boson, \( p_0(=2\pi n/\beta) \) and \( p_i(i=1,2,3) \), are set to be identically zero in the calculation in I.T.F. Such a case appears in the calculation of the effective potential by the Colemann-Weinberg method. The result in I.T.F. can be reproduced in T.F.D. by defining \( p_\mu = 0 \) as a space-like limit, \( p_0 = 0 \) and then \( p_i = 0 \), 'after' integration over the internal momenta in the diagrams given by

\(^*\) Rule 2 and rule 3 are stated in Ref. 9) where we investigated only singular diagrams. Here we state the rule in the complete form.
rule 1. However this calculation becomes involved at higher orders. Instead one may overcount the diagrams as this rule states and set $p_\nu=0$ 'before' integration. Then the calculation becomes a lot simpler. In other cases $p_\nu=0$ has to be defined as either a space-like limit or a time-like limit, $p_\tau=0$ and then $p_\eta=0$, 'after' integration. The two limits lead to different results due to the fact $T \neq 0$ perturbation destroys Lorentz invariance.

In the T.F.D. perturbation the integral to evaluate contains both real and imaginary parts explicitly. Therefore one must specify which to evaluate. Upon specification a lot of terms and diagrams may be discarded in the higher order calculation. (See Ref. 9 for various examples where real part is calculated.) This is to be contrasted with the case of I.T.F. In I.T.F. the integral is real. Therefore if the imaginary part is wanted one first calculates the real part and then takes the discontinuity with respect to the external momenta. On the other hand in T.F.D. the imaginary part can be obtained directly independent of the evaluation of the real part and in many cases the calculation becomes very simple.\(^{13}\) We present below relations on the imaginary part, $\text{Im} \Sigma$, of the two-point function.\(^{13}\) In case of scalar we find,

\[
\text{Im} \Sigma(k) = \frac{e^{\beta k_0} - 1}{e^{\beta k_0} + 1} \text{Im} \Sigma_{11}(k)_\tau, 
\]

\[
\quad = -\frac{1}{2} \frac{e^{\beta k_0} - 1}{e^{\beta k_2} + 1} \text{Im} \Sigma_{12}(k) \quad (2.12.1)
\]

In case of fermion we find,

\[
\text{Im} \Sigma(k) = \frac{e^{\beta k_0} + 1}{e^{\beta k_0} - 1} \text{Im} \Sigma_{11}(k) \quad (2.13.1)
\]

\[
\quad = -\frac{e^{\beta k_0} + 1}{e^{\beta k_2} - 1} \text{Im} \Sigma_{22}(k) \quad (2.13.2)
\]

\[
\quad = \frac{1}{2} \frac{e^{\beta k_0} + 1}{e^{\beta k_2} + 1} \text{Im} \Sigma_{12}(k) \quad (2.13.3)
\]

In the above $\Sigma_{11}$, $\Sigma_{12}$, $\Sigma_{21}$ and $\Sigma_{22}$ denote the self-energy of the two-point functions. In case of scalar they are respectively the self-energy of $\langle \phi \phi \rangle$, $\langle \phi \phi \rangle$, $\langle \bar{\phi} \bar{\phi} \rangle$ and $\langle \bar{\phi} \bar{\phi} \rangle$, likewise for fermion. $\text{Im} \Sigma(k)$ is the quantity related to the relaxation time $\tau$ by the following relation,

\[
\text{Im} \Sigma(k) = -\omega \tau^{-1}. \quad (\omega = (k^2 + m^2)^{1/2}) \quad (2.14)
\]

\section*{§ 3. Two-loop effective potential}

In this section we present the two-loop calculation of F.T.E.P. in a model involving a real scalar and a fermion. The Lagrangian is as given below,

\[
\tilde{L}(\phi, \psi) = L(\phi, \psi) - \tilde{L}(\bar{\phi}, \bar{\psi}) \quad (3.1)
\]

\[
L(\phi, \psi) = \frac{1}{2} \partial_{\mu} \phi \partial^{\mu} \phi - M_0^2 \phi^2 - \frac{\delta_0}{3!} \phi^3 - \frac{\lambda}{4!} \phi^4 + \bar{\psi} i \gamma \phi \psi - m_0 \bar{\psi} \phi - f \bar{\psi} \phi \phi \quad (3.2)
\]
After a shift in $\phi$, $\phi \to \phi + v$ one obtains a new Lagrangian,

$$L_{\text{shifted}}(\phi, \bar{\phi}) = L'(\phi, \bar{\phi}) - \bar{L}'(\bar{\phi}, \bar{\phi}),$$  \hspace{1cm} (3.3)

$$L'(\phi, \bar{\phi}) = \frac{1}{2} \partial_{\mu} \phi \partial^{\mu} \phi - M^2 \phi^2 - \frac{g}{3!} \phi^4 - \frac{\lambda}{4!} \phi^4 + \bar{\phi} i \gamma \phi - m \bar{\phi} \phi - f \bar{\psi} \psi \phi,$$  \hspace{1cm} (3.4)

where

$$M^2 = M_0^2 + g_0 v + \frac{\lambda}{2} v^2,$$

$$m = m_0 + f v,$$

$$g = g_0 + \lambda v.$$  \hspace{1cm} (3.5)

Propagators are given in (2.3) and (2.5).

1. **One-Loop F.T.E.P.**

One may obtain the one-loop F.T.E.P. by integrating the tadpole diagrams (Fig. 4) over $v$. (The integration constant is set to be zero.) The result is,

$$V_{\text{1-loop}} = \frac{1}{64 \pi^2} M^4 \ln \frac{M^2}{M_0^2} - \frac{1}{16 \pi^2} m^4 \ln \frac{m^2}{m_0^2} - \Delta V_{\text{1-loop}}$$

$$- \frac{M^4}{6 \pi^2} \int_1^\infty dx \frac{(x^2 - 1)^{3/2}}{e^{3mx} - 1} - \frac{2 m^4}{3 \pi^2} \int_1^\infty dx \frac{(x^2 - 1)^{3/2}}{e^{mx} + 1},$$  \hspace{1cm} (3.6)

where

$$\Delta V_{\text{1-loop}} = \frac{1}{64 \pi^2} (av + bv^2 + cv^3 + dv^4),$$

$$a = g_0 M_0^2 + 8 f m_0^3,$$

$$b = \frac{1}{2} (2 g_0^2 + \lambda M_0^2) + 7 f^2 m_0,$$

$$c = \frac{1}{6} \left( \frac{2 g_0^3}{M_0^2} + 9 g_0 \lambda \right) + \frac{104}{3} f^3 m_0,$$

$$d = \frac{1}{24} \left( \frac{2 g_0^5}{M_0^2} - \frac{2 \lambda^2}{M_0^4} + 9 \lambda^3 \right) + \frac{62}{3} f^4.$$  \hspace{1cm} (3.7)

In obtaining (3.7) we have imposed the following renormalization conditions:

![Diagrams](https://academic.oup.com/ptp/article/74/5/1105/1835610)

Fig. 4. One-loop tadpole diagrams. Broken (unbroken) line represents scalar (fermion).
\[
\frac{\partial V}{\partial v} \bigg|_{v=0} = 0, \quad \frac{\partial^2 V}{\partial v^2} \bigg|_{v=0} = M_0^2, \quad \frac{\partial^3 V}{\partial v^3} \bigg|_{v=0} = g_0, \quad \frac{\partial^4 V}{\partial v^4} \bigg|_{v=0} = \lambda. \quad (3.8)
\]

To prepare for two-loop calculation one needs mass and wave function counter terms.\(^)*\) They are obtained by calculating diagrams in Fig. 5 and imposing, for instance, the following renormalization conditions:

\[
\Delta_\beta^{-1}(k) \bigg|_{k=0, v=0} = -M_0^2, \quad S_F^{-1}(k) \bigg|_{k=0, v=0} = -m_0,
\]

\[
\frac{\partial \Delta_\beta^{-1}(k)}{\partial k^2} \bigg|_{k=0, v=0} = 1, \quad \frac{\partial S_F^{-1}(k)}{\partial k} \bigg|_{k=0, v=0} = 1, \quad (3.9)
\]

where

\[
\Delta_\beta^{-1} (S_F^{-1}) \text{ is the scalar (fermion) one-loop corrected inverse propagator including the counter terms.}
\]

![Fig. 5. One-loop self-energy diagrams for fermion and scalar.](image)

There are a variety of choices in the renormalization condition and we have made a specific choice as given above. If one denotes the counter terms as below,

\[
\delta Z_F k + \delta M_F m,
\]

\[
\delta Z_B k^2 + \delta M_B M^2, \quad (3.10)
\]

then one finds

\[
\delta Z_B = -\frac{i}{(4\pi)^2} f^2 \left( \frac{2}{\epsilon} - \gamma - \frac{2}{3} \ln m_0^2 \right) + \frac{i}{12(4\pi)^2} g^2 M_0^2,
\]

\[
\delta Z_F = -\frac{i}{(4\pi)^2} f^2 \left( \frac{1}{\epsilon} - \frac{\gamma}{2} - C_1 \right),
\]

\[
\delta M_B = -\frac{i}{(4\pi)^2} f^2 \left( 2g^2 + 2\lambda M \right) \left( \frac{2}{\epsilon} - \gamma + \frac{3}{2} \ln M_0^2 \right)
\]

\[
+ \frac{i}{(4\pi)^2} f^2 \left( 12f^2 m^4 \right) \left( \frac{2}{\epsilon} - \gamma + \frac{1}{3} \ln m_0^2 \right),
\]

\[
\delta M_F = -\frac{i}{(4\pi)^2} f^2 \left( \frac{2}{\epsilon} - \gamma - C_2 \right), \quad (3.11)
\]

where

\(^)*\) The scalar mass counter term is also obtained by using the second condition in (3.8).
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\[ C_1 = \frac{1}{2} \ln M_0^2 + \frac{1}{2} \left( \frac{m_0^2}{M_0^2 - m_0^2} \right)^2 \ln \frac{m_0^2}{M_0^2} - \frac{1}{4} + \frac{m_0^2}{2(M_0^2 - m_0^2)}, \]

\[ C_2 = \frac{1}{M_0^2 - m_0^2} \left( M_0^2 \ln M_0^2 - m_0^2 \ln m_0^2 \right) - 1. \]  

(3.12)

2. Two-Loop F.T.E.P.

We shall employ vacuum graph method since it requires less effort than tadpole method. The diagrams to calculate are as shown in Fig. 6. The bosonic sector has been evaluated in Ref. 8) and the result is as follows:\(^*)\)

\[ V_{\text{boson}}^{\text{two-loop}} = V(T=0) + V(T=0), \]

\[ V(T=0) = \frac{1}{8(4\pi)^4} \left( M^2 g^2 \left( \ln \frac{M^2}{M_0^2} - \ln \frac{M_0^2}{M_0^2} \right) + \lambda M^4 \left( \ln \frac{M^2}{M_0^2} + \ln \frac{M_0^2}{M_0^2} \right) \right. \]

\[ - M^2 \nu_1''(\nu) \ln \frac{M_0^2}{M_0^2} + \frac{1}{3} \frac{g_0^2}{M_0^2} M^4 \ln \frac{M_0^2}{M_0^2} - \nu_2(\nu) \}, \]  

(3.13.1)

where

\[ \nu_1''(\nu) = (2g_0^2 + M_0^2 \lambda) + \left( \frac{2g_0^2}{M_0^2} + 2g_0^2 \lambda \right) \nu + \frac{1}{2} \left( \frac{2g_0^2}{M_0^2} - \frac{2g_0^2}{M_0^4} + 9\lambda^2 \right) \nu^2, \]

\[ \nu_2(\nu) = a_2 \nu + b_2 \nu^3 + c_2 \nu^3 + d_2 \nu^4, \]

\[ a_2 = -\frac{11}{3} g_0^2, \quad c_2 = \frac{7}{9} \frac{g_0^2}{M_0^6} - \frac{19}{2} \frac{g_0^2}{M_0^2} - 9g_0^2 \lambda, \]

\[ b_2 = -\left( \frac{5}{2} \frac{g_0^2}{M_0^2} + \frac{65}{6} g_0^2 \lambda \right), \quad d_2 = \frac{7}{18} \frac{g_0^2}{M_0^6} - \frac{3}{4} \frac{g_0^2}{M_0^4} - \frac{49}{8} \frac{g_0^2}{M_0^2} - \frac{9}{4} \lambda^2, \]  

(3.13.3)

\[ V(T=0) = \frac{\lambda M^4}{32\pi^3} \left( F_1(\beta M)\right)^2 + \frac{g_0^2 M_0^2}{128\pi^4} \int_1^\infty dx \int_1^\infty dy F(\beta M) G(x, y) \]

\[ + \frac{\lambda M^4}{128\pi^4 \left( \frac{1}{2} + \ln \frac{M_0^2}{M_0^2} \right)} F_1(\beta M) + \frac{g_0^2 M_0^2}{128\pi^4 \left( -\left( \frac{1}{2} - \frac{\pi}{\sqrt{3}} \right) + \ln \frac{M_0^2}{M_0^2} \right)} F_1(\beta M) \]

\[ - \frac{\nu_1''(\nu) M^2}{256\pi^4} F_1(\beta M) + \frac{M^4}{768\pi^4} \frac{g_0^2}{M_0^2} F_1(\beta M), \]  

(3.14.1)

where

\[ \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \ quad
\[ F(\beta M) = \int_1^\infty dx \left( \frac{x^2 - 1}{e^{\beta Mx} - 1} \right)^{1/2}, \quad G(\beta M) = \frac{1}{e^{\beta Mx} - 1} - \frac{1}{e^{\beta My} - 1}, \]

\[ G(x, y) = \ln \left| X_+ \right| - \left| X_- \right|, \]

\[ X_\pm = (1 \pm 2\sqrt{x^2 - 1}\sqrt{y^2 - 1})^2 - 4x^2y^2. \quad (3.14.2) \]

Therefore we only have to calculate (C), (D) and (E) of Fig. 6. In (D) the counter terms (indicated by a blob) retained in the result below are those due to fermion contribution and are underlined in Eq. (3.11). One finds,

\[ I(C) = \frac{1}{2}(-if)^2(-1) \text{Tr} \int \frac{d^n k}{(2\pi)^n} \frac{d^n l}{(2\pi)^n} (k + m)(l + m) \]

\[ \times (\Delta_F(k) - 2\pi \delta(k^2 - m^2)\delta_{F}(k)) \times (\Delta_F(k - l) + 2\pi \delta((k - l)^2 - M^2)\delta_{F}(k - l)) \]

\[ \times (\Delta_F(l) - 2\pi \delta(l^2 - m^2)\delta_{F}(l)) , \quad (3.15) \]

\[ \equiv I_0(C) + I_{\beta-1}(C) + I_{\beta-2}(C) . \quad (3.16) \]

Since we are concerned with the real part, the term with three statistical factors has been dropped. A calculation\(^*\) gives the following result:

\[ I_0(C) = -\frac{if^2}{(4\pi)^4} \Gamma^2 \left( -1 + \frac{\epsilon}{2} \right) \left\{ m^4 \left( 1 - \frac{\epsilon}{2} \ln m^2 + \frac{\epsilon^2}{8} \ln^2 m^2 \right) \right. \]

\[ -2m^2M^2 \left( 1 - \frac{\epsilon}{2} \ln m^2 + \frac{\epsilon^2}{8} \ln^2 m^2 \right) \left( 1 - \frac{\epsilon}{2} \ln^2 m^2 + \frac{\epsilon^2}{8} \ln^2 M^2 \right) \]

\[ + \frac{2if^2}{(4\pi)^4} (4m^2 - M^2)(-1) \Gamma(1 + \epsilon) \]

\[ \left. \times \left[ \frac{1}{\epsilon} (4m^2 + 2M^2) + 2m^2 + M^2 - 4m^2\ln m^2 - 2M^2\ln M^2 \right] \right\] \[ + \epsilon \left\{ \left( \frac{3}{2} - \frac{\pi^2}{12} \right) M^2 - (2m^2\ln m^2 + M^2\ln M^2) \right. \]

\[ + (2m^2\ln^2 m^2 + M^2\ln^2 M^2) - M^2g(\alpha, x, y) + M^2C \} , \quad (3.17) \]

where

\[ g(\alpha, x, y) = \int_0^1 dx \int_0^1 dy \frac{1}{y} \ln \left[ 1 + y \left( \frac{1}{ax(1-x)} \right) \right], \]

\[ \alpha = \frac{M^2}{m^2} \]

\[ \quad (3.18) \]

and

\[ C \] is some constant independent of \( \alpha \).

\(^*\) The derivation of (3.17) is given in the Appendix.
Two-Loop Finite Temperature Effective Potential with Fermion

\[ I_{\beta-1}(C) = -2if^2 \frac{m^2}{32\pi^4} \left( \frac{2}{\epsilon} - \gamma + 1 - \ln m^2 \right) (m^2 F_1(\beta m) + M^2 F_1(\beta M)) \]

\[ + 2if^2 \frac{(4m^2 - M^2)}{32\pi^4} \left( \frac{2}{\epsilon} - \gamma + 1 - \ln M^2 \right) F_1(\beta M) \]

\[ - f^2 \frac{(4m^2 - M^2)}{32\pi^4} \left\{ 2I(m, M; m)m^2 F_1(\beta m) - I(m, m; M)M^2 F_1(\beta M) \right\}, \quad (3.19) \]

where

\[ F_1(\beta m) = \int_1^\infty dx \frac{(x^2 - 1)^{1/2}}{e^{\beta mx} + 1} \quad (3.20) \]

and

\[ I(m_1, m_2; m_3) = -i \left( \frac{2}{\epsilon} - \gamma - \int_0^1 dx \ln M(m_1, m_2; m_3) \right), \quad (3.21.1) \]

where

\[ \int_0^1 dx \ln M(m_1, m_2; m_3) = \int_0^1 dx \ln(x(x - 1)m_3^2 + (1 - x)m_1^2 + mx_2^2) \]

\[ = \ln m_3^2 - 2 + (1 - \alpha) \ln(1 - \alpha) + a \ln(-\alpha) \]

\[ + (1 - \beta) \ln(1 - \beta) + \beta \ln(-\beta), \]

\[ a = \frac{-b + (b^2 - 4c)^{1/2}}{2}, \quad \beta = \frac{-b - (b^2 - 4c)^{1/2}}{2}, \]

\[ b = \frac{m_2^2 - m_1^2 - m_3^2}{m_3^2}, \quad c = \frac{m_1^2}{m_3^2}, \quad (3.21.2) \]

\[ I_{\beta-2}(C) = -if^2 \frac{m^2}{4\pi^4} \left\{ m^2 F_2(\beta m) + 2M^2 F_1(\beta m) F_1(\beta M) \right\} \]

\[ - if^2 \frac{(4m^2 - M^2)}{32\pi^4} \left\{ 2mM \int_1^\infty dx \int_1^\infty dy \frac{F_2(x, y)}{(e^{\beta mx} + 1)(e^{\beta my} + 1)} \right\} \]

\[ - m^2 \int_1^\infty dx \int_1^\infty dy \frac{F_3(x, y)}{(e^{\beta mx} + 1)(e^{\beta my} + 1)} \}, \quad (3.22.1) \]

where

\[ F_2(x, y) = F(x, y, m_1, m_2; m_3) \bigg|_{m_1 = m_3 = m, \ m_3 = M}, \]

\[ F_3(x, y) = F(x, y, m_1, m_2; m_3) \bigg|_{m_1 = m_2 = m, \ m_3 = M}, \]

\[ F(x, y, m_1, m_2; m_3) \]

\[ = \ln \left| \frac{A + xy + (x^2 - 1)^{1/2}(y^2 - 1)^{1/2}}{A + xy - (x^2 - 1)^{1/2}(y^2 - 1)^{1/2}} \right| + \ln \left| \frac{A - xy + (x^2 - 1)^{1/2}(y^2 - 1)^{1/2}}{A - xy - (x^2 - 1)^{1/2}(y^2 - 1)^{1/2}} \right|, \]

\[ A = \frac{m_1^2 + m_2^2 - m_3^2}{2m_1 m_2}, \quad (3.22.2) \]
\[ I(D) = I_0(D) + I_{s-1}(D), \]  
\[ I_0(D) = \frac{i}{(4\pi)^3} f^2 \left( \frac{2}{\epsilon} - \gamma - \frac{2}{3} \ln m^2 \right) (-1) \Gamma \left( -1 + \frac{\epsilon}{2} \right) M^4 \left( 1 - \frac{\epsilon}{2} \ln M^2 + \frac{\epsilon^2}{8} \ln^2 M^2 \right) \]  
\[ - \frac{i}{(4\pi)^3} f^2 \left( \frac{2}{\epsilon} - \gamma + \frac{1}{3} \ln m^2 \right) (-1) \Gamma \left( -1 + \frac{\epsilon}{2} \right) M^2 m^2 \left( 1 - \frac{\epsilon}{2} \ln M^2 + \frac{\epsilon^2}{8} \ln^2 M^2 \right), \]  
\[ I_{s-1}(D) = -\frac{i f^2}{(4\pi)^3} \left( \frac{2}{\epsilon} - \gamma - \frac{2}{3} \ln m^2 \right) \frac{M^4}{2\pi^2} F_1(\beta M) \]  
\[ + \frac{6if^2}{(4\pi)^3} \left( \frac{2}{\epsilon} - \gamma + \frac{1}{3} \ln m^2 \right) \frac{M^2 m^2}{2\pi^2} F_1(\beta M), \]  
\[ I(E) = I_0(E) + I_{s-1}(E), \]  
\[ I_0(E) = \frac{4if^2}{(4\pi)^3} \left( \frac{1}{\epsilon} - \frac{\gamma - C_1}{2} \right) \Gamma \left( -1 + \frac{\epsilon}{2} \right) m^4 \left( 1 - \frac{\epsilon}{2} \ln m^2 + \frac{\epsilon^2}{8} \ln^2 m^2 \right) \]  
\[ + \frac{i f^2}{(4\pi)^3} \left( \frac{2}{\epsilon} - \gamma - C_2 \right) \Gamma \left( -1 + \frac{\epsilon}{2} \right) m^4 \left( 1 - \frac{\epsilon}{2} \ln m^2 + \frac{\epsilon^2}{8} \ln^2 m^2 \right), \]  
\[ I_{s-1}(E) = -\frac{4if^2}{(4\pi)^3} \left( \frac{1}{\epsilon} - \frac{\gamma - C_1}{2} \right) \frac{m^4}{2\pi^2} F_1(\beta m) - \frac{4if^2}{(4\pi)^3} \left( \frac{2}{\epsilon} - \gamma - C_2 \right) \frac{m^4}{2\pi^2} F_1(\beta m). \]  

In (3.23) – (3.26) the first (second) term represents the contribution due to the wave function (mass) counter term. Since our goal is to obtain effective potential, we may drop terms which are polynomials in \( m \) and \( M \) in \( T = 0 \) part. (They are absorbed in the counter terms.) Then one finds,

\[ I_0(C) = \frac{if^2}{(4\pi)^3} (4m^2 - M^2) \{ -\left( \frac{2}{\epsilon} - 2\gamma + 2 \right) (2m^2 \ln m^2 + M^2 \ln M^2) \]  
\[ - (2m^2 \ln m^2 + M^2 \ln M^2) + 2m^2 \ln^2 m^2 + M^2 \ln^2 M^2 - m^2 g(\alpha) \} \]  
\[ + \frac{2if^2}{(4\pi)^3} m^2 M^2 \left\{ - \left( \frac{2}{\epsilon} - 2\gamma + 2 \right) \ln m^2 \ln M^2 + \frac{1}{2} \ln^2 m^2 M^2 \right\} \]  
\[ - \frac{i f^2}{(4\pi)^3} m^4 \left\{ -2 \left( \frac{2}{\epsilon} - 2\gamma + 2 \right) \ln m^2 + \frac{1}{2} \ln^2 m^4 \right\} \]  
\[ = -\frac{if^2}{(4\pi)^3} \left( \frac{2}{\epsilon} - 2\gamma + 2 \right) (6m^4 \ln m^2 + M^2 \ln M^2) \]  
\[ - \frac{if^2}{(4\pi)^3} (4m^2 - M^2) (-2m^2 \ln m^2 - M^2 \ln M^2 + 2m^2 \ln^2 m^2) \]  
\[ + M^2 \ln^2 M^2 - m^2 g(\alpha, x, y) - m^2 M^2 \ln m^2 M^2 + m^4 \frac{M^4}{2} \ln^2 m^4), \]  
\[ I_0(D) + I_0(E) = \frac{if^2}{(4\pi)^3} \left( \frac{2}{\epsilon} - 2\gamma + 2 \right) (6m^4 \ln^2 m^2 - M^4) \ln M^2 + \left( \frac{1}{3} - \ln m^2 \right) M^4 \]
\[ +6 \left( -\frac{4}{3} \ln m_0^2 \right) M^2 m^2 \ln M^2 + \left( -3 M^2 m^2 + \frac{M^4}{2} \right) \ln^2 M^2 \]
\[ + \frac{i f^2}{4 \pi^4} \left\{ \left( \frac{2}{\varepsilon} - 2 \gamma + 2 \right) 6m^4 \ln m^2 + \left( 6 - 4 C_1 - 4 C_2 \right) m^4 \ln^2 m^2 
- 3m^4 \ln^2 m^2 \right\}, \]  
(3.28)

\[ I_{\beta^{-1}}(C) = \frac{i f^2}{32 \pi^4} \left( 6m^4 \left( \frac{2}{\varepsilon} - \gamma \right) + 2(4m^2 - M^2) m^2 P + 2m^4 \ln m^2 
+ 2m^2 M^2 (1 - \ln M^2) \right) F_1(\beta m) \]
\[ + \frac{i f^2}{32 \pi^4} \left( -6m^2 M^2 + M^4 \right) \left( \frac{2}{\varepsilon} - \gamma \right) + (4m^2 - M^2) M^2 Q 
- 2m^4 M^2 (1 - \ln m^2) \right) F_1(\beta M), \]  
(3.29)

\[ P = \int_0^1 dx \ln(x(x-1)m^2 + (1-x)m^2 + xM^2), \quad Q = \int_0^1 dx \ln(x(x-1)M^2 - m^2) \]

\[ I_{\beta^{-1}}(D) = -\frac{i f^2}{32 \pi^4} \left( -6m^2 M^2 + M^4 \right) \left( \frac{2}{\varepsilon} - \gamma \right) - 6 \left( \frac{1}{3} - \ln m_0^2 \right) M^2 m^2 
- \left( \frac{2}{3} - \ln m_0^2 \right) M^4 \right) F_1(\beta M), \]  
(3.30)

\[ I_{\beta^{-1}}(E) = -\frac{i f^2}{32 \pi^4} \left( 6m^4 \left( \frac{2}{\varepsilon} - \gamma \right) + 4m^4 (4C_1 + C_2) \right). \]  
(3.31)

Adding (3.27) and (3.28), one observes the cancellation of divergences logarithmic in mass, \( m \) and \( M \). Also the temperature dependent divergences in (3.29) \sim (3.31) cancel each other. In both respects T.F.D. perturbation is satisfactory. The two-loop finite temperature effective potential is given by multiplying overall \( i \) on (3.27) \sim (3.31). \(^*\) The finite renormalization of the temperature independent part, (3.27) and (3.28), can be done by imposing the condition, (3.8). We have left it out since a simple form of the finite counter term does not appear to follow because of the complicated term, \( g(\alpha, \kappa, \gamma) \).

Before closing the section we like to make a comment on the ITF calculation. Kapusta, Cook and Mahanthappa\(^*\) performed two-loop calculations in ITF in gauge theories. Although their models are different from ours, we can make a qualitative comparison with ours. Their result shows that terms proportional to one statistical factor in the two-loop integrals are "exactly" cancelled by the counter term contribution, while we find residual terms (see (3.29) \sim (3.31)). It is stated in their paper that their counter terms are prepared at \( T=0 \) like ours although they are not explicitly given. For the exact cancellation to occur their counter terms must be the entire one-loop self-energy contributions including the momentum dependence. It is hard to see how such a choice of the counter terms is justifiable. In other works quoted in Ref. 4) a few examples of the cancellation of temperature dependent divergences are given. However we have not seen

\(^*\) The contribution from scalar is given in (3.13-1) \sim (3.14-2).
a paper in which the remaining finite part is fully calculated. As stated in the Introduction and shown in this section the terms with one statistical factor should be present after the cancellation of temperature dependent divergences.*)

§ 4. The imaginary part in the four-fermion theory

In the last section we have calculated the real part of Feynman diagrams. In this section we provide an example of calculation of the imaginary part. As stated in the Introduction in T.F.D., the imaginary part can be calculated directly by the use of relations (2·12·1·2) and (2·13·1·3), without going through an analytic continuation necessary in I.T.F. Since we are concerned with diagrams involving fermion in this paper, let us take a purely fermionic theory, $f/2(\bar{\psi}\psi)^2$ theory in two-dimension, and calculate the imaginary part of the two-loop self-energy diagrams shown in Fig. 7.**)

\[
\text{Im} \tilde{\Sigma}(p) = \frac{1}{2} \frac{e^{sp_0 + 1}}{e^{\bar{s}p_0}} \left( I_a + I_b \right),
\]

\[
I_a = -f^2 \int \frac{d^2k_1}{(2\pi)^2} \frac{d^2k_2}{(2\pi)^2} \frac{d^2k_3}{(2\pi)^2} (2\pi)^2 \delta(p - k_1 - k_3 + k_2) \\
\times \varepsilon(k_{10}) \varepsilon(k_{20}) \varepsilon(k_{30}) (\not{k}_3 + m)(-\text{Tr}(\not{k}_1 + m)(\not{k}_2 + m)) \\
\times (2\pi)^2 \delta(k_1^2 - m^2) \delta(k_3^2 - m^2) g_F(k_1) g_F(k_2) g_F(k_3),
\]

\[
(g_F(k) \text{ is defined in (2·6).})
\]

\[
I_b = -f^2 \int \frac{d^2k_1}{(2\pi)^2} \frac{d^2k_2}{(2\pi)^2} \frac{d^2k_3}{(2\pi)^2} (2\pi)^2 \delta(p - k_1 - k_3 + k_2) \\
\times \varepsilon(k_{10}) \varepsilon(k_{20}) \varepsilon(k_{30}) (\not{k}_3 + m)(\not{k}_2 + m)(\not{k}_1 + m) \\
\times (2\pi)^2 \delta(k_1^2 - m^2) \delta(k_3^2 - m^2) g_F(k_1) g_F(k_2) g_F(k_3).
\]

Working out the Dirac algebra noting the symmetry under the interchange, $k_1 \leftrightarrow k_3$ and integrating over $k_{10}$ and $k_{30}$, one finds

\[
\text{Im} \tilde{\Sigma}(p) = -\frac{f^2}{2} \frac{e^{sp_0 + 1}}{e^{\bar{s}p_0}} \int \frac{dk_1}{(2\pi)^2} \frac{dk_3}{(2\pi)^2} \frac{dk_2}{(2\pi)^2} A \\
\times 2\pi \delta(k_2^2 - m^2)(2\pi)^2 \delta(p - k_1 - k_3 + k_2) \\
\times (\delta(p_0 - E_1 - E_3 + k_20) + \delta(p_0 + E_1 + E_3 + k_20) \\
- \delta(p_0 - E_1 + E_3 + k_20) - \delta(p_0 + E_1 - E_3 + k_20)),
\]

where

---

* In view of our result we feel that the earlier application of $T \neq 0$ perturbation to condensed matter system would have to be taken with caution.

** We make use of the identity (2·13·3), which is the simplest way of obtaining the imaginary part.
\[ E_i = (k_i^2 + m^2)^{1/2}, \quad (i = 1, 2, 3) \]
\[ A = -p \cdot k_2 \frac{1}{2} (p^2 - 3m^2) \hat{k}_2 \cdot m \frac{1}{2} (p^2 - 3m^2). \]  
\[ (4.5) \]

To avoid extra indices we have used a vector notation for the space component of momenta in (4.4). Then \( k_{20} \) integration yields
\[
\text{Im} \tilde{\Sigma}(p) = -\frac{f^2}{2} \frac{e^{\beta p_0} e^{\beta^2 p_0}}{\epsilon^{\beta p_0}} \int \frac{dk_1}{2\pi} \frac{dk_2}{2\pi} \frac{dk_3}{2\pi} \frac{1}{8E_1E_2E_3} (2\pi)^3 \delta(p - k_1 - k_2 + k_3) \\
\times \left[ -\left( p_0E_2 - p \cdot k_2 + \frac{1}{2} (p^2 - 3m^2) \right) (E_2 \gamma_0 - k_2 \cdot \gamma - m) \right. \\
\times \left\{ \delta(p_0 - E_1 + E_2 - E_3) + \delta(p_0 + E_1 + E_2 + E_3) - \delta(p_0 - E_1 + E_2 + E_3) \right. \\
\left. - \delta(p_0 + E_1 - E_3) - \delta(p_0 + E_1 + E_2 + E_3) \right. \\
\left. \times \delta(p_0 - E_1 + E_2 - E_3) + \delta(p_0 - E_1 + E_2 + E_3) \right. \\
\left. \left. - \delta(p_0 - E_1 - E_2 + E_3) - \delta(p_0 - E_1 - E_2 - E_3) \right] \right]. \quad (4.6) \]

To evaluate, for instance, the term proportional to \( \delta(p_0 - E_1 + E_2 - E_3) \) one proceeds as follows:
\[
I(p) \equiv \frac{e^{\beta p_0} e^{\beta^2 (E_1 + E_2 + E_3)}}{\epsilon^{\beta^2 (E_1 + E_2 + E_3)}} \delta(p_0 - E_1 + E_2 - E_3) \\
= \frac{e^{\beta(E_1 - E_2 + E_3)}}{\epsilon^{\beta^2 (E_1 - E_2 + E_3)}} f_F(E_1) f_F(E_2) f_F(E_3) \delta(p_0 - E_1 + E_2 - E_3) \\
= (e^{\beta(E_1 + E_2)} + e^{\beta E_3}) f_F(E_1) f_F(E_2) f_F(E_3) \delta(p_0 - E_1 + E_2 - E_3) \cdot \quad (4.7) \]

Then note
\[ e^{\beta(E_1 + E_2)} + e^{\beta E_3} = f_F^{-1}(E_1) f_F^{-1}(E_2) + f_F^{-1}(E_2) f_F^{-1}(E_1) - f_F^{-1}(E_1) - f_F^{-1}(E_2). \]  
\[ (4.8) \]

Therefore
\[ I(p) = \{ f_F(E_2)(1 - f_F(E_1) - f_F(E_2)) + f_F(E_1) + f_F(E_3) \} \delta(p_0 - E_1 + E_2 - E_3). \]  
\[ (4.9) \]

Other terms can be treated in a similar manner and in the end \( \text{Im} \tilde{\Sigma}(p) \) is brought to the standard form. We like to stress that even though the expression(4.5) is rather involved our evaluation is much simpler than the one in I.T.F.

The quantity obtained above is directly related to the relaxation time and thus is the fundamental quantity in the linear response theory. Since our work is of the technical nature we shall not discuss physical applications or the interpretation of the imaginary part. The readers are referred to Refs. 12) and 18) on these subjects.
§ 5. Summary

In this work we have given the summary of all the essential techniques and formulae involved in the T.F.D. perturbation. The machinery has been applied to evaluate both the real and the imaginary part of Feynman diagrams at the two-loop level in the basic theories with fermion. For the real part we have obtained the two-loop effective potential. And as for the imaginary part a very simple derivation has been presented. Through the examples presented in the text one would hopefully recognize the expediency of T.F.D. perturbation, in regularizing the integral and extracting the divergence and in calculating the imaginary part.

Acknowledgements

One of us (Y.F.) wishes to thank Professor A. Salam for his warm hospitality at the International Center for Theoretical Physics (Trieste, Italy) where the major part of the work was completed. He also thanks Yukawa Foundation for a financial support.

Appendix

In this appendix we provide the calculation of the two-loop diagram ((C) of Fig. 6),

\[
I(C) = \frac{f^2}{2} \text{Tr} \int \frac{d^n k}{(2\pi)^n} \frac{d^n l}{(2\pi)^n} (k + m)(\lambda + m) \\
\times (A_F(k) - 2\pi \delta((k^2 - m^2)f_F(k)))(A_F(k - l) + 2\pi \delta((k - l)^2 - M^2)f_F(k - l)) \\
\times (A_F(l) - 2\pi \delta((l^2 - m^2)f_F(l))).
\]

(A.1)

Using

\[
2k \cdot l + 2m^2 = -((k - l)^2 - M^2) + (k^2 - m^2) + (l^2 - m^2) + 4m^2 - M^2
\]

(A.2)

and

\[
(k^2 - m^2)\delta(k^2 - m^2) = 0,
\]

(A.3)

one can rewrite the integral as follows:

\[
I(C) = I_0(C) + I_{\lambda-1}(C) + I_{\lambda-2}(C),
\]

(A.4)

\[
I_0(C) = -if^2\int \frac{d^n k}{(2\pi)^n} \frac{d^n l}{(2\pi)^n} \\
\times \left\{ \frac{i}{k^2 - m^2 + i\epsilon} \frac{i}{l^2 - m^2 + i\epsilon} - 2\frac{i}{k^2 - m^2 + i\epsilon} \frac{i}{l^2 - M^2 + i\epsilon} \\
+ 2(4m^2 - M^2) \right\}.
\]

(A.5)

\[
I_{\lambda-1}(C) = 2if^2\int \frac{d^n k}{(2\pi)^n} \frac{d^n l}{(2\pi)^n}
\]
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\[
\times \left\{ \left( \frac{i}{k^2 - m^2 + i\epsilon} - \frac{i}{k^2 - M^2 + i\epsilon} \right) 2\pi \delta \left( l^2 - m^2 \right) f_R(l) \right. \\
+ \frac{i}{k^2 - m^2 + i\epsilon} 2\pi \delta \left( l^2 - M^2 \right) f_R(l) \left\} - f^2 (4m^2 - M^2) \int \frac{d^n k}{(2\pi)^n} \frac{d^n l}{(2\pi)^n} \\
\times \left\{ \frac{2i}{k^2 - m^2 + i\epsilon} (k - l) \frac{i}{(k - l)^2 - m^2 + i\epsilon} 2\pi \delta \left( l^2 - m^2 \right) f_R(l) \right. \\
\left. - \frac{i}{k^2 - m^2 + i\epsilon} (k - l) \frac{i}{(k - l)^2 - m^2 + i\epsilon} 2\pi \delta \left( l^2 - M^2 \right) f_R(l) \right\}, \quad \text{(A.6)}
\]

\[I_{s-2}(C) = -2if^2 \int \frac{d^n k}{(2\pi)^n} \frac{d^n l}{(2\pi)^n} (2\pi)^2 \delta (k^2 - m^2) \delta (l^2 - M^2) f_R(k) f_R(l) \]

\[- if^2 \left( \int \frac{d^n k}{(2\pi)^n} 2\pi \delta (k^2 - m^2) f_R(k) \right)^2 - if^2 (4m^2 - M^2) \int \frac{d^n k}{(2\pi)^n} \frac{d^n l}{(2\pi)^n} \]

\[\times \left\{ \frac{2}{(k - l)^2 - M^2} (2\pi)^2 \delta (k^2 - m^2) \delta (l^2 - M^2) f_R(k) f_R(l) \right. \\
\left. - \frac{1}{(k - l)^2 - M^2} (2\pi)^2 \delta (k^2 - m^2) \delta (l^2 - m^2) f_R(k) f_R(l) \right\}. \quad \text{(A.7)}
\]

The term with three statistical factors has been dropped since we are interested in the real part (after removing overall $-i$ from the integral). $I_{s-1}(C)$ and $I_{s-2}(C)$ are not difficult to calculate. We show below the integration of $I_0(C)$. The integration of the first two terms in (A.5) can be readily done. The result is

\[-if^2 \frac{1}{(4\pi)^2} \left[ \Gamma \left( -1 + \frac{\epsilon}{2} \right) \right]^2 m^2 \left( 1 - \frac{\epsilon}{2} \ln m^2 + \frac{\epsilon^2}{8} \ln^2 m^2 + O(\epsilon^3) \right)^2 \]

\[+ 2if^2 \frac{1}{(4\pi)^2} \left[ \Gamma \left( -1 + \frac{\epsilon}{2} \right) \right]^2 m^2 M^2 \left( 1 - \frac{\epsilon}{2} \ln M^2 + \frac{\epsilon^2}{8} \ln^2 M^2 + O(\epsilon^3) \right) \]

\[\times \left( 1 - \frac{\epsilon}{2} \ln M^2 + \frac{\epsilon^2}{8} \ln^2 M^2 + O(\epsilon^3) \right). \quad \text{(A.8)}
\]

To evaluate the last term of (A.5) one may proceed as follows:

\[I = \int \frac{d^n k}{(2\pi)^n} \frac{d^n l}{(2\pi)^n} \frac{1}{k^2 - m^2 + i\epsilon} \frac{1}{(k - l)^2 - m^2 + i\epsilon} \frac{1}{l^2 - M^2 + i\epsilon} \]

\[= \int_0^1 dx \int \frac{d^n k}{(2\pi)^n} \frac{d^n l}{(2\pi)^n} \frac{1}{k^2 - m^2 + i\epsilon} \frac{1}{(k + x(1 - x))^2 - m^2 + i\epsilon} \frac{1}{l^2 - M^2 + i\epsilon} \]

\[= \frac{i}{(4\pi)^2} \Gamma \left( \frac{\epsilon}{2} \right) (-x(1-x))^{-\epsilon/2} \int_0^1 dx \int \frac{d^n l}{(2\pi)^n} \frac{1}{l^2 - \frac{m^2}{x(1-x)^{\epsilon/2}}} \frac{1}{l^2 - M^2} \]

\[= \frac{i}{(4\pi)^2} \Gamma \left( 1 + \frac{\epsilon}{2} \right) \int_0^1 dx (-x(1-x))^{-\epsilon/2} \int_0^1 dy \int \frac{d^n l}{(2\pi)^n} \frac{3^{\epsilon/2-1}}{(l^2 - M^2)^{1+\epsilon/2}} \]

\[\left( M^2 = m^2 (x(1-x))^{-1} \left( y + \frac{M^2}{m^2} x(1-x)(1-y) \right) \right) \]
\[
= \frac{1}{(4\pi)^4} \Gamma(-1+\varepsilon)(m^2)^{1-\varepsilon} \int_0^1 dx \int_0^1 dy y^{\varepsilon/2-1}(x(1-x))^{1/2-1} \\
\times \left(y + \frac{M^2}{m^2}x(1-x)(1-y)\right)^{1-\varepsilon}.
\] (A.9)

Poles in the above integral are at \(x \to 0\), \(x \to 1\) and \(y \to 0\). Their contribution turns out to be

\[
I_{\text{pole}} = \frac{1}{(4\pi)^4} \Gamma(-1+\varepsilon)(m^2)^{1-\varepsilon} \frac{2}{\varepsilon}\left\{\left(1+\frac{\varepsilon}{2} + \frac{\varepsilon^2}{4}\right) + \left(1+\frac{\varepsilon}{2} + \frac{\varepsilon^2}{4}\right) + \left(1+\varepsilon + \frac{\varepsilon^2}{24}\right)\right\}.
\] (x \to 0) (x \to 1) (y \to 0) (A.10)

The integral can be evaluated by breaking it up into two parts,

\[
I = I_{\text{pole}} + (I - I_{\text{pole}}).
\] (A.11)

The evaluation of the second term in the above is lengthy but straightforward. In the end one finds

\[
I = \frac{1}{(4\pi)^4} \Gamma(-1+\varepsilon)(m^2)^{1-\varepsilon} \\
\times \left\{\frac{1}{\varepsilon} \left(4 + 2\frac{M^2}{m^2}\right) + 2 + \frac{M^2}{m^2} - 2\frac{M^2}{m^2}\ln\frac{M^2}{m^2} + \varepsilon\left(\frac{3}{2} - \frac{\pi^2}{12}\right)\frac{M^2}{m^2} \ln\frac{M^2}{m^2} + M^2\ln^2\frac{M^2}{m^2} - g(\alpha, x, y) + O(\varepsilon^2)\right\},
\] (A.12)

where

\[
g(\alpha, x, y) = \alpha \int_0^1 dx \int_0^1 dy \frac{1}{y} \ln\left\{1 + y\left(\frac{1}{ax(1-x)} - 1\right)\right\} \quad (\alpha = \frac{M^2}{m^2})
\] (A.13)

and \(C\) is a constant independent of \(\alpha\). Using \((m^2)^{1-\varepsilon} = m^2(1 - \varepsilon \ln m^2 + (\varepsilon^2/2)\ln^2 m^2)\), etc., one arrives at the final result

\[
I_o(C) = -\frac{if^2}{(4\pi)^4} \left\{-2m^2M^2\left(-\left(\frac{2}{\varepsilon} - 2\gamma + 2\right)(\ln m^2 + \ln M^2) + \frac{1}{2}\ln^2 m^2M^2\right) \\
+ m^2\left\{-2\left(\frac{2}{\varepsilon} - 2\gamma + 2\right)\ln m^2 + 2\ln^2 m^2\right\} \\
-(4m^2 - M^2)\left\{-\left(\frac{2}{\varepsilon} - 2\gamma + 2\right)(2m^2\ln m^2 + M^2\ln M^2)\right\} \\
-(2m^2\ln m^2 + M^2\ln M^2) + 2m^2\ln^2 m^2 + M^2\ln^2 M^2 - m^2 g(\alpha, x, y)\right\} \right\}.
\] (A.14)

References


\* We have retained only the logarithmic terms.
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3) Recently scheme dependence has been discussed by several people in the context of Q.C.D.

E. V. Shuryak, ibid. 61 (1980), 71.


L. V. Keldysh, JETP 20 (1965), 1018.
