Relative Motion in Stochastic Velocity and Force Fields

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Fundamental Fokker-Planck equations for the probability distribution function \(P(\xi, t)\) for the relative distance \(\xi\) of two test particles are derived exactly for Gaussian random velocity and force fields. The present formulation clarifies the validity of previous approximate theories on the relative diffusion in random fields. That is, almost all previous results are justified in higher dimensions. The case of one-dimensional velocity field is a remarkably exceptional one in which the relative diffusion \(\langle \xi^2(t) \rangle\) shows a fractional power-law behavior, namely \(\langle \xi^2(t) \rangle \sim t^{1/2}\) for long time \(t\), because a long-time velocity correlation exists between two adjacent particles.

§ 1. Introduction

Recently, an increasing interest has been attracted to relative diffusion in turbulent fluids and plasmas. One of the purposes of these works is to explain the strong correlation between particles moving in random fields. In plasmas, a long-time density granulation called clumps is observed. Dupree first studied this phenomenon theoretically.\(^1\) He showed that any two particles move together in the phase space with little deviation, if they are initially located very close to each other. The lifetime of this correlated motion is given by

\[
\tau_{cl} = \tau_0 \ln(1/x^2 \xi_0^2)
\]

(1.1)

with

\[
\tau_0 = (4x^2D)^{-1/3}.
\]

(1.2)

Here, \(x\) denotes the characteristic wave number of the turbulent electromagnetic field, \(D\) is the velocity diffusion constant, and \(\xi_0\) is the initial separation of the particles. For time \(t\) much longer than \(\tau_{cl}\), the clumps decay and the distance between two particles increases exponentially with \(t\).

More precise analyses were given by Misguich and Balescu.\(^2,3\) Their theory can describe the motion throughout the whole time region, i.e., initial granulation, exponential increasing of the distance, and final region where the two particles move independently so that the mean square of their distance \(y(t) = \langle (X_1(t) - X_2(t))^2 \rangle\) is proportional to \(t^3\). This is known as Richardson's law.

Very recently, one of the authors (M. S.) investigated\(^4,5\) this subject from a viewpoint of the scaling theory.\(^6,7\) He showed that the time evolution of \(y(t)\) is described in terms of a scaled parameter

\[
y(t) = x^2 \xi_0^2 \exp(t/\tau_0)
\]

(1.3)

as

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Here, \( f^{(\infty)}(\tau) \) is a scaling function whose asymptotic form is \( \tau \) or \((1/6)(1n\tau)^{3}\) according as \( \tau \to 0 \) or \( \tau \to \infty \).

In the previous works mentioned above, the equations for \( y(t) \) were derived by some approximate methods. Their forms are similar to, but slightly different from each other. One of the purposes of the present study is to re-derive the equation for \( y(t) \) in a strict way.

In the following sections, two cases are investigated: (I) random velocity field and (II) random force field. We derive an equation for the probability distribution function of two particles, from which we can immediately obtain the equation for \( y(t) \) and the diffusion constant of relative motion. It will be found that for case (I) \( y(t) \) grows exponentially with \( t \) in the initial region. The characteristic time of this exponential growth agrees with Eq. (1.1). For \( t \to \infty \), \( y(t) \) is proportional to \( t \), as is expected. One-dimensional case is an exception, where \( y(t) \) is proportional to \( t^{1/2} \) for \( t \to \infty \). This is because of the accumulation of the probability distribution \( P(y(t), t) \) at \( y = 0 \) as \( t \) goes to infinity. Namely, it is most probable that any two particles embedded in a one-dimensional random velocity field are infinitesimally close to each other as \( t \to \infty \).

§ 2. Velocity field case

We consider two test particles moving in a turbulent velocity field \( \mathbf{v}(\mathbf{x}, t) \) which is stochastic in space and time. We assume\(^{4,5}\) that \( \mathbf{v}(\mathbf{x}, t) \) is a Gaussian random variable whose mean and variance are given by

\[
\langle \mathbf{v}_{a}(\mathbf{x}, t) \rangle = 0, \tag{2.1}
\]

\[
\langle \mathbf{v}_{a}(\mathbf{x}, t) \mathbf{v}_{\beta}(\mathbf{x}', t') \rangle = 2\delta_{a\beta}D(\mathbf{x} - \mathbf{x}')\delta(t - t') \quad (a, \beta = x, y, z) \tag{2.2}
\]

The equations of motion are written as

\[
\dot{X}_{i}(t) = \mathbf{v}(X_{i}(t), t), \quad i = 1, 2, \tag{2.3}
\]

where \( X_{i}(t) \) is the position of the \( i \)-th particle.

What we are now interested in is the relative motion of the two particles. The two-particle probability distribution is defined by

\[
P(x_{1}, x_{2}; t) = \langle \delta(x_{1} - X_{1}(t)) \delta(x_{2} - X_{2}(t)) \rangle. \tag{2.4}
\]

In order to obtain the Fokker-Planck equation for \( P(x_{1}, x_{2}; t) \), we differentiate Eq. (2.4) with respect to time and substitute Eq. (2.3). Thus, we have

\[
\frac{\partial P}{\partial t} = -\frac{\partial}{\partial x_{1}} \langle \mathbf{v}(x_{1}, t) \delta(x_{1} - X_{1}(t)) \delta(x_{2} - X_{2}(t)) \rangle \]

\[
-\frac{\partial}{\partial x_{2}} \langle \mathbf{v}(x_{2}, t) \delta(x_{1} - X_{1}(t)) \delta(x_{2} - X_{2}(t)) \rangle. \tag{2.5}
\]

This equation can be rewritten by using Novikov's theorem\(^{6}\) as follows:

\[
\frac{\partial P}{\partial t} = D(0) \left( \frac{\partial^{2}}{\partial x_{1}^{2}} + \frac{\partial^{2}}{\partial x_{2}^{2}} \right) P + 2 \frac{\partial^{2}}{\partial x_{1} \partial x_{2}} D(x_{1} - x_{2}) P, \tag{2.6}
\]
where we have assumed that \( \partial_x D(0) = 0 \). Transforming the spatial coordinates from \( x_1 \) and \( x_2 \) to \( \xi \equiv x_1 - x_2 \) and \( \eta \equiv (x_1 + x_2)/2 \) and eliminating \( \eta \) by integration, we finally obtain

\[
\frac{\partial P(\xi)}{\partial t} = 2 \frac{\partial^2}{\partial \xi^2} [D(0) - D(\xi)] P(\xi, t),
\]  

where

\[
P(\xi, t) = \int P\left(\frac{1}{2} \xi + \eta, \frac{1}{2} \xi - \eta; t \right) d\eta.
\]  

Equation (2.7) is the Fokker-Planck equation in the relative coordinates, from which we can immediately derive the following relation:

\[
\frac{d}{dt} \langle \xi^2(t) \rangle = 4[D(0) - D(\xi)].
\]  

For small \( |\xi| \), we can approximate that

\[
D(\xi) \approx D(0)(1 - x^2 \xi^2).
\]

Then, Eq. (2.9) is written as

\[
\frac{d}{dt} \langle \xi^2(t) \rangle = 4D(0)x^2 \langle \xi^2(t) \rangle.
\]

Namely,

\[
\langle \xi^2(t) \rangle = \xi_0^2 \exp[4D(0)x^2 t],
\]

where \( \xi_0 \) is the initial value of \( |\xi| \). The time constant of this exponential growth is \( (4D(0)x^2)^{-1} \) which corresponds to Eq. (1.2) for the force field case.

If \( \langle D(\xi) \rangle \) is small, we may neglect it in Eq. (2.9) to obtain

\[
\frac{d}{dt} \langle \xi^2(t) \rangle = 4D(0),
\]

which shows that the distance increases linearly with \( t \), that is, the two particles move independently to each other.

The above statement is, however, not correct in one-dimensional cases. Then, \( \langle D(\xi) \rangle \) is not negligible, even if \( \langle \xi^2(t) \rangle \) is large, so that Eq. (2.13) does not hold.

In order to show this anomalous property in one dimension, we consider a simple example:

\[
\begin{align*}
D(\xi) &= D(0)(1 - x^2 \xi^2) \quad \text{for } |\xi| \leq x^{-1}, \\
D(\xi) &= 0 \quad \text{for } |\xi| > x^{-1}.
\end{align*}
\]  

Equation (2.7) is rewritten as

\[
\frac{\partial P(\xi, t)}{\partial t} = \frac{\partial^2}{\partial \xi^2} \xi^2 P(\xi, t) \quad \text{for } |\xi| \leq 1,
\]

\[
\frac{\partial P(\xi, t)}{\partial t} = \frac{\partial^2}{\partial \xi^2} P(\xi, t) \quad \text{for } |\xi| > 1,
\]
where $\xi$ and $t$ have been normalized as $\kappa\xi \to \xi$ and $2D(0)\kappa^2 t \to t$, respectively. With the initial condition

$$P(\xi, 0) = \delta(\xi - \xi_0),$$

Eqs. (2.15) and (2.16) are solved explicitly as follows,

$$\bar{P}(\xi, s) = \begin{cases} 0, & (\xi \leq 0) \\ a\xi^{-(3+\mu)/2}, & (0 < \xi \leq \xi_0) \\ b\xi^{-(3+\mu)/2} + c\xi^{-(3-\mu)/2}, & (\xi_0 < \xi \leq 1) \\ d\exp[-\sqrt{s}(\xi - 1)], & (1 < \xi) \end{cases}$$  \hspace{1cm} (2.17)

where $\bar{P}(\xi, s)$ is the Laplace transform of $P(\xi, t)$:

$$\bar{P}(\xi, s) = \int_0^\infty P(\xi, t)\exp(-st)\,dt$$  \hspace{1cm} (2.18)

and the coefficients $a, b, c$ and $d$ are given by

$$a = \mu^{-1}\xi_0^{(1-\mu)/2} + \xi_0^{(1+\mu)/2}\left\{1 + \frac{1}{2}s^{-1/2}(1-\mu) - \mu^{-1}\right\},$$  \hspace{1cm} (2.19)

$$b = a - \mu^{-1}\xi_0^{(1-\mu)/2},$$  \hspace{1cm} (2.20)

$$c = \mu^{-1}\xi_0^{(1+\mu)/2},$$  \hspace{1cm} (2.21)

$$d = \xi_0^{(1+\mu)/2}\left\{1 + \frac{1}{2}s^{-1/2}(1-\mu)\right\}. \quad (\mu \equiv (1+4s)^{1/2})$$  \hspace{1cm} (2.22)

The time evolution of $P(\xi, t)$ is illustrated in Fig. 1. In Eqs. (2.17)~(2.22), we have assumed that $0 < \xi_0 \leq 1$, for simplicity. It is easy to write down similar results in other cases of $\xi_0$. Conclusions stated below are not affected by the choice of $\xi_0$.

From Eq. (2.17), $\langle \xi(t) \rangle$ is calculated in the Laplace transform as

![Fig. 1. Probability distribution of $\xi$ at various times: (a) $t=0.1$, (b) 0.2, (c) 0.5, (d) 1.0, (e) 2.0, with the initial condition that $\xi_0=0.1$.](image-url)
\[ \langle \xi^2 \rangle_s = (s-2)^{-1} \xi_0^2 + 64 \left( \frac{-1}{(\mu+3)(\mu-3)(\mu+1)^2} + \frac{1}{(\mu^2-1)^{3/2}(\mu+3)} \right) \xi_0^{(1+\nu)/2}, \quad (2.23) \]

where

\[ \langle \xi^2 \rangle_s = \int_0^\infty \langle \xi^2(t) \rangle \exp(-st) \, dt. \quad (2.24) \]

Figure 2 shows the time evolution of \( \langle \xi^2(t) \rangle \). The asymptotic forms of \( \langle \xi^2(t) \rangle \) for small and large \( t \) are given by

\[ \langle \xi^2(t) \rangle \rightarrow \begin{cases} 
\xi_0^2 \exp(2t), & (t \to 0^+) \\
\sqrt{\pi} \xi_0 t^{1/2}, & (t \to \infty)
\end{cases} \quad (2.25) \]

Similarly we obtain

\[ \langle \xi^p(t) \rangle \rightarrow \begin{cases} 
\xi_0^p \exp[p(p-1)], & (t \to 0^+) \\
\xi_0 \Gamma(p+1) \Gamma \left( \frac{p+1}{2} \right) t^{(p-1)/2}. & (t \to \infty)
\end{cases} \quad (2.26) \]

The long time behavior does not agree with Eq. (2.13). This is because \( P(\xi, t) \) accumulates at \( \xi = 0 \) for \( t \to \infty \), as shown in Fig.1, so that \( \langle D(\xi) \rangle \) does not vanish even if \( \langle \xi^2(t) \rangle \) is large. The accumulation of \( P(\xi, t) \) at \( \xi = 0 \) means that it is most probable for the two particles to be adjacent infinitesimally as \( t \) goes to infinity. Indeed, even if a pair of particles are separated at a long distance, they will necessarily come close within any small distance some time, because their motion is restricted in one dimension. Once they come close to each other, their velocities correlate strongly, so that they do not easily separate again. This is the reason of the accumulation of \( P(\xi, t) \).

In higher dimensions, the probability that two particles approach within a small

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Fig. 2. Time evolution of \( \langle \xi^2(t) \rangle \) with the same initial condition as Fig. 1. Curve (a) is in the small \( \xi \) approximation, i.e., \( D(\xi) = D(0)(1-x^2 \xi^2) \); (b) is written on the assumption that \( D(\xi) = D(\sqrt{\xi^2}) \); (c) is by the present theory, which gives the true asymptotic form: \( y \sim t^{1/2} \). The dashed line indicates the clumps life-time given by Eq. (1.1).
distance is very small, if the initial separation is large. Consequently, there is no accumulation of \( P(\xi, t) \). It is easy to confirm this by an explicit calculation for such a simple case as (2.14). Thus, in higher dimensions, the previous decoupling approximation\(^4\) for the moment equation (2.9), namely,

\[
\frac{d}{dt} y(t) = 4 \{ D(0) - D(y(t)^{1/2}) \} \tag{2.27}
\]

with \( y(t) = \langle \xi^2(t) \rangle \), is allowed. Consequently, we obtain the scaling solution\(^4\)

\[
y(t) = x^{-2} f^{(0)}(\tau) = x^{-2} \log(1 + \tau) \tag{2.28}
\]

for \( D(\xi) = D(0) \exp(-x^2 \xi^2) \), where the scaling variable \( \tau \) is given by

\[
\tau = x^2 \xi^2 \exp(4D(0)x^2t). \tag{2.29}
\]

§ 3. Force field case

We consider in this section two test-particles in a turbulent force field. Equations of motion are written as

\[
\dot{X}_i(t) = V_i(t), \tag{3.1}
\]

\[
\dot{V}_i(t) = -\frac{q}{m} E(X_i(t), t). \tag{3.2}
\]

where the force field \( E(x, t) \) is assumed\(^4,5\) to be a Gaussian white random variable satisfying

\[
\langle E(x, t) \rangle = 0 \tag{3.3}
\]

and

\[
\frac{q^2}{m^2} \langle E_a(x, t) E_b(x', t') \rangle = 2\delta_{ab} S(x - x') \delta(t - t'). \tag{3.4}
\]

We deal with a probability distribution with respect to position and velocity of two particles:

\[
P(x_1, v_1, x_2, v_2; t) = \langle \prod_{i=1}^{2} \delta(x_i - X_i(t)) \delta(v_i - V_i(t)) \rangle. \tag{3.5}
\]

In a way similar to the derivation of (2.6) in § 2, we can derive the Fokker-Planck equation for \( P(x_1, v_1, x_2, v_2; t) \) as follows:

\[
\frac{\partial P}{\partial t} = - \left( v_1 \frac{\partial}{\partial x_1} + v_2 \frac{\partial}{\partial x_2} \right) P + \left[ S(0) \left( \frac{\partial^2}{\partial v_1^2} + \frac{\partial^2}{\partial v_2^2} \right) + 2S(x_1 - x_2) \frac{\partial^2}{\partial v_1 \partial v_2} \right] P. \tag{3.6}
\]

Reducing Eq. (3.6) to an equation with respect to the relative position and velocity, \( \xi = x_1 - x_2 \) and \( u = v_1 - v_2 \), we have

\[
\frac{\partial P(\xi, u; t)}{\partial t} = -u \frac{\partial P}{\partial \xi} + 2[S(0) - S(\xi)] \frac{\partial^2 P}{\partial u^2}. \tag{3.7}
\]

This is our desired Fokker-Planck equation. From Eq. (3.7), we obtain the following
moment equations

\[
\frac{d}{dt} \langle \xi^2(t) \rangle = 2 \langle \xi u \rangle ,
\]

\[
\frac{d}{dt} \langle \xi u \rangle = \langle u^2 \rangle ,
\]

\[
\frac{d}{dt} \langle u^2 \rangle = 4 \langle [S(0) - S(\xi)] \rangle
\]

or

\[
\frac{d^3}{dt^3} \langle \xi^2 \rangle = 8 \langle S(0) - S(\xi) \rangle .
\]

This is the starting equation for the relative diffusion.

If \( P(\xi, u; t) \) has a considerably finite value only around \( \xi = 0 \), we can make the following approximation

\[
\langle S(\xi) \rangle \approx S(0) \left[ 1 - x^2 \langle \xi^2 \rangle \right].
\]

Then, Eq. (3.11) becomes the linearized equation

\[
\frac{d^3}{dt^3} \langle \xi^2 \rangle = 8 S(0) x^2 \langle \xi^2 \rangle ,
\]

which reproduces Dupree’s equation.\(^1\)

Consequently, the distance of two particles grows exponentially, if the initial separation is very small \((\xi_0 \ll x^{-1})\). The time when the distance exceeds \( x^{-1} \), i.e., the clumps life-time, is given by \(^2, 3, 5, 8\)

\[
\tau_{\text{cl}} = \left[ 8 x^2 S(0) \right]^{-1/3} \ln \left( 1/\xi_0^3 x^2 \right),
\]

which reproduces Dupree’s result [Eq. (1.1)].

If \( t \) is large, \( \langle S(\xi) \rangle \) is very small and it can be neglected in Eq. (3.11). Then we have\(^3, 5, 8\)

\[
\langle \xi^2(t) \rangle \rightarrow \left( 4/3 \right) S(0) t^3, \quad (t \rightarrow \infty)
\]

which is so-called Richardson’s law.

Contrary to the velocity-field case discussed in § 2, there is no accumulation of \( P(\xi, u; t) \) near \( \xi = 0 \), so that \( \langle S(\xi) \rangle \) decreases to zero as \( t \rightarrow \infty \) in all dimensions. As for the velocity-field case, the nearer the two particles come to each other, the stronger their velocities are correlated. This is the reason of the anomalous long-time behavior in one dimension. For the force-field case, on the contrary, position and velocity are independent variables. Velocity correlation is not necessarily strong even when the two particles come close. Thus, no accumulation of \( P(\xi, u; t) \) occurs. Consequently the previous decoupling approximation,\(^3, 8\) namely,

\[
\frac{d^3}{dt^3} y(t) = 8 \{ S(0) - S(y(t)^{1/2}) \}
\]

for the relative diffusion \( y(t) = \langle \xi^2(t) \rangle \) is justified in the above sense that \( \langle S(\xi) \rangle \) goes to
zero in any dimensions. Thus, we arrive at the scaling solution $t^{3/2}$, as was given by one of the present authors.\textsuperscript{4,5}

\section{Summary and discussion}

We have obtained the fundamental Fokker-Planck equations for the relative motion of two particles in random velocity and force fields. It is confirmed that the distance between two particles grows exponentially at first with $t$, so long as the initial separation is much smaller than the correlation length of the field. The characteristic time of the exponential growth is related to the diffusion time of particles through the correlation length. After a long time, the two particles move without correlation. A remarkable exception is the velocity-field case in one dimension, where the mean square distance is proportional to $t^{1/2}$ for large $t$. This is due to the long-time velocity correlation between two adjacent particles. This is the most important result in the present paper.

The present study is made on the basis of the extrinsic stochasticity models, where the stochasticity of the field is given by an external source. In reality, turbulence is attributed to the intrinsic stochasticity such as a chaotic behavior of the Navier-Stokes equation. This is left out of consideration in this work.

The present analytical results will be compared with those of Monte Carlo simulations\textsuperscript{8} in the near future.

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\section*{References}