$$N^* = i\omega M^* + C_1 \exp\left(-y\sqrt{\frac{i\omega}{\nu}}\right) + C_2 \exp\left(y\sqrt{\frac{i\omega}{\nu}}\right) \quad \text{(B3)}$$

As y becomes large, N^* must approach $i\omega M^*$, so $C_2 = 0$. If the longitudinal motion of the cylindrical wall follows that of the base, then the wall velocity is given by $i\omega Y_L e^{i\omega t}$. The fluid velocity at the wall is the same, and therefore, $i\omega M^* + C_1 = i\omega Y_L$. With C_1 and C_2 determined, equation (B3) is rewritten as

$$N^* = i\omega M^* + i\omega(Y_L - M^*) \exp\left(-\sqrt{i}\,\frac{y}{\delta_0}\right) \quad \text{(B4)}$$

where δ_0 is the oscillation boundary-layer thickness, $\sqrt{\frac{\nu}{\omega}}$.

The wall shear stress, τ, is obtained by differentiating equation (B4) and evaluating $-\mu \frac{\partial u}{\partial y}$ at $y = 0$. The result, written in complex notation, is

$$\tau^* = (i)^{3/2} \left(\frac{\mu\omega}{\delta_0}\right)(Y_L - M^*). \quad \text{(B5)}$$

Equation (A7) is now modified to include the shear force acting on the fluid at the wall. The net force becomes

$$F_x = \rho A dx \left(g - \frac{1}{\rho}\frac{\partial P}{\partial x}\right) + \tau\pi D dx. \quad \text{(B6)}$$

The last term in this equation is obtained from equation (B5).

To account for viscoelastic behavior, the complex modulus of elasticity [17] of the cylinder wall is written as $E^* = E' + iE''$. For small amounts of damping the real portion is essentially the same as the ordinary modulus of elasticity. In this case $E^* = E(1 + i\phi)$ where ϕ is the loss factor $\frac{E''}{E'}$. The oscillating components of stress and strain are obtained from the development given in Appendix A, but with E replaced by E^* to account for the damping effect. Equation (A5) is rewritten as

$$E_x^* \equiv \frac{B}{\left[1 + \frac{B}{\frac{s}{D}E^*}\right]} = \left(\frac{1 + i\phi}{1 + i\gamma\phi}\right) E_x. \quad \text{(B7)}$$

The oscillating component of force is obtained from equation (B6) and is equated to the inertia term as in equation (A6). With $Y_1(x, t)$ expressed as $M^*(x)e^{i\omega t}$, the result is

$$\rho A dx (i\omega)^2 M^* = -A dx \frac{d}{dx}\left(-E_x^* \frac{dM}{dx}\right) + \tau^*\pi D dx. \quad \text{(B8)}$$

Inserting τ^* from equation (B5) gives

$$c^{*2}\frac{d^2M^*}{dx^2} + \omega^2 M^* + 4(i)^{3/2}\left(\frac{\delta_0}{D}\right)\omega^2(Y_L - M^*) = 0 \quad \text{(B9)}$$

where $c^* = \sqrt{\frac{E_x^*}{\rho}}$, the complex wave velocity of the system. Note that the last term is negligibly small when $\frac{\delta_0}{D} \ll 1$, provided that M^* is the same order of magnitude as Y_L. For experiments carried out in this investigation, $\frac{\delta_0}{D}$ was of the order of 10^{-3}. It is concluded, therefore, that the damping effect due to viscous fluid shear at the wall was very small.[5] For this reason the last term in equation (B9) is neglected. The solution for the complex pressure amplitude follows essentially the same procedure as was used in obtaining equation (11). The result is identical in form and is written as

[5] There is another source of energy dissipation associated with viscous action in the core of the fluid due to its periodic compression and expansion. In the present study this effect was found to have a magnitude even smaller than that of the wall friction.

$$\frac{\frac{\Delta P^*}{\rho g L}}{G} = \frac{\sin \Omega^* \frac{x}{L}}{\Omega^* \cos \Omega^*} \quad \text{(B10)}$$

In this formulation the oscillating pressure is given by $\Delta P^* e^{i\omega t}$. Ω^* is the dimensionless complex frequency defined as $\frac{\omega L}{c^*}$. Using the expression given in equation (B7), it can be shown that $\Omega^* = \Omega\sqrt{\frac{1 + i\gamma\phi}{1 + i\phi}}$.

The magnitude of the pressure oscillations is obtained from equation (B10) and is written as

$$\frac{\left|\frac{\Delta P^*}{\rho g L}\right|}{G} = \left|\frac{\sin\left(\Omega\sqrt{\frac{1 + i\gamma\phi}{1 + i\phi}}\,\frac{x}{L}\right)}{\Omega\sqrt{\frac{1 + i\gamma\phi}{1 + i\phi}}\cos\left(\Omega\sqrt{\frac{1 + i\gamma\phi}{1 + i\phi}}\right)}\right| \quad \text{(B11)}$$

DISCUSSION

George Rudinger[6]

I should like to ask the authors whether they have established that the bubbles consist of vapor of the liquid or of dissolved air, or other gases. It would seem that the onset of bubble formation should be different for a pure liquid and for one that contains dissolved gas.

Authors' Closure

Dr. Rudinger has raised a question of considerable importance in connection with the physical processes accompanying bubble formation and growth in an oscillating pressure field. In the present study it was found that bubbles having sufficient time to become fully grown, in a given vibratory environment, did contain air. This was evidenced by the fact that when vibration was suddenly terminated the larger bubbles did not immediately collapse, but remained suspended in the liquid, rose slowly to the surface under the influence of buoyancy, and escaped. If these bubbles had consisted of pure vapor, they most surely would have collapsed rapidly as soon as the vibration had ceased. The air most probably came out of solution from the surrounding liquid, although at low frequencies those bubbles migrating downward from the liquid surface may have entrained air at the surface before moving downward. Despite the presence of air in these fully developed bubbles it is not clear whether it diffused into the bubbles, subsequent to their formation as vapor cavities, or actually had a part in the initial formation itself. In connection with this point it is important to note that the onset of formation was predicted fairly well on the basis of the liquid saturation pressure alone. This would indicate that vaporization is the most important single factor, as opposed to phenomena associated with dissolved gases, at the condition of incipient formation. This concept is also presented in standard texts on cavitation which correlate the cavitation threshold in terms of the liquid saturation pressure, although the minimum liquid pressure at the threshold is known to be considerably below the saturation pressure if the frequency is well into the kilocycle range. Despite the importance of the saturation pressure in establishing the condition of incipient bubble formation, dissolved gases have been found to have a significant effect also. In almost all cases which have been experimentally studied, dissolved gases have caused the onset of bubble formation to occur more readily. For a more thorough discussion of the items mentioned in the foregoing, the reader is referred to the treatment provided by Hueter and Bolt [9].

[6] Principal Physicist, Cornell Aeronautical Laboratory, Inc., Buffalo, N. Y. Mem. ASME.