American Journal of Physics

NOTE | JANUARY 01 2023

Concerning classical forces, energies, and potentials for accelerated point charges FREE

Timothy H. Boyer

Am. J. Phys. 91, 74–78 (2023)
https://doi.org/10.1119/5.0094457

Concerning classical forces, energies, and potentials for accelerated point charges

Timothy H. Boyer[a)]
Department of Physics, City College of the City University of New York, New York, New York 10031

(Received 3 April 2022; accepted 14 October 2022)

Although expressions for energy densities involving electric and magnetic fields are exactly analogous, their connections to forces and electromagnetic potentials are vastly different. For electrostatic situations, changes in the *electric* energy can be related directly to *electric* forces and to the electrostatic potential. In contrast, discussions of magnetic forces and energy changes involve two fundamentally different situations. For charged particles moving with constant velocities, the changes in both electric and magnetic field energies are provided by the external forces that keep the particles' velocities constant; there are no Faraday acceleration electric fields in this situation. However, for particles that change speed, the changes in *magnetic* energy density are related to acceleration-dependent Faraday *electric* fields. Current undergraduate and graduate textbooks deal only with highly symmetric situations, where the Faraday electric fields are easily calculated from the time-changing magnetic flux. However, in situations that lack high symmetry, such as the magnetic Aharonov–Bohm situation, the back (Faraday) acceleration electric fields of point charges may seem unfamiliar. In this article, we present a simple unsymmetric example and analyze it using the Darwin Lagrangian. In *all* cases involving changing velocities of the current carriers, it is the work done by the back (Faraday) acceleration *electric* fields that balances the *magnetic* energy changes.
https://doi.org/10.1119/5.0094457

I. INTRODUCTION

The expression $\mathbf{E}^2/(8\pi)$ for the energy density associated with electric fields is exactly analogous to the energy density $\mathbf{B}^2/(8\pi)$ for magnetic fields. However, the connections between the energy expressions and the forces exerted by the fields are vastly different in the electric and magnetic cases. The work done by electrostatic forces leads directly to changes in electric field energy. On the other hand, magnetic fields do no work, so the association between changes in magnetic energy and work is more subtle. In the quasistatic regime, *electric* field energies depend only upon the relative *positions* of charges, whereas *magnetic* field energies depend upon both the relative *positions* of charges and also their *velocities*. Current electromagnetism textbooks discuss quasistatic magnetic energy changes for only two situations: (1) charges that move with constant speed and (2) charges which change their speed but are in highly symmetric configurations. This limited perspective leaves out magnetic energy changes for situations that lack high symmetry. Here, we present a simple point-charge example lacking high symmetry, which illustrates the connections between forces, energies, and potentials in quasistatic classical electrodynamics when radiation fields are excluded.

Reference to an analogy may clarify the purpose of the present article. Magnetostatics, like electrostatics, involves only two of Maxwell's equations: $\nabla \cdot \mathbf{B} = 0$ and $\nabla \times \mathbf{B} = (4\pi/c)\mathbf{J}$. However, if one considers only situations of high symmetry, then one can apply only Ampere's law and ignore the divergence requirement for $\mathbf{B}$. Thus, if a current density $\mathbf{J}(\boldsymbol{r})$ has axial symmetry, Ampere's law plus symmetry is sufficient to determine the magnetic field $\mathbf{B}$, without any need for the divergence equation. However, these highly symmetric situations give a limited and false impression of the theory. When the situation lacks high symmetry, then the Biot–Savart law must be used to determine $\mathbf{B}$. The Biot–Savart law is a particular solution requiring *both* Maxwell equations for $\mathbf{B}$ and may be complemented by a solution of the homogeneous equations. Similarly, highly symmetric situations involving axial symmetry give the impression that Faraday's law $\nabla \times \mathbf{E} = -(1/c)\partial\mathbf{B}/\partial t$ can be used without the need for the rest of Maxwell's equations, because the confounding terms are suppressed by the symmetry. Once again, the symmetric situations give only a limited understanding of slowly varying electromagnetic fields.

With their understanding limited to symmetric situations, physicists may be unprepared to analyze situations that lack high symmetry such as the simple example in the present article or such as the Aharonov–Bohm situation. Although the azimuthally symmetric vector potential associated with a circular solenoid is often treated in junior-level courses in classical electromagnetism,[1] the experimentally realized situation of the magnetic Aharonov–Bohm effect (where electrons pass on both sides of a long solenoid) involves time-dependent interactions whose classical electromagnetic aspects are *not* azimuthally symmetric. It is often claimed in electromagnetism texts[2] and in quantum texts[3] that there is no classical electromagnetic interaction of the solenoid back on the passing electrons. This claim is made despite the magnetic energy changes associated with the overlap of a passing electron's magnetic field with the solenoid's magnetic field, which changes depending on the side of the solenoid that the electron passes. Such no-interaction claims do not take into account the fact that charges in the solenoid accelerate in response to the passing charge, resulting in Faraday acceleration fields that are related to the changes in magnetic energy. Faraday acceleration fields associated with magnetic energy changes are treated in the present article; further discussion of the Aharonov–Bohm situation is presented elsewhere.[4]

II. THE DARWIN–LAGRANGIAN APPROXIMATION

A. The Darwin Lagrangian

The classical electrodynamics of charged particles with arbitrary changing velocities is enormously complicated, particularly because electromagnetic fields depend upon sources at retarded times. In order to simplify the situation, we will consider the behavior of just two point charges that move at speeds small compared to c, the speed of light in vacuum. This situation corresponds to that described by the Darwin Lagrangian. This approximation, which excludes radiation and involves no retarded times, is an enormous simplification.

The Darwin Lagrangian for two point charges e and q with positions $\mathbf{r}_e$, $\mathbf{r}_q$, and velocities $\mathbf{v}_e$, $\mathbf{v}_q$ is given by[5,6]

$$\begin{aligned} L &= -m_e c^2\sqrt{1-v_e^2/c^2} - m_q c^2\sqrt{1-v_q^2/c^2} - \frac{eq}{|\mathbf{r}_e-\mathbf{r}_q|} \\ &\quad + \frac{eq}{2c^2}\left[\frac{\mathbf{v}_e\cdot\mathbf{v}_q}{|\mathbf{r}_e-\mathbf{r}_q|} + \frac{[\mathbf{v}_e\cdot(\mathbf{r}_e-\mathbf{r}_q)][\mathbf{v}_q\cdot(\mathbf{r}_e-\mathbf{r}_q)]}{|\mathbf{r}_e-\mathbf{r}_q|^3}\right] \\ &= -m_e c^2\sqrt{1-v_e^2/c^2} - m_q c^2\sqrt{1-v_q^2/c^2} \\ &\quad - \left[\frac{e\Phi_q(\mathbf{r}_e,t)+q\Phi_e(\mathbf{r}_q,t)}{2}\right] \\ &\quad + \left[\frac{e\mathbf{v}_e\cdot\mathbf{A}_q(\mathbf{r}_e,t)+q\mathbf{v}_q\cdot\mathbf{A}_e(\mathbf{r}_q,t)}{2c}\right], \end{aligned} \tag{1}$$

where we have implicitly defined the potentials $\Phi_e, \Phi_q, \mathbf{A}_e, \mathbf{A}_q$; explicit expressions are provided in Subsection II B. It is understood that the square roots are to be expanded in powers of $1/c^2$ through order $1/c^4$. Here, we have chosen not to expand the square roots because maintaining these exact forms leads to familiar expressions for the particle mechanical energies derived from the Lagrangian.

B. Potentials and fields in the Darwin–Lagrangian approximation

For a point charge q located at $\mathbf{r}_q(t)$ and moving with small velocity $\mathbf{v}_q(t)$ and acceleration $\mathbf{a}_q(t)$, through order $1/c^2$, the Lagrangian in Eq. (1) can be described as involving a scalar potential

$$\Phi_q(\mathbf{r},t) = \frac{q}{|\mathbf{r}-\mathbf{r}_q(t)|}, \tag{2}$$

a vector potential[6]

$$\mathbf{A}_q(\mathbf{r},t) = \frac{q}{2c}\left[\frac{\mathbf{v}_q}{|\mathbf{r}-\mathbf{r}_q|} + \frac{[\mathbf{v}_q\cdot(\mathbf{r}-\mathbf{r}_q)](\mathbf{r}-\mathbf{r}_q)}{|\mathbf{r}-\mathbf{r}_q|^3}\right], \tag{3}$$

an electric field[7]

$$\begin{aligned} \mathbf{E}_q(\mathbf{r},t) &= -\nabla\Phi_q(\mathbf{r},t) - \frac{1}{c}\frac{\partial\mathbf{A}_q(\mathbf{r},t)}{\partial t} \\ &= q\frac{\mathbf{r}-\mathbf{r}_q}{|\mathbf{r}-\mathbf{r}_q|^3}\left[1+\frac{v_q^2}{2c^2} - \frac{3}{2}\left(\frac{\mathbf{v}_q}{c}\cdot\frac{\mathbf{r}-\mathbf{r}_q}{|\mathbf{r}-\mathbf{r}_q|}\right)^2\right] \\ &\quad - \frac{q}{2c^2}\left[\frac{\mathbf{a}_q}{|\mathbf{r}-\mathbf{r}_q|} + \frac{[\mathbf{a}_q\cdot(\mathbf{r}-\mathbf{r}_q)](\mathbf{r}-\mathbf{r}_q)}{|\mathbf{r}-\mathbf{r}_q|^3}\right], \end{aligned} \tag{4}$$

and a magnetic field

$$\mathbf{B}_q(\mathbf{r},t) = \nabla\times\mathbf{A}_q(\mathbf{r},t) = q\frac{\mathbf{v}_q(t)}{c}\times\frac{\mathbf{r}-\mathbf{r}_q(t)}{|\mathbf{r}-\mathbf{r}_q(t)|^3}. \tag{5}$$

In Eqs. (3) and (4), we have suppressed the argument t in $\mathbf{r}_q(t)$, $\mathbf{v}_q(t)$, $\mathbf{a}_q(t)$ for brevity.

C. Comments on the electromagnetic fields in the Darwin approximation

We notice immediately that the scalar potential in Eq. (2) is the same as that for a point charge in electrostatics, except that the position $\mathbf{r}_q(t)$ is now time dependent. Similarly, one might be tempted to use the Biot–Savart law in magnetostatics for a point charge with $\mathbf{J}_q(\mathbf{r},t) = q\mathbf{v}_q(t)\delta^3(\mathbf{r}-\mathbf{r}_q(t))$ resulting in Eq. (5). However, these electrostatic and magnetostatic expressions are now only approximations; by employing the Darwin Lagrangian, we have moved beyond electrostatics and magnetostatics into electrodynamics. Just how different the theory is from electrostatics can be seen in the expression (4) for the electric field, which now involves velocities and even accelerations of the particle. The scalar potential appears in connection with the electric field $\mathbf{E}_q(\mathbf{r},t)$ in Eq. (4) but not in connection with the magnetic field $\mathbf{B}_q(\mathbf{r},t)$ in Eq. (5). The scalar potential $\Phi_q(\mathbf{r},t)$ depends on the relative positions $|\mathbf{r}-\mathbf{r}_q|$ of the charge and the field point. On the other hand, the vector potential $\mathbf{A}_q(\mathbf{r},t)$ is connected to *both* $\mathbf{E}_q(\mathbf{r},t)$ and $\mathbf{B}_q(\mathbf{r},t)$. The vector potential $\mathbf{A}_q(\mathbf{r},t)$ depends not only on the relative displacement $(\mathbf{r}-\mathbf{r}_q)$ between the charge and the field point but also on the velocity $\mathbf{v}_q$ of the charge q. The magnetic field $\mathbf{B}_q(\mathbf{r},t)$ depends upon *spatial* derivatives of $\mathbf{A}_q(\mathbf{r},t)$. In contrast, the electric field $\mathbf{E}_q(\mathbf{r},t)$ in Eq. (4) depends upon the *time* rate of change of the vector potential $\mathbf{A}_q(\mathbf{r},t)$, which may involve changes in the charge's position $\mathbf{r}_q$ and/or the charge's velocity $\mathbf{v}_q$. The terms in Eq. (4) for $\mathbf{E}_q(\mathbf{r},t)$ involving v_q^2 arise from *position*-dependent changes in $\mathbf{A}_q(\mathbf{r},t)$ with time. The terms involving acceleration $\mathbf{a}_q$ arise from time-dependent changes associated with *velocity*. The acceleration-dependent terms give the Faraday acceleration fields of the charged particle. The acceleration-dependent terms for the electric field in Eq. (4) appear in an electromagnetism textbook[7] published in 1940 but do not seem to appear in more recent textbooks.

D. Equations of motion from the Darwin Lagrangian

The Euler–Lagrange equations of motion for the charges e and q can be obtained in terms of the canonical momentum of a particle. However, these equations can be rewritten in the conventional form for Newton's second law as $d\mathbf{p}_{\text{mechanical}}/dt = q\mathbf{E} + q(\mathbf{v}/c)\times\mathbf{B}$, where $\mathbf{E} = -\nabla\Phi - (1/c)\partial\mathbf{A}/\partial t$ and $\mathbf{B} = \nabla\times\mathbf{A}$.

E. The total energy

The system's total energy follows directly from the Lagrangian in Eq. (1), giving a sum of mechanical, electric, and magnetic energies

$$U^{(\text{total})} = U^{(\text{m})} + U^{(\text{E})} + U^{(\text{B})}, \tag{6}$$

where the mechanical energy is

$$U^{(\mathrm{m})} = \frac{m_e c^2}{\sqrt{1 - v_e^2/c^2}} + \frac{m_q c^2}{\sqrt{1 - v_q^2/c^2}}, \tag{7}$$

the electric field energy is

$$U^{(\mathrm{E})} = \frac{1}{2}\left[e\Phi_q(\mathbf{r}_e, t) + q\Phi_e(\mathbf{r}_q, t)\right] = \frac{eq}{|\mathbf{r}_e - \mathbf{r}_q|}, \tag{8}$$

and the magnetic field energy is

$$U^{(\mathrm{B})} = \frac{1}{2}\left[e\frac{\mathbf{v}_e}{c}\cdot\left[\mathbf{A}_q(\mathbf{r}_e, t)\right] + q\frac{\mathbf{v}_q}{c}\cdot\left[\mathbf{A}_e(\mathbf{r}_q, t)\right]\right]$$
$$= \frac{eq}{2c^2}\left[\frac{\mathbf{v}_e\cdot\mathbf{v}_q}{|\mathbf{r}_e - \mathbf{r}_q|} + \frac{[\mathbf{v}_e\cdot(\mathbf{r}_e - \mathbf{r}_q)][\mathbf{v}_q\cdot(\mathbf{r}_e - \mathbf{r}_q)]}{|\mathbf{r}_e - \mathbf{r}_q|^3}\right]. \tag{9}$$

III. FIELD-POTENTIAL-ENERGY CONNECTIONS FOR ELECTRIC AND MAGNETIC FIELDS

A. Electric energy changes

In the Darwin approximation, the changes in electric field energy are directly analogous to those familiar from electrostatics. The electric energy in Eq. (8) involves only the scalar potentials, which take the same form as in electrostatics. The electric forces associated with the electric potential are directly associated with energy in the electric field. The electric energy involves only the separations $|\mathbf{r}_e - \mathbf{r}_q|$ of the charged particles.

B. Magnetic energy changes

In contrast to electric field energy changes, magnetic energy changes involve both changes in the charges' separation and also changes in their velocities, as indicated in Eq. (9). It seems convenient to separate magnetic energy changes into two separate situations—those involving constant velocity and those involving accelerating charges.

1. Charged particles moving with constant velocity

The situation of charges moving with constant velocities turns out to be uncomplicated, and the connections between forces and energy changes are *exactly* soluble. It is possible to change the magnetic field energy while keeping the particle velocities $\mathbf{v}_e$ and $\mathbf{v}_q$ constant if the charges' relative displacement $(\mathbf{r}_e - \mathbf{r}_q)$ changes. In such a case, the change in magnetic field energy is provided by the external forces, which maintain these constant velocities.[8] There are no kinetic energy changes and no Faraday acceleration electric fields.

A charged particle at rest in an inertial frame does not cause any *emf* around any closed curve in that space. However, a charged particle moving with constant velocity will indeed cause an *emf* around a general closed curve in the inertial frame. The moving charge has an associated magnetic field, and the resulting *emf* agrees with the changing magnetic flux through the closed curve. Thus, depending upon the speed of the charge in an inertial frame, a charge q may or may not cause an *emf*. Indeed, if we are dealing with only two charges moving with constant velocities, we can always go to the rest frame of one of the charges leaving us with only changes in electric field energy and no magnetic field energy changes at all.

Situations involving *motional emfs* are often treated as though the relevant charge carriers were moving with constant velocity, and so can be understood in terms of energy changes being balanced by the work done by external forces.[9]

2. Accelerating charges

Particularly troubling magnetic energy situations involve groups of charges where particles change their speeds. These cases generally involve Faraday acceleration electric forces and may include particle kinetic energy changes and/or changes in electromagnetic field energy and/or work done by external forces. We will discuss magnetic energy changes in two different examples: one highly symmetric situation and one unsymmetric situation.

IV. SITUATIONS INVOLVING A LONG SOLENOID

A. Azimuthally symmetric solenoid with increasing currents

The most familiar situation of a back (Faraday) acceleration electric field occurs for the highly symmetric situation of a circular solenoid. When the currents in the solenoid are constant, the charge carriers are *accelerating* towards the central axis since they are moving in a circle. A single charge moving in a circle does not have just electrostatic and magnetostatic fields but also has higher order terms in $1/c$, including radiation terms. However, the situation changes for a many-particle current, which can be approximated as a steady current. Even though the charges are still accelerating when the current is constant, all the higher-order contributions to the fields cancel, leaving only the electrostatic and magnetostatic contributions.[10] For a circular solenoid, the vector potential $\mathbf{A}$ may be chosen to be in the azimuthal $\hat{\phi}$-direction with $\mathbf{A} = \hat{\phi}Br/2 = \hat{\phi}2\pi nevr/c$ inside the solenoid and $\mathbf{A} = \hat{\phi}2\pi nevR^2/(cr)$ outside the solenoid, where n is the number of current carriers per unit area, e and v are the charge and speed of each current carrier, and R is the radius of the long solenoid. This corresponds to the Coulomb gauge for $\mathbf{A}$.

When the currents in a solenoid are changing in an azimuthally symmetric pattern, the charge carriers in the solenoid have a tangential acceleration associated with the same change in *speed* v for each charge. The agent changing the speed of the current carriers must provide the power against the back (Faraday) acceleration electric fields[11] corresponding to the acceleration terms in Eq. (4). The magnetic energy balance for quasistatic systems requires the existence of Faraday electric forces associated with the accelerations of the charged particles.

The back electric fields associated with the changing speeds of the charge carriers can be described in terms of the changing vector potential. Thus, for a long solenoid with slowly increasing currents, the *changes of speed* of all the current carriers give a changing vector potential with nonzero $\partial\mathbf{A}/\partial t$ and so a back *emf* from the non-vanishing (Faraday) acceleration electric field appears from $\mathbf{E} = -\nabla\Phi - (1/c)\partial\mathbf{A}/\partial t$. Outside the solenoid, even though the magnitude of the vector potential is increasing in $\mathbf{A} = \hat{\phi}2\pi nevR^2/(cr)$ due to the increase in v, the $1/r$ fall-off of the vector potential with distance from the central axis still gives $\mathbf{B} = \nabla \times \mathbf{A} = 0$. To the surprise of many students, there is an acceleration-dependent *electric* field outside the solenoid from $\partial \mathbf{A}/\partial t$, but still no *magnetic* field outside.

In elementary electromagnetism classes where examples and problems are highly symmetric, it is easy to relate the back *emf* of the solenoid to Faraday's law involving a changing magnetic field

$$emf = \oint d\mathbf{r} \cdot \mathbf{E} = -\frac{1}{c}\frac{d}{dt}\left(\int d\mathbf{S} \cdot \mathbf{B}\right), \tag{10}$$

where $\mathbf{S}$ is the area and to take advantage of the high symmetry to calculate $\mathbf{E}$. The textbooks emphasize that changing currents in a solenoid lead to changing magnetic fields and, therefore, to Faraday electric fields. They do not emphasize that this back *emf* is due directly to the electric fields produced by the *accelerating* current carriers. There is no need to go through a changing magnetic field to obtain the *emf*. The same current carriers that produce the Faraday acceleration electric field also produce the magnetic field of the solenoid in the first place.[12]

B. Unsymmetric situations

The Darwin Lagrangian is rarely mentioned in junior-level electromagnetism texts. Although the full Lienard–Wiechert fields involving retarded times are often given, the quasistatic approximation through $1/c^2$ for the electric fields in Eq. (4) goes unmentioned even in graduate-level texts. Thus, many physicists are unfamiliar with these acceleration-dependent terms, and so they are unprepared to understand the classical electromagnetic aspects of an unsymmetric situation such as that proposed by Aharonov and Bohm where electrons pass a long solenoid.

When an electron passes alongside a long solenoid, the magnetic field of the passing charge will overlap with the magnetic field of the solenoid, which changes the total magnetic energy, a relativistic $1/c^2$ effect. Depending upon which side the electron passes, the change in total magnetic field energy can be positive or negative. Naturally, one asks, "What provides the energy that balances the magnetic energy change?" If the solenoid is a superconductor, there is no external source of energy. In other cases, there is a battery providing current to the solenoid and one sometimes hears the suggestion that the battery provides the necessary energy changes. However, this side-dependent magnetic energy change does *not* correspond to a highly symmetric situation; the forces on the charged particles in the solenoid due to the passing charge's fields will be in *different directions in different locations*, some solenoid charges increasing their speeds and some decreasing their speeds.

The classical electromagnetic situation of charged particles passing a long solenoid lacks high symmetry, and therefore, we cannot use symmetry and the integral form of Faraday's law to calculate the induced electric field, or the force on the passing electrons. One must turn to a more fundamental analysis such as one through the Darwin Lagrangian.[13]

V. EXAMPLE OF TWO POINT CHARGES MOVING SIDE-BY-SIDE

A. Side-by-side charges

For the simplest possible illustration of magnetic energy changes in a situation lacking high symmetry, we consider two point charges e and q moving side-by-side with velocity $\mathbf{v}_e = \mathbf{v}_q \equiv \mathbf{v}$, along parallel frictionless rods. The rods are oriented parallel to the z-axis and are separated by a distance $l = |\mathbf{r}_e - \mathbf{r}_q|$. This system will have both electric and magnetic fields and associated energies. For motion with *constant velocity*, the electric and magnetic forces of one charge on the other are perpendicular to the frictionless rods. This same system, where the forces of constraint introduce neither energy nor net linear momentum, was used previously[14] as an illustration of Einstein's mass-energy relation $\mathcal{E} = mc^2$.

In Darwin's $1/c^2$ approximation for the energy (given in Eqs. (6)–(9)), the electric field energy is $U^{(\mathrm{E})}(\mathbf{r}_e, \mathbf{r}_q) = eq/|\mathbf{r}_e - \mathbf{r}_q| = eq/l$ as in Eq. (8), whereas the magnetic field energy is given in Eq. (9). For our simple side-by-side situation, where $\mathbf{r}_e - \mathbf{r}_q$ is perpendicular to $\mathbf{v}$, we have the magnetic field energy

$$U^{(\mathrm{B})}(\mathbf{r}_e, \mathbf{r}_q, \mathbf{v}) = \frac{eqv^2}{2c^2 l}. \tag{11}$$

B. Accelerating the system of two point charges

Suppose that external forces $\mathbf{F}_{\mathrm{ext}}^{(e)}$ and $\mathbf{F}_{\mathrm{ext}}^{(q)}$ parallel to the z-axis are applied to the two charges so as to give the particles a small common acceleration $\mathbf{a} = \hat{z}a$ for a short time τ, thus changing the particles' common speed by $\Delta v = a\tau$. The kinetic energy $mc^2(\gamma - 1)$ of each particle changes due to the change in speed. The electric field energy does not change according to the Darwin approximation in Eq. (8). However, the magnetic field energy changes from that given in Eq. (11) over to that with v replaced by $v + \Delta v$, so that

$$\Delta U^{(\mathrm{B})} = \frac{eq(v + \Delta v)^2}{2c^2 l} - \frac{eqv^2}{2c^2 l} \approx \frac{eqv\Delta v}{c^2 l}. \tag{12}$$

During the acceleration of the parallel-moving charges e and q, the vector potential $\mathbf{A} = \mathbf{A}_e + \mathbf{A}_q$ changes as the speed changes, leading to changes in the magnetic field and in the magnetic field energy. The acceleration of the charges and the change in the vector potential $\mathbf{A}$ also leads to a change in the electric field through $\mathbf{E} = -\nabla\Phi - (1/c)\partial\mathbf{A}/\partial t$. Thus, the electric field has a new *acceleration-related* component *parallel* to the velocity $\mathbf{v}$ of the particles. Therefore, from Eq. (4), each charge exerts an additional (*acceleration-dependent*) electric force

$$e\mathbf{E}_{acc-q} = q\mathbf{E}_{acc-e} = -\frac{eq\mathbf{a}}{2c^2|\mathbf{r}_e - \mathbf{r}_q|} = -\frac{eq\mathbf{a}}{2c^2 l}, \tag{13}$$

a *retarding* force, on the other charge. However, now Newton's second law for the *mechanical* momentum change in the z-direction for each particle involves *two* forces, both the external force and the electric force of one charge on the other

$$\frac{d\mathbf{p}_e}{dt} = \mathbf{F}_{\mathrm{ext}}^{(e)} - \frac{eq\mathbf{a}}{2c^2 l} \quad \text{and} \quad \frac{\mathrm{d}\mathbf{p}_\mathrm{q}}{\mathrm{dt}} = \mathbf{F}_{\mathrm{ext}}^{(\mathrm{q})} - \frac{\mathrm{q}e\mathbf{a}}{2\mathrm{c}^2\mathrm{l}}. \tag{14}$$

Therefore, the external forces (which provided the acceleration $\mathbf{a}$ of the charges) have to be larger in magnitude than these retarding forces. Rewriting Newton's second-law equations in Eq. (14), we require

$$\mathbf{F}_{\text{ext}}^{(e)} = \frac{d\mathbf{p}_e}{dt} + \frac{eq\mathbf{a}}{2c^2 l} \quad \text{and} \quad \mathbf{F}_{\text{ext}}^{(q)} = \frac{d\mathbf{p}_q}{dt} + \frac{qe\mathbf{a}}{2c^2 l}. \tag{15}$$

The change in the energy of the system due to the external forces is

$$\begin{aligned} \Delta U^{(\text{total})} &= \int_0^\tau dt \left(\mathbf{F}_{\text{ext}}^{(e)} + \mathbf{F}_{\text{ext}}^{(q)} \right) \cdot \mathbf{v} \\ &= \int_0^\tau dt \left(\frac{d\mathbf{p}_e}{dt} + \frac{d\mathbf{p}_q}{dt} + \frac{eq\mathbf{a}}{c^2 l} \right) \cdot \mathbf{v} \\ &\approxeq \Delta U^{(\mathrm{m_e})} + \Delta U^{(\mathrm{m_q})} + \frac{eq\mathbf{v}\cdot\mathbf{a}\tau}{c^2 l} \\ &= \Delta U^{(\mathrm{m_e})} + \Delta U^{(\mathrm{m_q})} + \frac{eqv\Delta v}{c^2 l}, \end{aligned} \tag{16}$$

where we have used $\Delta\mathbf{v} = \mathbf{a}\tau$ and have kept only first-order terms. Thus, the external forces indeed account for the energy change, both the *mechanical* energy change $\Delta U^{(\mathrm{m_e})} + \Delta U^{(\mathrm{m_q})}$ (from the time-integrations over the changes in momentum) and also the *magnetic* energy change (from the time-integration over the Faraday acceleration electric fields) corresponding to Eq. (12).

We emphasize again: *The magnetic energy balance for quasistatic systems requires the existence of Faraday electric forces associated with the accelerations of the charged particles.* These Faraday electric fields from accelerating charges become important in groups of closely spaced charges, and failure to recognize them in unsymmetric cases can lead to paradoxes.[4]

This example involving two point charges illustrates some of the contrasts involving forces, energies, and potentials for the electric and magnetic fields. In the electric case, the work associated with the electric field appears directly as energy stored in the electric fields, as is familiar in electrostatics. In the magnetic case, external or internal forces can change the speeds of the particles that are producing the magnetic fields which, in turn, leads to changes in magnetic energy. However, the change of charged particles' speeds also causes back (Faraday) *acceleration* electric fields, which make it harder for the external or internal forces to accelerate the charges. It is the additional work against the back (Faraday) acceleration electric fields that balances the change in magnetic field energy.

VI. CLOSING SUMMARY

The Darwin energy expressions in Eqs. (6)–(9) give the energy of two low-speed interacting charges. The electric field energy is the familiar expression from electrostatics. The magnetic field energy involves the separations and velocities of the charged particles. There are no Faraday acceleration electric fields for charges moving with constant velocities, and any change in magnetic field energy is provided by the external forces, which keep the particles' velocities constant.[8] On the other hand, the magnetic field energy can also change due to accelerations $\mathbf{a}_e$ and $\mathbf{a}_q$ of the charges. In this case, the change in magnetic field energy is associated with the Faraday acceleration terms in the electric field given in Eq. (4).

Particularly for situations where the speed of current carriers varies, the vector potential $\mathbf{A}(\mathbf{r}, t)$ serves as an intermediate connection between magnetic fields and electric fields through $\mathbf{B} = \nabla \times \mathbf{A}$ and $\mathbf{E} = -\nabla\Phi - (1/c)\partial\mathbf{A}/\partial t$. Any external or internal agent that tries to accelerate the current carriers produces a time-changing vector potential, which creates an electric field that (fitting with Lenz's law) tries to balance the energy change against work done by the agent. The change in the speed of the current carriers may produce a change in the *magnetic* field energy; this change is consistent with energy conservation because the external or internal agent causing the change had to do additional work against the back Faraday *electric* fields of the accelerating charges.

ACKNOWLEDGMENTS

This article was stimulated by a manuscript of Dr. Hanno Essén discussing magnetic energy changes, "A classical Aharonov-Bohm effect arises when one goes beyond the test particle approximation." The author is grateful to two anonymous referees whose extensive and helpful comments improved this article.

AUTHOR DECLARATIONS

Conflict of Interest

The author has no known conflict of interest.

[a)]ORCID: 0000-0003-2267-6290.

[1]D. J. Griffiths, *Introduction to Electrodynamics*, 4th ed. (Pearson, New York, 2013), pp. 247–248.

[2]A. Shadowitz, *The Electromagnetic Field* (Dover, New York, 1988), pp. 197, 208–209, 517–522; A. Garg, *Classical Electromagnetism in a Nutshell* (Princeton U. P., Princeton, NJ, 2012), pp. 107–108.

[3]See, for example, D. J. Griffiths, *Introduction to Quantum Mechanics*, 2nd ed. (Pearson Prentice Hall, Upper Saddle River, NJ, 2005), pp. 384–391 or L. E. Balentine, *Quantum Mechanics* (Prentice Hall, Englewood Cliffs, New Jersey, 1990), pp. 220–223.

[4]T. H. Boyer, "The Aharonov–Bohm phase shift as a relativity "paradox" and "A classical electromagnetic basis for the Aharonov–Bohm phase shift" " (submitted).

[5]C. G. Darwin, "The dynamical motions of charged particles," Phil. Mag. **39**, 537–551 (1920).

[6]J. D. Jackson, *Classical Electrodynamics*, 2nd ed. (John Wiley & Sons, New York, 1975), pp. 593–595 and p. 616, Problem 12.12.

[7]L. Page and N. I. Adams, *Electrodynamics* (Van Nostrand, New York, 1940), p. 175. See also, L. Page and N. I. Adams, "Action and reaction between moving charges," Am. J. Phys. **13**, 141–147 (1945).

[8]T. H. Boyer, "Energy and momentum in electromagnetic field for charged particles moving with constant velocities," Am. J. Phys. **39**, 257–270 (1971).

[9]See Ref. 1, pp. 373–378.

[10]See Ref. 6, p. 697, Problems 14.12 and 14.13.

[11]If there were no change in magnetic field energy, the agent causing the carriers to change speed would need to provide power so as to change only the kinetic energy of the current carriers. This change in kinetic energy of the current carriers is so small as to escape any mention in classical electromagnetism textbooks. It is the change in *magnetic field energy* that is overwhelmingly dominant.

[12]T. H. Boyer, "Faraday induction and the current carriers in a circuit," Am. J. Phys. **83**, 263–271 (2015).

[13]T. H. Boyer, "Classical electromagnetic deflections and lag effects associated with quantum interference pattern shifts: Considerations related to the Aharonov–Bohm effect," Phys. Rev. D **8**, 1679–1693 (1973); "The Aharonov–Bohm effect as a classical electromagnetic lag effect: An electrostatic analogue and possible experimental test," Il Nuovo Cimento **100**, 685–701 (1987); "Does the Aharonov-Bohm effect exist?" Found. Phys. **30**, 893–905 (2000); "Classical electromagnetism and the Aharonov–Bohm phase shift," Found. Phys. **30**, 907–932 (2000); "Darwin–Lagrangian analysis for the interaction of a point charge and a magnet: Considerations related to the controversy regarding the Aharonov–Bohm and Aharonov–Casher phase shifts," J. Phys. A: Math. Gen. **39**, 3455–3477 (2006).

[14]T. H. Boyer, "Electrostatic potential energy leading to an inertial mass change for a system of two point charges," Am. J. Phys. **46**, 383–385 (1978).