The total cross section for the process \( e^- e^+ \to e^- t \bar{b} \bar{\nu}_e \) is calculated below the threshold energy of the pair production of top quarks. We use the real photon approximation at the lowest order in the minimal gauge theory of \( SU(2)_L \times U(1) \). Estimated total cross sections are \( 0.11 \times 10^{-7} \) and \( 0.29 \times 10^{-5} \) pb at the center-of-mass energies \( \sqrt{s} = 50 \) and 65 GeV, respectively, when the top quark mass is 40 GeV.

§ 1. Introduction

The top quark was introduced by Kobayashi and Maskawa with the bottom quark in order to accomodate the CP violation in the \( K \) meson within the Weinberg-Salam model. The discovery of the bottom quark encouraged experimentalists to search for the top quark. Recently its candidates have been reported by UA1 group in \( S p p S \) collider. They say that the mass of the top quark is \( 30 \text{ GeV} < m_t < 50 \text{ GeV} \). Experimental difficulties, however, are pointed out in proton-antiproton collision. For example, the UA2 group cannot find top quark candidates. On the other hand, the electron-positron colliding machine will certainly be able to make more detailed experiments on the existence of top quark and precise measurement of its mass. If energy reaches the toponium mass, one can see the resonance like \( \Psi \) and \( \Upsilon \). Also if it reaches the threshold of pair production of top quarks, one can easily see changes in \( R \) ratio or event topology through the thrust and sphericity. Energy of the present machine PEP and PETRA is 45 GeV or less, so that these states of top quarks cannot be produced. Next project for the electron-positron machine is the TRISTAN where its maximum energy is expected to be 80 GeV.

There has not been a study on the possibility to observe the top quark, when the beam energy is less than its mass. The single top quark production cross section in electron-positron scattering is supposed to be small in this energy region, however, the importance of the top quark forces us to answer the question whether the single top quark is visible or not below the threshold of the toponium.

With this motivation we will investigate the process

\[
e^- e^+ \to e^- t \bar{b} \bar{\nu}_e \ .
\]

To estimate the cross section for this process, one would have to do a tedious work even at the lowest order if one did the calculations without any approximation. In this paper, we use the real photon approximation, which greatly reduces our labor to calculate the cross section. We will comment later that, from the evaluation of top pair production,
Calculation of Single Top Quark Production

$e^-e^+ \rightarrow \gamma, Z \rightarrow t\bar{b}e\bar{\nu}_e$, the contribution from the other diagrams is indeed smaller. The real photon approximation allows us to pick up only the $t$-channel photon exchange diagrams which are dominant when the electron scatters forward. In § 2, the formulae needed for computing the cross section are given, and the numerical results are presented in § 3.

§ 2. Formulation

We shall neglect the Feynman diagrams which have no $t$-channel photon pole. The diagrams considered here are shown in Fig. 1, which are expected to give the dominant contributions to the process (1) except around the threshold energy of the $t$-quark pair production.

The $S$ matrix element is defined as follows:

$$S = i e \bar{u}(P_\mu) \gamma_\mu u(P_\nu)\frac{g^{\mu\nu}}{k^2}M_\nu,$$

where $P_\mu(E, p_\mu)$ and $Q_\mu(E', q_\mu)$ (also other momentum $k, P_\tau, Q_\tau(Q'_\tau, q'_\tau), P_t(E_t, p_t)$ and $P_b(E_b, p_b)$) are indicated in Fig. 2. We denote the magnitude of the 3 dimensional momentum $q_-, q_+, p_t$ and $p_b$ by $q_-, q_+, p_t$ and $p_b$, respectively. And $M_\nu$ represents the amplitude for the process $\gamma e^+ \rightarrow t\bar{b}\bar{\nu}_e$.

The production cross section for $e^-e^+ \rightarrow e^-t\bar{b}\bar{\nu}_e$ integrated over the final electron phase space can be written as

$$\frac{d\sigma_{e^-e^+ \rightarrow e^-t\bar{b}\bar{\nu}_e}}{d^3 \vec{q}_e} = \frac{\alpha}{2\pi^2} \int \frac{d^3 q_-}{EE'} \frac{g_{\mu\nu}}{k^2}g_{\alpha\beta} \left(2P_-P_\alpha + \frac{1}{2}k^2g_{\mu\alpha}\right)\frac{1}{4}M_\nu M_\nu' d\vec{r},$$

where

![Feynman diagrams](https://example.com/feynman_diagrams.png)

Fig. 1. Feynman diagrams having the $t$-channel photon pole. The wavy line represents a photon and the spiral one does a $W$ boson.

![Definition of the momentum](https://example.com/momentum_definition.png)

Fig. 2. Definition of the momentum.
The real photon approximation leads to

$$d\sigma_{e^+e^-\rightarrow t\bar{b}v_e} = \int_{\omega_{\min}}^{\omega_{\max}} \frac{d\omega}{\omega} N(\omega) d\sigma_{e^+e^-\rightarrow t\bar{b}v_e}(\omega) ,$$

where \(\omega = E - E'\), \(\tilde{s} = 2\omega\sqrt{s}\), \(\sigma_{e^+e^-\rightarrow t\bar{b}v_e}\) is a cross section for \(e^+e^-\rightarrow t\bar{b}v_e\), \(\omega_{\min}\) and \(\omega_{\max}\) are the threshold and the maximal photon energies, respectively, and \(N(\omega)\) is the luminosity function given by

$$N(\omega) = \frac{\alpha}{\pi} \left[ \ln \frac{E}{m_e} - \frac{1}{2} \right] \left[ \frac{E^2 + E'^2}{E^2} \right] + \cdots .$$

What we have to do is, therefore, to calculate the cross section for the process

$$e^+e^-\rightarrow t\bar{b}v_e .$$

We define the amplitudes \(A^\mu\sim D^\mu\) corresponding to the Feynman diagrams \(A\sim D\) in Fig. 1, respectively, as follows:

$$A^\mu = \frac{2\pi\alpha M_z^2}{M_z^2 - M_w^2} [\bar{u}_L(P_t) \gamma^\mu v_L(P_b)] [\bar{u}_L(P_+) \gamma^\mu v_L(Q_+)] \times \Gamma^{\mu\alpha\beta} \times \frac{-g_{a\gamma} + Q a Q_\beta / M_w^2}{Q^2 - M_w^2 - iM_w \Gamma w (k - Q)^2 - M_w^2} ,$$

where \(Q = P_t + P_b\) and

$$\Gamma^{\mu\alpha\beta} \equiv (Q - 2k)^a g_{\beta\mu} - (2Q - k)^a g_{\alpha\beta} + (k + Q)^a g_{\alpha\beta} ,$$

$$B^\mu = \frac{2\pi\alpha M_z^2}{M_z^2 - M_w^2} [\bar{u}_L(P_t) \gamma^\mu \frac{-k + P_t + M_t}{(k - P_t)^2 - M_t^2} \gamma^\alpha v_L(P_b)]$$

$$\times \left[ \bar{v}_L(P_+) \gamma^\beta v_L(Q_+) \right] \frac{-g_{a\beta}}{(k - Q)^2 - M_w^2} ,$$

$$C^\mu = \frac{2\pi\alpha M_z^2}{M_z^2 - M_w^2} [\bar{u}_L(P_t) \gamma^\mu \frac{k - P_b + M_b}{(k - P_b)^2 - M_b^2} \gamma^\alpha v_L(P_b)]$$

$$\times \left[ \bar{v}_L(P_+) \gamma^\beta v_L(Q_+) \right] \frac{-g_{a\beta}}{(k - Q)^2 - M_w^2} ,$$

$$D^\mu = \frac{2\pi\alpha M_z^2}{M_z^2 - M_w^2} [\bar{u}_L(P_t) \gamma^\mu \frac{-k + P_+}{(k + P_+)^2 - M_w^2} \gamma^\alpha v_L(P_b)]$$

$$\times \left[ \bar{v}_L(P_+) \gamma^\beta v_L(Q_+) \right] \frac{-g_{a\beta} + Q a Q_\beta / M_w^2}{Q^2 - M_w^2 - iM_w \Gamma w} .$$

The whole amplitude is given by

$$M^\mu = A^\mu + B^\mu + C^\mu + D^\mu .$$

We can check our amplitudes using the gauge invariance of the photon, \(k \mu M^\mu = 0\).**

---

*1 We represent the Weinberg angle by the gauge boson masses \(M_w\) and \(M_z\) as \(\sin^2 \theta_w = 1 - M_w^2 / M_z^2\).

** This is true of course in the limit \(M_w / M_w \rightarrow 0\).
After the photon polarization sum, the matrix element square $-g_{\mu\nu}M^{\mu}M^{\nu}$ is expressed in terms of the inner products $(k\cdot P_+), (k\cdot P_t), (k\cdot P_b), (P_+\cdot P_t), (P_+\cdot P_b)$ and antisymmetric combination $\varepsilon_{a\beta\gamma} k^a P^\beta P_t^\gamma P_b^\delta$. The $\varepsilon$ term, however, which always appears with the decay width of the weak boson $\Gamma_w$ does not contribute when we integrate over the azimuthal angles $\phi_t$ and $\phi_b$. (The angles $\phi_t$ and $\phi_b$ will be defined in § 3.) Therefore, we will omit the $\varepsilon$ term hereafter.

We write

$$-g_{\mu\nu}M^{\mu}M^{\nu} = \left( \frac{4\pi\alpha}{M_w^2} \right)^3 \left( \frac{M_z^2}{M_z^2 - M_w^2} \right)^2 f,$$

where $f$ is a function of the inner products $(k\cdot P_+) \cdots (P_+\cdot P_b)$, which is calculated by using the REDUCE.\(^3\) Using this function the invariant differential cross section is given by

$$E_t E_b Q_+ \frac{d\sigma_{\gamma^* \rightarrow t\bar{b}\bar{\nu}_e}}{d^3 p_t d^3 p_b d^3 q_+} = 3C \frac{1}{2(k\cdot P_+)} f\delta^4(k + P_+ - P_t - P_b - Q_+),$$

where

$$C = \frac{\alpha^3}{32\pi^2} \frac{1}{M_w^2} \left( \frac{M_z^2}{M_z^2 - M_w^2} \right)^2,$$

and the factor 3 comes from the color degrees of freedom. The total cross section for $\gamma^* \rightarrow t\bar{b}\bar{\nu}_e$ is, therefore, written as

$$\sigma_{\gamma^* \rightarrow t\bar{b}\bar{\nu}_e} = 3C \frac{1}{(k\cdot P_+)} \int \frac{d^3 p_t}{E_t} \frac{d^3 p_b}{E_b} f\delta[(k + P_+ - P_t - P_b)^2].$$

### § 3. Numerical integrations

We choose the center-of-mass system of the photon and the positron, i.e., $k + P_+ = (\sqrt{s}, 0)$. We define the scattering angles $\theta_t, \theta_b$ and $\phi_t, \phi_b$ as shown in Fig. 3. If we eliminate the momentum for neutrino, the total cross section at the center-of-mass energy $\sqrt{s}$ is given by

$$\sigma_{\gamma^* \rightarrow t\bar{b}\bar{\nu}_e} = 3C \times (2\pi)^2 \int_0^1 \frac{d\cos \theta_t}{1} \frac{d\cos \theta_b}{1} \int_0^{2\pi} \frac{d(\phi_t - \phi_b)}{1} \int_{E_t^{\min}}^{E_t^{\max}} \frac{dE_t}{k \cdot P_+} \frac{p_t p_b}{|h|} f,$$

where

$$E_t^{\min} = M_t + M_b,$$

$$E_t^{\max} = (\sqrt{s} + M_t^2 + M_b^2) / 2\sqrt{s},$$

![Fig. 3. Definitions of the angles $\theta_t, \theta_b$ and $\phi = \phi_t - \phi_b$.](https://academic.oup.com/ptp/article-abstract/75/1/92/1844884)
and $\Theta$ stands for the angle between top and bottom quarks.

The 4 dimensional integrations are carried out numerically by using the adoptive Monte Carlo integration routine, BASES. The set of parameters used in the calculations is as follows:

$$M_Z = 93.0 \text{ GeV}, \quad M_W = 81.0 \text{ GeV}, \quad M_b = 5.0 \text{ GeV}, \quad 30.0 \text{ GeV} \leq M_t \leq 60.0 \text{ GeV},$$

and $\Gamma_W$ is calculated by the formula

$$\Gamma_W = \frac{a}{12} \frac{M_Z^2 - M_W^2}{M_W^2} M_W \left[ 9 + 3 \left( \frac{M_t^2 + M_b^2}{M_W^2} - \frac{1}{2} \left( \frac{M_t^2 - M_b^2}{M_W^2} \right)^2 \right) \right.$$

$$\times \left\{ \left( \frac{M_t^2 + M_b^2}{M_W^2} \right)^2 - 4 \left( \frac{M_t M_b}{M_W^2} \right)^2 \right\}^{1/2} \right].$$

This formula is derived by taking into account three generations of quark and lepton and by neglecting the masses of $u, d, c, s$ quarks and leptons. The results are shown in Fig. 4, where the 6 dotted curves are the total cross sections for $\gamma e^+ \rightarrow t \bar{b} \nu_e$ for $M_t=30, 35, 40, 45, 50$ and $60 \text{ GeV}$ from the top. We give three comments on the results in Fig. 4. (1) We can integrate over the momentum of top and bottom quarks analytically, therefore, we can get the total cross section by doing the two dimensional integrations numerically. The numerical results are checked by this way. (2) For $M_t=40 \text{ GeV}$ ($\Gamma_W=2.225 \text{ GeV}$) we parametrize the result as follows:

$$\sigma_{\gamma e^+ \rightarrow t \bar{b} \nu_e} = (\ln \bar{x})^2 \left[ 0.0235 - 0.117/\bar{x} + 0.230/\bar{x}^2 - 0.206/\bar{x}^3 + 0.0690/\bar{x}^4 \right]$$

$$+ \theta(\sqrt{s} - M_W) Br(W \rightarrow tb) \times \sigma_{\gamma e^+ \rightarrow W^+ \nu_e},$$

in pb unit, where $\bar{x} \equiv s/(M_t + M_b)^2$, $\sigma_{\gamma e^+ \rightarrow W^+ \nu_e}$, is the total cross section$^7$ for the process $\gamma e^+ \rightarrow W^+ \nu_e$ and $Br(W \rightarrow tb)$ is the branching ratio of $W$ boson decay into top and bottom quarks. This formula reproduces the results roughly also for other $M_t$. (3) The calculated cross sections are useful if one wants to estimate the event rate of single top production by tagging an electron.

Embedding the cross sections into formula (5), we obtain the total cross sections for the process $e^-e^+ \rightarrow e^-t \bar{b} \nu_e$ as shown also in Fig. 4 by the solid curves from the top in cases $M_t=30, 35, 40, 45, 50, 60 \text{ GeV}$.

Corresponding to Eq. (23) we get the parametrized form of the cross section for the process $e^-e^+ \rightarrow e^-t \bar{b} \nu_e(M_t=40 \text{ GeV})$ as

$$\sigma_{e^-e^+ \rightarrow e^-t \bar{b} \nu_e(M_t=40 \text{ GeV})} = \left( \ln \bar{x} \right)^2 \left[ 0.0235 - 0.117/\bar{x} + 0.230/\bar{x}^2 - 0.206/\bar{x}^3 + 0.0690/\bar{x}^4 \right]$$

$$+ \theta(\sqrt{s} - M_W) Br(W \rightarrow tb) \times \sigma_{e^-e^+ \rightarrow W^+ \nu_e},$$

in pb unit, where $\bar{x} \equiv s/(M_t + M_b)^2$, $\sigma_{e^-e^+ \rightarrow W^+ \nu_e}$, is the total cross section$^7$ for the process $e^-e^+ \rightarrow W^+ \nu_e$ and $Br(W \rightarrow tb)$ is the branching ratio of $W$ boson decay into top and bottom quarks. This formula reproduces the results roughly also for other $M_t$. (3) The calculated cross sections are useful if one wants to estimate the event rate of single top production by tagging an electron.

Fig. 4. Total cross sections for $\gamma e^+ \rightarrow t \bar{b} \nu_e$ and for $e^-e^+ \rightarrow e^-t \bar{b} \nu_e$ are shown by the dotted and solid curves, respectively. Each curve from the top is obtained when the top quark mass is assumed to be 30, 35, 40, 45, 50 and 60 GeV.
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\[
\sigma_{e^-e^+\to e^-e^-\bar{t}x_e} = \frac{\alpha}{\pi} \left( \ln \frac{\sqrt{s}}{2m_e} - \frac{1}{2} \right) \times \left[ \frac{1-x}{x} \{0.467 + 0.968/x + 0.328/x^2 - 0.113/x^3\} \right. \\
+ (\ln x) \{0.0822 + 0.702/x + 0.691/x^2 + 0.297/x^3 - 0.0211/x^4\} \\
+ (\ln x)^2 \{-0.0352 + 0.117/x + 0.230/x^2 + 0.137/x^3 - 0.0230/x^4\} \\
+ (\ln x)^3 \{0.0157 + 0.0780/x + 0.0768/x^2\} \\
\left. + \theta(\sqrt{s} - M_w) Br(W \to tb) \times \sigma_{e^-e^+\to e^-e^-}\nu_x(y)\right], \tag{24}
\]

where

\[
\sigma_{e^-e^+\to e^-e^-\bar{t}x_e}(y) = \frac{\alpha^2}{16} \frac{M_x^2}{M^2 - M_w^2} \frac{1}{M^2 W^2} \left( \ln \frac{\sqrt{s}}{2m_e} - \frac{1}{2} \right) \left[-\frac{920}{9}\right. \\
+ 176/y - 104/y^2 + (272)/9/y^3 \\
+ (\ln y) \{32 + 24/y - 8/y^2 + 32/3/y^3\} + (\ln y)^2 \{32/y - 8/y^2\}\right], \tag{25}
\]

and \(x = s/(M_t + M_b)^2\), \(y = s/M_w^2\). This formula is also checked by integrating Eq. (5) directly which contains 5 variables with Eq. (17).

The contribution from \(e^-e^+\to \gamma, Z\to e^-\bar{t}\bar{t}\nu\) at \(M_t = 40\) GeV is less than \(10^{-11}\) pb and \(4\times10^{-8}\) pb for \(\sqrt{s} = 50\) and 70 GeV, respectively.\(^*)\)

Our results show that the total cross sections for single top quark production in electron-positron scattering are strongly dependent on top quark mass, but the magnitude is fairly small as shown in Fig. 4. Even if we have \(L = 10^{33}\) cm\(^2\) sec\(^{-1}\), the expected production rate is only one event per 1500 days, for \(\sqrt{s} = 70\) GeV and \(M_t = 40\) GeV.

References

6. C. Weiszacker and E. J. Williams, ibid. 88 (1934), 612.
7. I. Landau and E. Lifshitz, Phys. Z. Sowjetunion 6 (1934), 244.

\(^*)\) We estimated these values by the following approximation formula,

\[
\frac{d\sigma}{dM^2} = \frac{1}{\pi} \frac{M_t\Gamma(M^2)}{(M^2 - M_t^2)^2 + M_t^2\Gamma^2(M^2)} \frac{2|q_1|/W}{\pi^2} r(s, M_t^2),
\]

where \(\Gamma(M^2) = (M - M_b)/(M - M_t)^3 \Gamma_1, \Gamma_1 = 2.3\) keV (real top decay width), \(2|q_1|/W = (2\sqrt{s} - B^2 - M^2)(2\sqrt{s} + M_t^2 - M^2)/s\) is the phase space factor and \(r(s, M_t^2)\) is the matrix element squared for real top pair production.