Thermoelastic bending of the lithosphere: implications for basin subsidence

Bruce G. Bills Jet Propulsion Laboratory, California Institute of Technology, Pasadena, CA 91103, USA

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Summary. Thermoelastic stresses are capable of producing significant lithospheric deflection. A systematic approach to modelling this effect is presented, in which the lithosphere is presumed to behave as a thin, elastic (or viscoelastic) plate on a fluid substrate, and lateral variations in basal heat flow induce both vertical buoyant loads and thermoelastic bending moments. The amount of uplift or subsidence produced by a given heat source depends on a number of factors, including the strength, duration and lateral extent of the thermal anomaly, and the thickness, density, rigidity and viscosity of the plate.

The mechanical response of the plate is characterized by two distinct length scales (one for shearing, the other for bending) and the deformation produced depends critically on the scaled width of the heat source. The plate acts as a low pass spatial filter in response to thermal loads, but has a narrow band pass filter response to applied moments. The temporal response to an abrupt change in basal temperature or heat flow also depends on the ratio of the thermal diffusion time for the plate versus the viscoelastic relaxation time of the material.

A rapid, localized increase in basal heat flow will initially produce a central depression and a peripheral uplift. Thus, while it is often assumed that the main depositional phase in platform basins is associated with the cooling and contraction of the lithosphere following an earlier thermal doming event, it is shown that, if conditions are right, a basin may also form in direct response to the initial heating event. Either way, subsequent thermal equilibration and viscous stress relaxation will tend to modify the initial deflection profile.

1 Introduction

A locally increased heat flow into the base of the lithosphere produces uplift in two ways: vertical expansion causes a simple thickening of the plate, while horizontal expansion induces isostatic uplift by reducing the weight per area of a column through the plate. However, this uneven heating of the plate will also produce thermoelastic bending moments which tend to induce local subsidence rather than uplift. The relative magnitudes of these
two competing effects will depend on a number of factors, including the strength, duration
and spatial extent of the thermal anomaly and the thickness, rigidity, viscosity and density
of the plate.

Thermally induced uplift and subsidence are recognized as a major source of large-scale
topographic and structural features in both oceanic and continental regions. In particular,
the gradual subsidence of oceanic plates as they move away from ridges is adequately
explained in terms of conductive heat loss, leading to lithospheric densification and isostatic
subsidence (Sclater & Francheteau 1970; Parker & Oldenburg 1973; Davis & Lister 1974).
Furthermore, the gradual subsidence of mid-plate rises or swells as they move away ‘down-
stream’ from mantle hot spots is explained by much the same mechanism (Detrick & Crough
1978; Crough 1978; Sandwell 1982). In both of these situations, lateral motion of the litho-
spheric plate past the heat source makes the response highly anisotropic. A thermal origin
has also been proposed for sedimentary basin subsidence (Sleep & Snell 1976; Haxby,
Turcotte & Bird 1976; Beaumont 1978) and continental plateau uplift (Suppe, Powell &
Berry 1975; Crough & Thompson 1976; Withjack 1979; Mareschal 1981). These latter effects
are significant only when the plate is essentially at rest with respect to the heat source (Gass
et al. 1978; Pollack et al. 1981). Despite the attention these models have received, the
flexural effect of thermal stresses has been largely ignored.

The primary objective of this paper is to examine the role of thermoelastic stresses in
producing lithospheric deformation. It will be assumed throughout that the plate is at rest
with respect to the underlying heat source. We will first develop a simple elastic response
model for the lithosphere and discuss the effective loads and moments induced by a
characteristic thermal anomaly. Then, to gain a better understanding of the basic thermo-
elastic response modes of this system, we examine the case of an axisymmetric, steady state
thermal anomaly and its associated lithospheric deflection. After that, we consider the effects
of time-dependent heat sources and viscoelastic plate response.

We find that the deformatinal responses to loads and moments have rather different
dependencies on lateral temperature variations. In fact, there are two distinct length scales
which characterize the mechanical response (one for shearing, the other for bending), and
the deformation produced depends critically on the scaled width of the source. Further-
more, though the moment response can significantly alter the load induced deformation, it
does not change the total volume displaced.

The effective thermal loads and moments also sample the vertical temperature structure
quite differentially and this leads to different temporal responses to time-dependent thermal
anomalies. The bending moment is more sensitive to basal temperature and thus responds
more rapidly to changes at the lower thermal boundary.

Viscous stress relaxation may considerably diminish the effectiveness of thermal bending
moments in producing lithospheric deformation, but does not completely suppress it. In
fact, because of the strong wavelength dependence of the viscoelastic relaxation time, the
short wavelength features of the initial elastic response will be left essentially intact, whereas
the long wavelength features are exponentially attenuated.

Finally we discuss the applicability of these theoretical results to actual cases of plateau
uplift and basin subsidence. It is seen that a number of features of sedimentary basin
development, in particular, can be satisfactorily explained as manifestations of thermoelastic
bending of the lithosphere.

2 Thermal stress and system response

The primary objective of this section is to determine the deformatonal response of the
Earth to changes in the thermal state of the crust and upper mantle. Rather than attempt to
solve the fully coupled thermoelastic problem we will instead use the plane strain, thin plate approximation. The problem thus separates into two essentially independent parts. They are: (1) determining the effective normal loads \((P)\), bending moments \((M)\) and horizontal membrane stress resultants \((N)\) applied to the plate due to its thermal state (or other causes) and (2) separately determining the response of the plate and its foundation to the applied forces and moments. We will first discuss some simple models of the deformational response.

2.1 DEFORMATION MODELS

2.1.1 Foundation

The simplest response model is that for a bare Winkler foundation (Kerr 1964). In this model the plate itself is purely passive and the foundation material (asthenosphere) has no resistance to shearing or bending, but resists deformation due to applied normal loads \((P)\) with a reaction proportional to the vertical displacement \((w)\) produced. The force balance is thus
\[\Delta k w = P.\]
The foundation modulus is
\[\Delta k = g \Delta \rho\]
where \(g\) is the gravitational acceleration and \(\Delta \rho\) is the density contrast between the material below (asthenosphere) and above (air, water, sediments, etc.) the plate. This model produces a local isostatic response to imposed loads and the Green's function, or response to a point load
\[P(r) = p \delta(r)\]
has the simple form
\[w(r) = \frac{p}{\Delta k} \delta(r).\]

2.1.2 Membrane

At the next level of approximation, the lithosphere is assumed to act as a membrane which resists shearing but not bending. The force balance in this case becomes (Kerr 1964)
\[\Delta k w + (N - C) \nabla^2 w = P\]
where the shearing resistance for a membrane of thickness \(h\), Young's modulus \(E\) and Poisson's ratio \(\nu\) is
\[C = \frac{Eh}{6(1 - \nu)}\]
and the membrane stress resultants due to horizontal stresses \((\sigma_x, \sigma_y)\) are
\[(N_x, N_y) = \int_0^h [\sigma_x(z), \sigma_y(z)] dz.\]
The general anisotropic response \((N_x \neq N_y)\) is rather complex but generally the deformation
contours around a point load are elongated in the direction of deviatoric tension (Dundurs & Jahanshahi 1965). In the isotropic case, the Green’s function is simply

$$w(r) = \frac{p}{\Delta k} K_0(\gamma r)$$

where $K_0$ is a modified Bessel function of order zero and

$$\gamma^2 = \frac{\Delta k}{C - N}$$

determines the intrinsic length scale of the deflection. For a given load, the total volume displaced is the same as before, but the membrane causes the deformation to be spread out over a wider area. Note that, whereas one part ($C$) of the effective shearing resistance is intrinsic to the membrane and is always positive, the other part ($N$) is a result of the imposed stress state and may be of either sign. In principle, even a very flexible membrane, with no appreciable intrinsic shear resistance, can be made quite resistant by placing it under planar tension. Or, conversely, an intrinsically strong plate can be effectively weakened, in its response to normal loads, by subjecting it to simultaneous lateral compression (Kerr 1972, 1979). However, the stress levels required to make a significant contribution to lithospheric shear resistance greatly exceed the strength of the rock.

The deflection of the membrane also produces a characteristic gravity anomaly. An axially symmetric density perturbation, concentrated at depth $d$,

$$\rho(r, z) = A K_0(\gamma r) \delta(z - d)$$

gives rise to a gravitational perturbation at the surface of magnitude

$$\Delta g(r) = 2\pi G A K_0(\gamma r) \exp(-\gamma d).$$

This relationship may be used to construct linear transfer functions connecting the observed gravity anomalies with surface deformation (McKenzie & Bowin 1976; Banks, Parker & Huestis 1977).

2.1.3 Plate

In addition to their shear resistance, elastic plates have the ability to resist bending and, in fact, for thin plates the bending resistance is usually more significant (Frederick 1955). The stress balance for plate and foundation together when subjected to both loads and moments becomes (Timoshenko & Woinowsky-Krieger 1959)

$$\Delta k w + (N - C) \nabla^2 w + D \nabla^4 w = \rho - \nabla^2 M$$  \hspace{1cm} (1)

where the bending resistance or flexural rigidity is

$$D = \frac{E h^2}{12(1 - \nu^2)}.$$

For an incompressible plate ($\nu = \frac{1}{2}$) the shearing and bending resistances reduce to

$$C = \mu h$$

$$D = \mu h^3 / 3$$

respectively, where $\mu$ is the intrinsic shear modulus of the plate material.
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The solutions of the general plate equation (1) can also be expressed in terms of complex Bessel functions (Yu 1957; Kerr 1978; Takagi 1979) and are applicable to certain tectonic problems (Withjack 1979; Turcotte et al. 1981). However, our main interest will be in the somewhat simpler equation

$$\Delta k w + D \nabla^4 w = P - \nabla^2 M \quad (2)$$

where the (usually small) effects of membrane stresses and intrinsic shear resistance are neglected. In this case the Green's functions are (Brotchie & Sylvester 1969; Kerr 1978)

$$w_p(r) = \frac{-p \beta^2}{2\pi \Delta k} \text{kei}(\beta r)$$

for point loads and

$$w_m(r) = \frac{m}{2\pi D} \text{ker}(\beta r)$$

for point moments, where

$$\beta^4 = \frac{\Delta k}{D}$$

determines the flexural length scale. The Kelvin (Thompson) functions ker and kei satisfy the differential equation (Abramowitz & Stegun 1972)

$$(\nabla^2 - i \beta^2) [\text{ker}(\beta r) + i \text{kei}(\beta r)] = 0$$

and are thus obviously just the real and imaginary parts of a complex Bessel function (Kerr 1978; Yu 1979).

Note that normal loads applied to the plate are ultimately supported by the underlying foundation. The role of a membrane or plate is merely to spread the deformation over a larger area. Thus, though a load may appear to be partially supported by the lithosphere, the total volume displaced

$$2\pi \int_0^{\infty} w_p(r) r \, dr = \frac{2\pi}{\Delta k} \int_0^{\infty} P(r) r \, dr$$

is the same as if there were no plate at all. Furthermore, an applied moment can significantly modify the pattern of lithospheric deformation, but does not change the total volume displaced

$$2\pi \int_0^{\infty} w_m(r) r \, dr = 0.$$  

Just as in the membrane model, the deflection of an elastic plate and its foundation produces a characteristic gravity anomaly. A density distribution of the form

$$\rho(r, z) = A [\text{ker}(\beta r) + i \text{kei}(\beta r)] \delta(z - d)$$

produces a surface gravity anomaly of strength

$$\Delta g(r) = 2\pi GA [\text{ker}(\beta r) + i \text{kei}(\beta r)] \exp\left[-(1-i)bd\right]$$

where

$$2b^2 = \beta^2.$$
For example, the deformation produced jointly by a point load of magnitude $\beta^2 P$ and a point moment of magnitude $M$ has the form

$$w(r) = \frac{1}{4\pi D} \left[ M \text{ker}(\beta r) - P \text{kei}(\beta r) \right]$$

and, if compensated at depth $d$, produces the gravity anomaly

$$\Delta g(r) = A \text{ker}(\beta r) + B \text{kei}(\beta r)$$

where

$$A = \frac{G}{2D} [M \cos(bd) - P \sin(bd)] \exp(-bd)$$

$$B = \frac{G}{2D} [P \cos(bd) + M \sin(bd)] \exp(-bd).$$

Linear transfer functions connecting gravity and surface deformation may be derived for an elastic plate (Banks et al. 1977) and they will differ somewhat from those pertaining to a thin membrane. However, for normal loads the difference will be rather subtle and it is unlikely that gravity data alone would suffice to distinguish between them. It is also obvious, though, that failure to include the effects of bending moments will seriously bias estimates of plate parameters obtained from observed deformation and gravity anomalies (Parsons & Molnar 1976; Watts & Cochran 1974).

### 2.2 Thermal Stresses

We assume that the lithosphere is initially undeformed and that there are no lateral temperature variations or residual thermal stresses. A local increase in temperature throughout the plate will tend to produce uplift of the free surface. Part of this uplift (roughly $1/3$) is due to direct vertical expansion and consequent thickening of the plate. The remainder is due to horizontal expansion which simply reduces the weight per unit area of a column through the plate. The net result is essentially equivalent to applying an upward directed normal load (Sleep & Snell 1976; Pollack 1980; Mareschal 1981)

$$P(x, y) = 3\alpha k \int_0^h T(x, y, z) \, dz \quad (3)$$

to the plate. Here $T$ is the temperature perturbation (relative to the initial state), $\alpha$ is the linear thermal expansion coefficient and

$$k = \frac{g\rho}{\alpha}$$

where $\rho$ is the unperturbed mean density of the plate.

A temperature perturbation also produces isotropic horizontal stresses (Boley & Wiener 1960; Nowacki 1961)

$$\sigma_x = \sigma_y = \frac{\alpha E T}{1 - \nu}.$$

The ensuing mean membrane stress resultant and bending moment are thus

$$N(x, y) = \frac{\alpha E}{1 - \nu} \int_0^h T(x, y, z) \, dz$$

$$M(x, y) = \frac{\alpha E}{1 - \nu} \int_0^h T(x, y, z)(z - h/2) \, dz. \quad (4)$$
Note particularly that, of these effects \((P, N, M)\), the bending moment is most sensitive to changes in temperature at the base of the plate. The others essentially reflect changes in mean temperature. This difference in sensitivity to vertical temperature structure can lead to significant differences in temporal response to time dependent thermal anomalies.

The membrane force resultant will usually be partially relieved by uniform lateral expansion (Turcotte 1974) and will thus contribute to the effective thermal load \(P\). The remainder will somewhat modify the (generally inconsequential) shear resistance of the plate, as per equation (1). In any event, it makes no direct contribution to plate deformation, and no further explicit mention will be made of it. On the other hand, the bending moment \(M\) contributes to lithospheric deflection in a unique and distinctive way. In fact, one of the chief objectives of this paper is to examine the effects of this contribution.

The thin plate approximation involved in deriving equation (2) causes some difficulties in modelling the response to thermal anomalies since the plate itself varies in thickness. Sleep & Snell (1976) have examined this problem in conjunction with the thickness variations caused by direct thermal expansion and find the model to be quite adequate for thermal loads.

A potentially more serious problem is related to the fact that the lithosphere is essentially a thermal boundary layer whose thickness is determined by the position of some characteristic isotherm \(T_f\) (Pollack & Chapman 1977). Thus, a local increase in vertical temperature gradient throughout the lithosphere

\[
T(z) = A \frac{z}{h}
\]

has two rather disparate effects on bending. The thermal stresses induce a bending moment

\[
M = \frac{\alpha A h}{2} \frac{E h}{6(1-\nu)}
\]

but the temperature increase also causes a local decrease in lithospheric thickness

\[
\Delta h = -(A/T_f) h
\]

and thus a possibly significant decrease in flexural rigidity. However, examination of the moment–curvature relation (Timoshenko & Woinowsky-Krieger 1959)

\[
\nabla^2 w = -\frac{M}{D} = -\frac{\alpha(1+\nu)A}{h(1-A/T_f)}
\]

indicates that changes in \(M\) and \(D\) tend to cancel. Thus, as long as lateral temperature variations are small compared to \(T_f\), the changes in plate thickness may be ignored.

Another apparent limitation of the present approach is the failure to distinguish explicitly between a ‘thermal’ lithosphere and a ‘mechanical’ lithosphere. The thicknesses of the thermal and mechanical boundary layers are often taken as the depths to the 1200 and 600°C isotherms, respectively. For example, Sandwell (1982) postulates that, below the mechanically strong elastic lithosphere, there is a ‘plastic layer that can support stresses arising from thermal buoyancy but cannot support the larger deviatoric stresses associated with lithospheric flexure’. While it is certainly plausible to assume a gradual decrease in long term strength with depth, the \textit{ab initio} distinction between mechanical and thermal lithospheres seems unwarranted.
Many of the observations which have been used in support of separate thermal and mechanical definitions of the lithosphere can be explained equally well as simple manifestations of viscoelastic behaviour. For example, the response of a viscoelastic plate to a sustained load does give the impression of a gradual decrease in equivalent elastic thickness with time. However, if an additional load is subsequently applied, the incremental deformation will still be determined by the initial, unrelaxed 'elastic' thickness. Thus, while the purely elastic model is really only applicable to the initial stages of deformation, the viscoelastic model accommodates both early and late stages without ad hoc assumptions.

3 Static elastic deformation

We initially consider the static response of a thin elastic plate to axially symmetric, time-invariant thermal loads and bending moments. Axial symmetry is assumed for simplicity and because plateau uplift and basin subsidence often exhibit roughly circular planforms. A cylindrical coordinate system \((r, z)\) is used. Taking the zero-order Hankel transform (Sneddon 1972) on the radial variable of the plate equation (2) yields

\[
(\Delta k + u^4D) \tilde{w}(u) = \tilde{P}(u) + u^2\tilde{M}(u)
\]

where

\[
\tilde{f}(u) = \int_0^\infty f(r) J_0(ur) \, r \, dr
\]

\[
f(r) = \int_0^\infty \tilde{f}(u) J_0(ur) \, u \, du
\]

define the transform pair. The transform of the load induced deformation is thus

\[
\tilde{w}_p(u) = \frac{\tilde{P}(u)}{\Delta k + u^4D}
\]

and the corresponding response to an applied moment is

\[
\tilde{w}_m(u) = \frac{u^2\tilde{M}(u)}{\Delta k + u^4D}.
\]

Of course these results are quite general and are not restricted to loads and moments of thermal origin, but rather represent the purely mechanical response of the plate and its foundation.

In order to explicitly include thermal effects without unduly obscuring matters, we will initially consider a rather idealized situation. The steady temperature perturbation induced in the plate by a symmetric heat flow anomaly at its base will have the approximate form

\[
T(r, z) = A f(r) z/h
\]

where \(A\) is the amplitude of the basal anomaly and

\[
f(r) \equiv f(0) = 1
\]

describes its radial variation. Substituting this into equations (3) and (4) and Hankel transforming yields

\[
\tilde{P}(u) = \frac{\alpha Ah}{2} 3k \tilde{f}(u)
\]

\[
\tilde{M}(u) = -\frac{\alpha Ah}{2} C \tilde{f}(u).
\]
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The quantity \( \alpha Alr/2 \) is simply the amount by which the plate thickens due to direct vertical expansion and it makes a convenient normalization factor.

The Hankel transformed lithospheric deformation due to a simple, steady state thermal anomaly is thus

\[
\bar{w}(u) = \frac{\alpha Ah}{2} \frac{3k-u^2C}{\Delta k+u^4D} \tilde{f}(u).
\]

Alternatively, this may be written

\[
\bar{w}_p(u) = \frac{\alpha Ah}{2} \frac{k}{\Delta k} \left[ \frac{3}{1+(u/\beta)^2} \right] \tilde{f}(u) \tag{9}
\]

\[
\bar{w}_m(u) = -\frac{\alpha Ah}{2} \frac{k}{\Delta k} \left[ \frac{(u/\gamma)^2}{1+(u/\gamma)^4} \right] \tilde{f}(u) \tag{10}
\]

where \( \beta \) and \( \gamma \) denote the two characteristic length scales of the deformation and are given by

\[\beta^4 = \Delta k/D \quad \text{(bending)}\]
\[\gamma^2 = k/C \quad \text{(shearing)}\]

The bracketed factors in (9) and (10),

\[\psi_p(u) = 3/[1+(u/\beta)^4]\]
\[\psi_m(u) = (u/\gamma)^2/[1+(u/\beta)^4]\]

are the normalized spectral response functions for the two basic modes of lithospheric deformation. They are plotted in Fig. (1a and b) for a wide range of wavelengths \((2\pi/u)\) and a variety of plate thicknesses, with assumed values of \( E = 6.0 \times 10^{11} \text{ g cm}^{-2} \), \( \nu = 1/2\), \( \Delta k = 7.85 \times 10^3 \text{ g s}^{-2} \) and \( k = 3.24 \times 10^3 \text{ g s}^{-2} \).

The lithospheric response to normal loads is seen to act as a low pass \((u < \beta)\) spatial filter, with short wavelengths (high spatial frequencies) rapidly attenuated. In contrast, the response to bending moments is essentially a narrow band pass \((u \sim \beta)\) filter, with selective amplification of a band of wavelengths centred around \(2\pi/\beta\). Thus, for example, the width of a flexurally induced swell need not accurately reflect the lateral extent of the causative thermal anomaly. In particular, a sharply peaked thermal anomaly will produce a flexural response with characteristic wavelength close to \(2\pi/\beta\). It is also noteworthy that flexural deformation due to thermal bending moments can match or even exceed the direct thermal loading effects, and that the flexural advantage is greater for thin plates.

The plate parameters \( \beta \) and \( \gamma \) play rather different roles in determining the spectral response curve \( \psi_m(u) \). The bending parameter \( \beta \) alone determines the spatial frequency of the spectral peak \((u = \beta)\) and thus establishes the wavelength of the physical deformation. The shearing parameter \( \gamma \) on the other hand influences the amplitude of the spectral peak \( \psi_m(\beta) = (\gamma/\beta)^2 \), without changing its frequency.

The mechanical response to an arbitrary distribution of loads and moments may, of course, be obtained by convolution with the appropriate Green's functions. In the present formulation, they are found by taking

\[f(r) = \delta(r)\]

so that

\[\tilde{f}(u) = -1/2\pi\]
Figure 1. Spectral response curves for point thermal loads and moments applied to a floating elastic plate. The normalization is such that the local isostatic response to an applied thermal load corresponds to a deflection of 3. The different curves represent various plate thicknesses. Note that the plate acts as a low pass spatial filter in response to loads, but has a narrow band pass filter response to applied moments.
and then inverse transforming equations (6) and (7). Thus the response to a point load of magnitude $P$ is simply

$$w_p(r) = -\frac{P}{2\pi \Delta k} \int_0^\infty \left[ \frac{1}{1 + (u/\beta)^4} \right] J_0(ur) u \, du$$

$$= -\frac{P}{\beta^2 \Delta k} \text{kei} (\beta r)$$

and the response to a point bending moment of magnitude $M$ is likewise

$$w_m(r) = \frac{M}{2\pi \Delta k} \int_0^\infty \left[ \frac{u^2}{1 + (u/\beta)^4} \right] J_0(ur) u \, du$$

$$= \frac{M}{\beta^4 \Delta k} \text{ker} (\beta r).$$

However, rather than calculate the spatial convolution of mechanical Green's functions with the thermal loads and moments, we will generally find it easier to obtain the thermo-mechanical response by simple multiplication in the transform domain and performing the inverse transform. This is particularly true if the thermal anomalies and their corresponding loads and moments are expressed in terms of Gauss–LaGuerre functions: a set of orthogonal functions obtained as the product of a Gaussian times a Laguerre polynomial

$$G_n(r, \sigma) = L_n(2r^2/\sigma^2) \exp(-r^2/\sigma^2).$$

Figure 2. Gauss-LaGuerre functions. These comprise an orthogonal set of eigenfunctions of the zero-order Hankel transform. The lowest order function is a simple Gaussian. A linear combination of the functions with order $n \leq 4$ can be made to approximate the plate response to a Gaussian heat source.
The LaGuerre polynomials are defined as (Szegö 1959)

\[ L_n(x) = \exp(x) \frac{d^n}{dx^n} [\exp(-x)x^n]. \]

Fig. 2 illustrates the form of the first few of these Gauss–LaGuerre (G-L) functions. Their

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**Figure 3.** Spectral response curves for Gaussian thermal loads and moments, normalized as in Fig. 1. The separate curves in (a) and (b) correspond to different plate thicknesses while the normalized source width is kept constant at \( \beta \sigma = 1/2 \). For comparison, the curves in (c) and (d) illustrate the effects of varying the source width while keeping the plate thickness fixed at \( h = 40 \text{ km} \).
usefulness in the present context is largely a result of the fact that they are eigenfunctions of the Hankel transform. Thus the transform of a G-L function of degree \( n \)

\[
\tilde{G}_n(u, \sigma) = (-1)^n \sigma^3/2 G_n(u, 2/\sigma)
\]

is again a G-L function of degree \( n \). A closely related property is that the G-L functions,
and \( G_1(r, \sigma) = \exp(-r^2/\sigma^2) \) in particular, are very likely forms for a thermal anomaly. Also, more complex sources can be expanded in a series of G-L functions:

\[
f(r) = \sum_{n=0}^{\infty} F_n G_n(r, \sigma)
\]

**Figure 4.** Static elastic response to increased heat flow through the plate for various source widths. For sufficiently narrow sources \((\beta \sigma \lesssim 1)\) the central region is depressed and a peripheral bulge is raised. For wider sources \((\beta \sigma \sim 2)\), the central region is uplifted and the peripheral rim is less prominent. For sufficiently broad sources \((\beta \sigma \gtrsim 4)\), the full isostatic deflection is attained.
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combined response: \( w_p(r, \sigma) + w_m(r, \sigma) \)

\( \beta \sigma = 2 \)

\( \beta \sigma = 1 \)

\( \beta \sigma = 1/2 \)

Figure 4 – continued

\[ F_n = \frac{4}{\sigma^2} \int_0^{\infty} f(r) G_n(r, \sigma) r \, dr. \]

The application of these functions to Hankel transform problems is further discussed by Cavanagh & Cook (1979).

The remainder of this section will focus on simple Gaussian sources and will attempt to illustrate the effect on the resultant deflection profiles of varying some of the source and/or plate parameters. The Hankel transformed load response (9) now becomes

\[ \bar{w}_p(u, \sigma) = \frac{\alpha Ah}{2} \frac{k}{\Delta k} \left[ \frac{3 \bar{G}(u, \sigma)}{1 + (u/\beta)^4} \right] \]

and is characterized by a single dimensionless length scale \( \beta \sigma \) which is simply the heat source width normalized by the flexural length of the plate. Thus, changes in source and/or plate parameters which leave the product \( \beta \sigma \) unchanged will only change the scale or size of the deflection profile, but not its shape. On the other hand, the transformed moment response (10) becomes

\[ \bar{w}_m(u, \sigma) = -\frac{\alpha Ah}{2} \frac{k}{\Delta k} \left[ \frac{(u/\gamma)^2 \bar{G}(u, \sigma)}{1 + (u/\beta)^4} \right] \]

which has two normalized length scales \( \beta \sigma \) and \( \gamma \sigma \). Consequently, the class of parameter changes which leave the response geometrically similar is much more restrictive.

Fig. 3 shows the bracketed terms in equations (11) and (12) and illustrates the effect, in the Hankel transform domain, of variations in plate thickness and source width. All other relevant parameters have the same values as in Fig. 1. Radial deflection profiles are given in Fig. 4 for positive heat sources with normalized widths of \( \beta \sigma = 1/2 \), 1 and 2. They are
obtained by a numerical Hankel transformation of equations (11) and (12). The numerical algorithm employed is similar to that discussed by Beaumont (1978).

The load-induced deflection has a nearly Gaussian shape, particularly for the wider sources. The region of significant deflection extends out to a radial distance of roughly \( \sigma \) or \( 1/\beta \) whichever is greater. For narrow sources, the actual shape of the load is largely immaterial (Menke 1981). In fact, Haxby et al. (1976) and Brotchie & Sylvester (1969) have both discussed deflection profiles due to cylindrical loads

\[
P(r) = \begin{cases} 
1 & r < a \\
0 & r > a 
\end{cases}
\]

and, for sufficiently narrow loads, the cylindrical deflection profile is very similar to that produced by an equivalent \((\sigma = a)\) Gaussian load.

The deflection profile induced by an applied moment distribution is somewhat more complex and is rather more sensitive to the shape of the source. In the example of Fig. 4, the central region is depressed and there is a raised rim in the vicinity of \( r = \sigma \) (where \( \nabla^2 M \) is greatest).

For the widest sources illustrated \((\beta \sigma = 2)\), the combined deflection profile \( w_P + w_m \) closely resembles the load-induced deflection alone. However, for sufficiently narrow sources the central region is actually depressed below the reference level and the only raised region is on the periphery.

From the examples just given, it might appear that the amplitude of the central deflection is a monotonic, rapidly increasing function of source width for both loads and moments. However, as illustrated in Fig. 5 this is not so. In estimating the central deflection from the

![Figure 5](https://academic.oup.com/gji/article-abstract/75/1/169/613110 by guest on 10 December 2018)

Figure 5. Variation of central deflection with source width and plate thickness. The normalized load response increases with source width until the isostatic value is attained. The moment response is greatest for thin plates, and normalized source widths near unity.
Hankel inverses of (11) and (12), we note that the Bessel factor \( J_0(\alpha r) \) becomes a constant \( (J_0(0) = 1) \), and we thus find

\[
\int_0^\infty \left[ \frac{G(u, \sigma)}{1 + (u/\beta)^4} \right] u \, du = b \left[ \sin(b) \, ci(b) - \cos(b) \, si(b) \right]
\]

\[
\int_0^\infty \left[ \frac{(u/\gamma)^2 G(u, \sigma)}{1 + (u/\beta)^4} \right] u \, du = -\frac{b^2}{c} \left[ \cos(b) \, ci(b) + \sin(b) \, si(b) \right]
\]
\[ b = (\beta \alpha/2)^2 \]
\[ c = (\gamma \alpha/2)^2 \]
and
\[ \text{ci}(x) = - \int_x^{\infty} \frac{\cos(t)}{t} \, dt \]
\[ \text{si}(x) = - \int_x^{\infty} \frac{\sin(t)}{t} \, dt \]
are the cosine integral and sine integral, respectively.

Returning to Fig. 5 we see that the load-induced central deflection \( w_p(0, \sigma) \) is in fact a monotonic function of normalized load width \( \beta \alpha \) but the only real variation occurs in the range \( 10^{-1} \leq \beta \alpha \leq 10^1 \). For narrower loads there is virtually no deflection (the plate supports the load) and for wider loads the central deflection does not increase beyond the isostatic value. For applied moments, we require two parameters \( \beta \alpha \) and \( \gamma \sigma \) (or alternatively, \( \beta \alpha \) and plate thickness \( h \)) to characterize the response completely. However, for any value of \( \gamma \) (or \( h \)) the central deflection is greatest when \( \beta \alpha \approx 2 \).

4 Time-dependent elastic deformation

In the last section, the lithospheric deformation modes \( w_p \) and \( w_m \) were seen to have significantly different responses to spatial variations in the source. The purpose of this section is to explore differences in response to temporal source variations. In particular, we will examine the lithospheric temperature and concomitant deformation following a rapid change in the basal temperature regime. The mechanical response of the plate is still presumed to be elastic. The next section will deal with complications due to a viscoelastic plate.

The initial problem is to determine the temperature perturbation throughout the lithosphere due to changes in the lower boundary conditions. The temperature perturbation satisfies the diffusion equation (Carslaw & Jaeger 1959)

\[ \frac{\partial T}{\partial t} = \kappa \nabla^2 T \]

where \( \kappa \) is the thermal diffusivity, which is assumed to be constant throughout. The boundary condition at the free surface is

\[ T(r, z, t) \mid_{z=0} = 0 \]

and at the lower boundary, either the temperature perturbation

\[ T(r, z, t) \mid_{z=h} = A(t) G_n(r, \sigma) \] (13)

or the change in the thermal gradient (or, equivalently, the basal heat flow)

\[ \frac{\partial T}{\partial t} (r, z, t) \mid_{z=h} = B(t) G_n(r, \sigma) \] (14)

is specified. In either case, the initial condition is that the lithosphere was in thermal equilibrium prior to the specified perturbation.
The problem can be formally solved by taking the Hankel transform
\[ H[f: r \to u] \]
on the radial variable and the Laplace transform
\[ L[f: t \to s] \]
on the time variable (Sneddon 1972; Mareschal 1981). The transformed diffusion equation is
\[ s \bar{T} = \kappa \left[ \frac{\partial^2 \bar{T}}{\partial z^2} - u^2 \bar{T} \right] \]
and it has solutions of the form
\[ \bar{T}(u, z, s) = \bar{A}(s) \bar{G}_n(u, 0) \left[ \frac{\sinh(az)}{\sinh(ah)} \right] \]
corresponding to the temperature boundary condition (13) and
\[ \bar{T}(u, z, s) = \bar{B}(s) \bar{G}_n(u, 0) \left[ \frac{\sinh(az)}{a \cosh(ah)} \right] \]
corresponding to the heat flow boundary condition (14). The vertical length scale (thermal skin depth) is determined by
\[ a^2 = u^2 + s/\kappa. \]
The perturbation in the lower boundary condition will be assumed to occur abruptly at the time \( t' \). The Laplace transformed amplitudes are thus
\[ \bar{A}(s) = A/s \exp(-st') \]
\[ \bar{B}(s) = B/s \exp(-st'). \]

The different boundary conditions (13) and (14) give rise to rather different temperature distributions. In both cases the temperature approaches a new equilibrium state which has the approximate form (neglecting lateral heat conduction) which was assumed in equation (8). However, the time dependence of the resultant thermal loads and moments is somewhat different. Mareschal (1981) has recently discussed the surface heat flow, thermal loads and surface deformation (albeit neglecting lithospheric bending) due to a basal heat flow disturbance (14). The discussion here will focus on the other case (13) of a basal temperature perturbation, not so much because it is more appropriate physically, but rather because it more clearly illustrates the possible differences in temporal development between the thermal loads and moments. However, it also provides a reasonable approximation to the situation that might develop as a lithospheric plate comes to rest and over a mantle hot spot (Burke & Wilson 1972; Briden & Gass 1974).

The time domain temperature distribution corresponding to (15) is (Carslaw & Jaeger 1959)
\[ T(u, z, t) = A \ G_n(u, 0) \left[ z/h + F(z, t) \exp(-\kappa u^2 t) \right] \]
where
\[ F(z, t) = \sum_{m=1}^{\infty} (-1)^m \left( \frac{2}{m\pi} \right) \sin(m\pi z/h) \exp(-m^2 t/\tau) \]
represents the primary time dependence of the vertical structure and

$$\tau = \frac{h^2}{\kappa \pi^2}$$

defines the diffusional time-scale. The corresponding thermal loads and moments may be expressed in the form

$$\bar{P}(u, t) = \frac{\alpha Ah}{2} 3k \left[ \bar{G}_n(u, \sigma) + F_p(t) (\sigma/\xi)^2 \bar{G}_n(u, \xi) \right]$$

$$\bar{M}(u, t) = -\frac{\alpha Ah}{2} C \left[ \bar{G}_n(u, \sigma) + F_m(t) (\sigma/\xi)^2 \bar{G}_n(u, \xi) \right]$$

where the time-dependent variance

$$\xi^2 = \sigma^2 + 4\kappa t$$

Figure 6. Time-dependent elastic deflection profiles following a sudden change in basal temperature. Separate curves represent different times after the thermal event, normalized by the diffusion time of the plate. The normalized times are $t/\tau = 0.01, 0.03, 0.1, 0.3, 1$ and $3$. There is very little additional change after $t/\tau = 3$, as can be seen by comparing these profiles with those in Fig. 4. For a narrow source ($\beta \sigma \leq 1$), the initial central deflection is downward, reflecting the early dominance of the moment response. Later, the central region moves upward as the load response becomes more significant. Upward deflection near the peripheral bulge increases monotonically, but the locus of peak deflection moves radially inward in the late stages. For wider sources ($\beta \sigma \approx 2$), the initial central depression is less pronounced but the peripheral bulge still makes the initial profile concave upward. For sufficiently broad sources ($\beta \sigma \geq 4$), the response is monotonic uplift, with only a trace of a peripheral bulge evident in the early stages.
indicates a (usually slight) diffusional widening of the original Gauss–LaGuerre distribution. However, the main time dependence resides in the factors

\[ F_p(t) = 2/h \int_0^h F(z, t) \, dz \]

\[ = -8/\pi^2 \sum_{m=0}^{\infty} \frac{\exp\left[-(2m+1)^2 t/\tau\right]}{(2m+1)^2} \]

\[ = -\exp(-t/\tau) f_p(t) \]
and

\[
F_m(t) = \frac{12}{h^2} \int_0^h F(z, t) (z - h/2) \, dz
\]

\[
= -\frac{24}{\pi^2} \sum_{m=1}^\infty \frac{\exp \left[ -\left(\frac{2m}{\tau}\right)^2 t/\tau \right]}{(2m)^2}
\]

\[
= -\exp(-4t/\tau) f_m(t)
\]

The subsidiary factors \(f_p\) and \(f_m\) are of order unity

\[
8/\pi^2 \lesssim f_p(t) \lesssim 1
\]

\[
6/\pi^2 \lesssim f_m(t) \lesssim 1
\]

and may frequently be ignored. Thus to a very good approximation (especially for wide sources, i.e. \(\sigma \gg h\)) the thermal loads and moments induced by a sudden change in temperature at the base of the lithosphere (13), are

\[
P(u, t) \approx \frac{\alpha Ah}{2} 3k G_n(u, \sigma) \left[1 - \exp(-t/\tau)\right]
\]

\[
M(u, t) \approx \frac{-\alpha Ah}{2} C G_n(u, \sigma) \left[1 - \exp(-4t/\tau)\right].
\]

**Figure 7.** Time-dependent elastic central deflection following a sudden change in basal temperature. The load response is a monotonic uplift, whose ultimate amplitude depends only on normalized source width (compare with Fig. 5). The moment response is a monotonic subsidence and it depends on both source width and plate thickness. The combined response simply reflects a balance between these competing influences and generally exhibits a moment dominated early phase followed by a load-dominated relative uplift.
The reason for the faster response of the moment is, as previously mentioned, that is is more sensitive to temperatures near the base of the plate.

In the Laplace transform domain these approximate formulae take the form

\[ P(u, s) = \frac{\alpha Ah}{2} 3k G_n(u, \sigma) \left[ s_p / s (s + s_p) \right] \]

\[ M(u, s) = \frac{-\alpha Ah}{2} C G_n(u, \sigma) \left[ s_m / s (s + s_m) \right] \]
where

\[ l/s_p = \tau \]
\[ l/s_m = 4\tau \]

are their respective time constants.

The elastic lithospheric response to a time-dependent thermal anomaly is easily obtained by substituting the appropriate Hankel transformed loads and moments, such as (17) and (18), into equations (6) and (7), respectively, and taking the inverse transform. For example, radial deflection profiles corresponding to positive Gaussian anomalies (13) of normalized widths \( \beta_0 = 1, 2 \) and 4 are given in Fig. (6a, b and c, respectively). The deflections are shown at normalized times \( t/\tau = 0.01, 0.03, 0.1, 0.3, 1 \) and 3 after the change in boundary conditions.

For the narrower sources (\( \beta_0 = 1, 2 \)), the initial central deflection is downwards and there is a slight peripheral bulge in the vicinity of \( r = a \). The central deflection continues downward for a normalized time on the order of unity, and then reverses sign. Similarly, the peripheral bulge first moves outward, stops at around \( t/\tau = 1 \), and then subsequently moves inward. Fig. 7 further illustrates the time dependence of the central deflection for a variety of plate thicknesses and a normalized source width of \( \beta_0 = 2 \), and shows the contributions of the thermal loads and moments separately. For a thin plate, the final central deflection is downward whereas, for a thick plate, the bending moments only succeed in somewhat delaying the ultimate central uplift.

\section*{5 Viscoelastic deformation}

In addition to the basic elastic response, which we have examined in previous sections, the lithosphere exhibits a broad range of anelastic behaviours. A simple Maxwell viscoelastic model is capable of duplicating much of the observed (or inferred) anelastic response (Walcott 1970; Peltier 1974, 1980; Sleep & Snell 1976; Beaumont 1978, 1979) and has the advantage of analytical tractability. Other, more complex lithospheric models have been proposed, including elastic—plastic rheology (Turcotte, McAdoo & Caldwell 1978; McAdoo, Caldwell & Turcotte 1978; Chapple & Forsyth 1979) or laminates of elastic and viscoelastic plates (Lambeck & Nakiboglu 1981). However, in the present context they do not seem justified.

A Maxwell material behaves on a short time-scale just as an elastic solid, but has the long-term response characteristics of a viscous fluid. The addition of a viscous response mode does not necessarily suggest that the actual local deformation mechanism is simple Newtonian viscous flow. Rather, if time- and/or length-scales are chosen large enough, then temporal and/or spatial fluctuations in either the elastic or plastic deformation mechanisms can cause the averages of stresses and strain rates over these scales to be related linearly, even though the non-averaged quantities are related non-linearly (Hibler 1977). It may well be that the effective viscosity of the lithosphere is actually a manifestation of this sort of stochastically averaged elastic—plastic behaviour rather than a micro-scale viscous behaviour.

In the remainder of this section, we will use the correspondence principle to derive the viscoelastic plate equations from their elastic counterparts and then explore some aspects of the associated mechanical and thermomechanical problems of lithospheric deformation. The main objective will be to determine how stress relaxation modifies the basic repertoire of thermoelastic responses which we established in the previous section. Perhaps the most significant modification involves the introduction of another intrinsic time-scale (in addition to the thermal diffusion time). Just as the presence of two length-scales (\( \beta \) and \( \gamma \)) complicates the elastic response to applied moments, so the addition of a rheological time-scale broadens the range of possible lithospheric response to simple thermal perturbations.
The correspondence principle (Biot 1954; Lee 1955; Peltier 1974) simply states that, in the case of zero initial conditions, the Laplace transformed viscoelastic field equations and boundary conditions are formally identical to the corresponding elastic equations for a body of the same geometry. The only difference is that the elastic moduli are no longer constants. In particular, for an incompressible Maxwell plate of rigidity $\mu$ and effective viscosity $\eta$, the shear and flexural parameters become

$$
\dot{C}(s) = Cs/(s + s_0)
$$

$$
\dot{D}(s) = Ds/(s + s_0)
$$

where

$$
1/s_0 = \eta/\mu
$$

is the Maxwell relaxation time of the material. The response of such a viscoelastic lithosphere to normal loads has been discussed by Nadai (1963), Walcott (1970), Beaumont (1978), Peltier (1980) and Lambeck & Nakiboglu 1981.

In general, the deformational response of the system to imposed loads and moments is simply obtained by substituting the Laplace transformed viscoelastic moduli into the elastic equations (5) to obtain

$$
\tilde{w}(u, s) = \frac{(s + s_0) \tilde{p}(u, s) + (s + s_0) u^2 \tilde{m}(u, s)}{(s + s_0) \Delta k + s u^4 D}
$$

and then taking the inverse transforms. The order in which the inverse transforms (Hankel & Laplace) are taken is immaterial if the wavelength of the applied loads and moments is independent of time (Lambeck & Nakiboglu 1981).

The introduction of an additional variable, via the substitution

$$
s_4 = \frac{s_0}{1 + (u/\beta)^4}
$$

considerably simplifies the Laplace inversion (Nadai 1963; Beaumont 1978) and allows us to rewrite (19) in the form

$$
\tilde{w}(u, s) = \frac{(s + s_0)}{(s + s_4)} \left[ \frac{\tilde{p}(u, s) + u^2 \tilde{m}(u, s)}{\Delta k + u^4 D} \right].
$$

Written this way, the term in brackets formally corresponds to the initial elastic response and the factor in parentheses is a viscoelastic flexural operator which governs the transformation from this initial elastic response into the final viscous response. An important aspect of this transformation is the strong wavelength dependence of the relaxation time $1/s_4$. For long wavelengths ($u/\beta < 1$) the relaxation time is essentially independent of wavelength and is just $1/s_0$, the intrinsic Maxwell time of the lithospheric material. However, for short wavelength ($u/\beta > 1$), the relaxation time is inversely proportional to the fourth power of the wavelength. Thus, within a few Maxwell times ($s_0 t > 1$) after the imposition of arbitrary loads and moments, only the short-wavelength components of the initial elastic response remain.

For short time intervals ($s_0 t < 1$) and long wavelengths ($u/\beta < 1$), the instantaneous
configuration of a viscoelastic system corresponds very nearly to that of an equivalent elastic system with effective plate parameters (Walcott 1970)

\[ C(t) = C(0) \exp(-s_0 t) \]
\[ D(t) = D(0) \exp(-s_0 t). \]

However, short-wavelength features in the viscoelastic response are actually much more persistent than this approximation would suggest. In fact, some misgivings about the use of viscoelastic models (versus models with finite yield strength, for example) seem to originate from inappropriate use of this approximation. Furthermore, an elastic analysis applied to a viscoelastic plate tends to underestimate the flexural rigidity \( D \) for large \( (\beta \sigma > 1) \), old \( (s_0 t > 1) \) loads. These points are discussed further by Beaumont (1978) and Lambeck & Nakiboglu (1981).

The time dependence of the response to applied moments is further complicated by the presence of the factor \( C(t) \), which represents a relaxation of the shear resistance of the plate after the moment is applied. For example, if a thermal perturbation (which induces loads \( P \) and moments \( M \)) is suddenly applied at time \( t = 0 \) and subsequently maintained at a fixed value, the loads and moments will be

\[ \bar{P}(u, s) = \bar{P}(u)/s \]
\[ \bar{M}(u, s) = \bar{M}(u)/(s + s_0). \]

The Laplace transformed response functions are thus

\[ \bar{w}_p(u, s) = \frac{\bar{P}(u)}{\Delta k + u^4 D} \left[ \frac{s + s_0}{s(s + s_4)} \right] \]
\[ \bar{w}_m(u, s) = \frac{u^2 \bar{M}(u)}{\Delta k + u^4 D} \left[ \frac{1}{s + s_4} \right] \]

and the corresponding time domain forms are

\[ \bar{w}_p(u, t) = \frac{\bar{P}(u)}{\Delta k} \left[ \frac{\exp(-s_4 t)}{1 + (u/\beta)^4} + [1 - \exp(-s_4 t)] \right] \]
\[ \bar{w}_m(u, t) = \frac{u^2 \bar{M}(u)}{\Delta k} \left[ \frac{\exp(-s_4 t)}{1 + (u/\beta)^4} \right] . \]

The initial response \( (t = 0) \) to both loads and moments is, of course, just the previously determined elastic response. However, this elastic deformation is subsequently attenuated, via the factor \( \exp(-s_4 t) \), with long wavelength components disappearing first. The load response has an additional term which represents a growing viscous deformation, whereas the moment response consists only of the selectively attenuated elastic response. The total volume displaced remains constant

\[ 2\pi \int_0^\infty w(r, t) r dr = 2\pi \int_0^\infty \frac{P(r)}{\Delta k} r dr \]
\[ = 2\pi \frac{P(u)}{\Delta k} \bigg|_{u=0} \]

and, as in the elastic case, the applied moments make no net contribution.
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It is interesting to note the similarity between the time-dependent part of the viscoelastic load response
\[ \Delta \bar{w}_p(u, t) = \bar{w}_p(u, t) - \bar{w}_p(u, 0) \]
\[ = \frac{(u/\beta)^3 \bar{p}(u)}{\Delta k + u^4D} [1 - \exp(-s_4t)] \]
and the time-dependent elastic moment response
\[ \Delta \bar{w}_m(u, t) = \frac{-u^2 \bar{M}(u)}{\Delta k + u^4D} [1 - \exp(-s_4t)]. \]
Characteristics which they share include:

1. neither changes the total volume displaced by a stationary load;
2. for Gaussian sources, their primary influence on deflection is in peripheral regions;
3. both are capable of causing a net spectral shift to shorter wavelengths, such as would occur during the inward displacement of a peripheral bulge.

The observation that the viscous loading response has a \( u^4 \) factor versus a \( u^2 \) factor for the flexural response implies that, if the source and response parameters were sufficiently well known, it would be possible to distinguish between the two effects. However, in actual practice, their behaviour is similar enough to be easily confused, one for the other. In fact, it appears likely that in some cases at least part of the lithospheric response which has been previously interpreted in terms of viscous relaxation (Walcott 1970; Kunze 1980) is actually a manifestation of thermoelastic flexure.

We will now examine the thermomechanical response of a viscoelastic lithosphere to a sudden change in basal temperature. The present analysis parallels that for the dynamic elastic case considered previously. Substituting (15) into (20) and (21) via equations (3) and (4), respectively, we obtain

\[ \bar{w}_p(u, s) = \bar{W}_p(u) \left[ \frac{2 [\cosh(ah) - 1]}{sah \sinh(ah)} \right] \left( \frac{s + s_0}{s + s_4} \right) \]
\[ \bar{w}_m(u, s) = \bar{W}_m(u) \left[ \frac{12 \sinh(ah) - 6 ah [\cosh(ah) + 1]}{s (ah)^2 \sinh(ah)} \right] \left( \frac{s}{s + s_4} \right) \]
where \( \bar{W}_p(u) \) and \( \bar{W}_m(u) \) are the appropriate static elastic response functions (11) and (12). Retaining only the dominant terms in the thermal response, we find

\[ \bar{w}_p(u, s) = \bar{W}_p(u) \left[ \frac{s_p}{s(s + s_p)} \right] \left( \frac{s + s_0}{s + s_4} \right) \]
\[ \bar{w}_m(u, s) = \bar{W}_m(u) \left[ \frac{s_m}{s(s + s_m)} \right] \left( \frac{s}{s + s_4} \right). \]
The inverse Laplace transforms of this latter pair yield (Nadai 1961; Sleep & Snell 1976)

\[ \bar{w}_p(u, t) = \bar{W}_p(u) \left[ \frac{s_p - s_0}{s_p - s_4} \right] [1 - \exp(-s_p t)] + \frac{s_p (s_0 - s_4)}{s_4 (s_p - s_4)} [1 - \exp(-s_4 t)] \]
\[ \bar{w}_m(u, t) = \bar{W}_m(u) \left[ \frac{s_m}{s_m - s_4} \right] [\exp(-s_4 t) - \exp(-s_m t)]. \]
In discussing these results it will be helpful to distinguish three separate categories of response, dependent upon the relative lengths of the thermal versus rheological time-scales. If the thermal diffusion time is very short compared to the Maxwell relaxation time ($s_p, s_m > s_0$), the response is initially very similar to the elastic case considered above. The static elastic equilibrium deformation is essentially attained within a few thermal time-scales. Only much later does the viscous relaxation become evident. This is actually the simplest category of response in that the two time-scales are clearly separated. The elastic deformation and subsequent viscous relaxation can be treated as essentially separate problems.

If, instead, the thermal excitation is very slow compared to the viscous relaxation ($s_p, s_m \ll s_0$), the initial elastic response will be completely suppressed at long wavelengths, and it is essentially only the final viscous behaviour that survives. However, because of the strong wavelength dependence of the relaxation time $1/s_4$, any reasonable set of plate parameters will leave the short-wavelength elastic features essentially intact. There is thus no clean separation of time-scales as there was in the first case.

Finally, if the excitation and relaxation time-scales are comparable ($s_p, s_m \approx s_0$), the full viscoelastic behaviour is evident. If, in fact, the time-scales are equal ($s_p = s_0$), the loading response simplifies considerably

$$\overline{w}_p(u, t) = \overline{w}_p(u) \left( \frac{s_0}{s_4} \right) [1 - \exp(-s_4 t)]$$

$$- \frac{p(u)}{\Delta k} [1 - \exp(-s_4 t)].$$

This differs from a purely elastic loading response in that the spatial dependence is fully relaxed (locally compensated) at all times, but also differs from a purely viscous response in that the elastic plate influences the time dependence via the factor $\exp(-s_4 t)$.

6 Discussion

The most obvious applications of this theory are to problems of plateau uplift and basin subsidence. While some plateaus are obviously related to compressional tectonic regimes (Molnar & Tapponier 1975; Gupta, Rao & Singh 1982), others appear to have a more purely thermal origin (Thompson & Zoback 1979; Gass et al. 1978; Crough 1981). Mareschal (1981) has recently presented a theoretical analysis of the surface heat flow anomalies and uplift which would follow a sudden change in lithospheric basal heat flow. Unfortunately, erosion tends to obscure the details of plateau uplift history so that it is difficult to find clear evidence of even such obvious things as lithospheric flexure, and it would appear extremely difficult to obtain the resolution necessary to distinguish between the effects of viscous relaxation versus thermal bending moments.

Sedimentary basins, on the other hand, preserve in their stratigraphy a much better record of vertical crustal movement over time-scales of $10^7$ to $10^8$ yr. The main dynamic role of the sedimentary load itself is merely to amplify the subsidence caused by some other primary mechanism (Watts & Ryan 1976). The subsidence of marginal basins appears to be related to rifting episodes which cause both extensional crustal thinning and subsequent thermal contraction (Sleep 1971; McKenzie 1978). However, for intraplate basins, there appears to be very little associated horizontal motion and the subsidence is presumably of more directly thermal origin (Bott 1976; Haxby et al. 1976; Turcotte & Ahern 1977).

The history of a major depositional cycle in one of these basins typically includes a widespread, gentle uplift, accompanied by subareal erosion, and followed by initially rapid subsidence of a broad, roughly Gaussian basin and a subsequent gradual decrease in both rate of
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subsidence and basin width (Sloss & Scherer 1971; Sleep & Snell 1976). A smooth, exponential decay of subsidence rate with a time constant of roughly 50 Myr would be consistent with a thermal origin (Sleep 1971). Most of the apparent irregularities in observed subsidence rate and gaps in the sedimentary record can be adequately explained by eustatic sea-level changes (Sleep 1976).

The observed decrease in basin width with time is rather more peculiar in that there are at least two obvious processes which tend to oppose it. First and foremost is simple sedimentary infilling which tends both to deepen and widen the initial depression. The sedimentary accumulation itself constitutes an additional load on the lithosphere and thus produces additional deflection which extends somewhat beyond the edge of the causative load (Walcott 1972; Lambeck & Nakiboglu 1981). Furthermore, if the subsidence is associated with thermal recovery from a previous doming event, any thickening of the lithosphere with time due to conductive cooling will tend to increase the flexural length (Haxby et al. 1976; Caldwell & Turcotte 1979) and thus widen the basin corresponding to a fixed load. The generally accepted resolution of this dilemma involves viscoelastic relaxation of the lithosphere which, as Sleep & Snell (1976) and Beaumont (1978) have clearly shown, does indeed produce a gradual narrowing of sedimentary basins. However, as we have previously seen, thermal bending moments can produce very similar effects. In fact, we suggest that thermal flexural effects may significantly modify sedimentary basin development. Thus, any attempt to estimate lithospheric viscosity from the time-dependant response to thermal loads, without proper consideration of thermoelastic bending moments (Sleep & Snell 1976; Beaumont 1978), may be badly biased.

It is often assumed that the main depositional phase in old intraplate basins, such as the Palaeozoic Michigan basin, is associated with the cooling and contraction of the lithosphere following an earlier thermal doming event (Sleep & Snell 1976; Haxby et al. 1976). However, Burke (1976) has suggested, by analogy with the active intracontinental Chad basin, that the central depression may instead have been formed in response to loading by sediments eroded off a peripheral rim. Central subsidence, in response to this loading, would provide a continually renewed basin. Thus, once initiated, this process would keep the basin going as long as the periphery continues to be elevated. It is tempting to speculate that the raised region along the Chad basin watershed corresponds to a flexurally induced peripheral uplift as shown in Fig. 4.

In this model, a fairly broad positive thermal anomaly would produce an initially uplifted region with an even higher peripheral rim. If the rim has sufficient continuity to form a closed drainage basin, the central region may accumulate enough sediments eventually to subside below the initial base level. For sufficiently narrow sources, or other combinations of source and/or plate parameters which favour thermal flexure, the central deflection may even be initially downward, prior to significant sediment accumulation. For sufficient broad thermal anomalies, the result would be a simple plateau uplift.

It has been assumed throughout that the plate is at rest with respect to the heat source. For moving plates the assumption of axial symmetry breaks down and the different time dependencies for loads and moments translate into characteristic patterns of relative uplift and subsidence at various distances 'downstream' from the source. This will be the topic of a subsequent investigation.

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