Microscopic Derivation of
the Spin-Orbit Potential of Mass-Three Nucleus

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A formulation how to derive the spin-orbit potential of mass-three nucleus microscopically is given on the basis of the resonating group method. The calculated spin-orbit potentials of $^3$He for two target nuclei, $^{16}$O and $^{40}$Ca are analysed. Various exchange components and their energy-dependences are discussed. The renormalization of the kinetic exchange potential and central-interaction exchange potential into the spin-orbit potential is found to be large, which makes the total spin-orbit potential larger even three times than the double-folding spin-orbit potential in the tail region for the low incident energy.

§ 1. Introduction

In recent decade, the systematic scattering experiments of $^3$He and $^6$ have been performed by using the polarized beams and the analyses of these data by the optical potentials have given us rich information on the spin-orbit ($l\cdot s$) potentials of these $3N$ (three-nucleon) particles. According to these studies, although the derived parameters of the $l\cdot s$ potentials for $^3$He and for $^6$ are fairly different from each other, both $l\cdot s$ potentials are found a few times stronger than those expected by the folding model. Therefore, it is important to derive and study microscopically the $l\cdot s$ potentials for $3N$ particles.

In recent years, we have been studying the inter-nucleus interaction microscopically on the basis of the resonating group method (RGM). In these studies, the non-local inter-nucleus potentials of the RGM are transformed into the equivalent local potentials (ELP) with very high accuracy. Analyses of the localized potentials have revealed many interesting properties of the inter-nucleus potential. However these studies are limited to the investigations of the central inter-nucleus potentials. The purpose of this paper is to derive the $l\cdot s$ potential of two systems of $3N + ^{16}$O and $3N + ^{40}$Ca on the basis of the RGM by the same method used to derive the central inter-nucleus local potential.

The non-central two-nucleon force adopted in the microscopic Hamiltonian is the two-nucleon spin-orbit force $v^{LS}$ and this force is the origin of the $3N$ $l\cdot s$ potential. We will see that the formation of the $3N$ $l\cdot s$ potential from $v^{LS}$ is made through the following two mechanisms. One is, of course, due to the folding of the direct and exchange matrix elements of $v^{LS}$. The other is due to the renormalization contribution of the two-nucleon central potential $v^c$ and the kinetic energy. The latter component of the $3N$ $l\cdot s$ potential due to the renormalization of $v^c$ and the kinetic energy will be shown to be of sizable magnitude and to occur from the fact that the local momentum for $j = l + 1/2$ is different from that for $j = l - 1/2$. Here $l$ is the orbital angular momentum of the relative motion between $3N$ and the target, and $j$ is the added angular momentum of $l$ and the spin of $3N$.

As for the former component of the $3N$ potential coming directly from $v^{LS}$, we will show that it consists of three terms: On account of the Pauli principle, the three nucleons of the $3N$ particle are accommodated in the valence orbits around the target nucleus. Thus the $v^{LS}$ force contributes through three energies; the Hartree potential energy, the Fock potential energy and the mutual interaction energy among valence nucleons. These three kinds of contribution of $v^{LS}$ constitute the above-mentioned three terms. Since the valence orbits have non-zero angular momenta, one may at first consider that the contribution from the mutual interaction energy between valence nucleons is not small. But we will show that this contribution is usually negligibly small.

The construction of this paper is as follows. In § 2, we explain the calculation of the GCM kernel of $v^{LS}$ where as usual GCM is the abbreviation of “generator coordinate method”. The calculation is done by the valence-orbit method. Here we investigate the three contributions of $v^{LS}$ through the Hartree potential, the Fock potential and the mutual interaction among valence nucleons. In § 3 we transform the GCM kernel of $v^{LS}$ to the corresponding RGM kernel. By calculating the Wigner transforms of the RGM kernels we get the Hamilton-Jacobi equation to determine the local momentum. Formulas are given by which we calculate the central and $l \cdot s$ $3N$-potentials using the local momenta for $j = l \pm 1/2$. In § 4, results of the numerical calculations for $3N + ^{16}O$ and $3N + ^{40}Ca$ are reported. Exchange contributions are analysed by comparing them with the double folding $l \cdot s$ potential. The renormalization of $v^e$ and the kinetic energy in the $l \cdot s$ potential is also analysed here. Finally in §5, we give a summary.

§ 2. GCM kernel of two-nucleon spin-orbit force

2.1. Derivation of GCM kernel of $v^{LS}$ by the valence-orbit method

We denote the projectile as $P$ and the target as $T$. Here $P$ is a $3N$ particle ($^3$He or $t$) or a nucleon (proton or neutron), and $T$ is $^{16}O$ or $^{40}Ca$. According to the valence-orbit method, the GCM kernel of the microscopic Hamiltonian for the “P + T” system is given by

$$M_{GCM}(\sigma, R; \sigma', R') = \langle \Gamma (r - R) \phi_\sigma (P) \phi_\sigma (T) | \left( \sum_{i=1}^{A_p + A_T} t_i - T + \sum_{i>j} v_{ij} \right) \left\{ \left[ \Gamma (r - R') \phi_{\sigma'} (P) \phi_\sigma (T) \right] \right\rangle $$

$$= \exp \left( -\frac{A_p^2}{2(A_p + A_T)} \nu (R - R')^2 \right)$$

$$\times \left[ \langle \phi_{R \sigma} (P) | \phi_{R \sigma'} (P) \rangle \left\{ E(T) - \frac{3}{4} \hbar \omega + \frac{A_p^2 \hbar \omega}{4(A_p + A_T)} \nu (R - R')^2 \right\} \right. $$

$$+ \left\langle \phi_{R \sigma} (P) | \sum_{i=1}^{A_p} (t + U_{HF})_i + \sum_{i>j} v_{ij} \right\rangle \phi_{R \sigma'} (P) \rangle \right] ,$$

(2·1)

where $\phi_\sigma (P)$ and $\phi (T)$ are the internal wave functions of $P$ and $T$, respectively, with $\sigma$ of $\phi_\sigma (P)$ denoting the $z$-component of the spin of $P$, and $A_p$ and $A_T$ are the mass numbers of $P$ and $T$, respectively. The definitions of other symbols in Eq. (2·1) are as follows:
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\[ \Gamma (r - R) \equiv \left( \frac{2\gamma}{\pi} \right)^{3/4} e^{-\gamma(r - R)^2}, \quad \gamma = \frac{A_P A_T}{A_P + A_T} \nu, \]

\[ E(T) = \langle \psi(T) | \left[ \sum_{i=1}^{A_T} t_i + \sum_{i>j}^{A_T} v_{ij} \right] \psi(T) \rangle, \]

\[ \psi(T) = \left( \frac{2A_T \nu}{\pi} \right)^{3/4} e^{-\gamma(x - R)^2} \phi(T) \]

\[ = \frac{1}{\sqrt{A_T}} \det [\varphi_1^{4\times} \varphi_{A_T^{4\times}}], \]

\[ \tilde{\varphi}_R(P) = \begin{cases} \frac{1}{\sqrt{3!}} \det [\tilde{\varphi}_R(x_1) \eta_{n\sigma} (1) \tilde{\varphi}_R(x_2) \eta_{p\sigma} (2) \tilde{\varphi}_R(x_3) \eta_{n\sigma} (3)], & \text{when } P = ^3\text{He}, \\
\end{cases} \]

\[ \tilde{\varphi}_R(x) = (1 - \sum_{j=1}^{A_T/4} |\varphi_j\rangle \langle \varphi_j|) \left( \frac{2\nu}{\pi} \right)^{3/4} e^{-\nu(x - R)^2}, \]

\[ \langle f | U_{HF} | g \rangle = \sum_{j=1}^{A_T/4} \langle f, (\varphi_j \eta_{r \sigma}) | v | g, (\varphi_j \eta_{r \sigma}) \rangle - \langle f, (\varphi_j \eta_{r \sigma}) | v (\varphi_j \eta_{r \sigma}) \rangle \]

where \( \eta_{r \sigma} \) stands for the isospin-spin function with \( r \) and \( \sigma \) denoting the \( z \)-components of the isospin and spin, respectively.

Now we study the components of the GCM kernel that come from the two-nucleon spin-orbit force \( v^{LS} \) of the form

\[ v^{LS} = \{ u(^3O) P(^3O) + u(^3E) P(^3E) \} e^{-\nu(x - R)^2} L \cdot (S_1 + S_2), \]

\[ L = x_1 x_2 - \frac{1}{2} (x_1 - x_2) \times \left( \frac{\partial}{\partial x_1} - \frac{\partial}{\partial x_2} \right), \]

where \( P(^3O) \) and \( P(^3E) \) are projection operators onto the \( ^3O \) (triplet odd) and \( ^3E \) (triplet even) states, respectively, and \( u(^3O) \) and \( u(^3E) \) are strength parameters. The actual \( v^{LS} \) we use is a sum of terms of the above form. First, we note that \( E(T) \) has no contribution from \( v^{LS} \) since

\[ \langle \psi(T) | \sum_{i>j}^{A_T} v_{ij} \phi(T) \rangle = 0. \]

Next, when we denote the component of \( U_{HF} \) (Hartree-Fock potential) that comes from \( v^{LS} \) as \( U^{LS}_{HF} \), we have

\[ \langle \tilde{\varphi}_R(P) | \sum_{j=1}^{A_T/4} (U^{LS}_{HF})_{|\varphi_j\rangle \langle \varphi_j|} \tilde{\varphi}_R \sigma(P) \rangle \]

\[ = \delta^{A_T-1} \langle \eta_{\sigma} | S | \eta_{\sigma} \rangle \left[ \{ 3 u(^3O) + u(^3E) \} \sum_{j=1}^{A_T/4} \langle \tilde{\varphi}_R \varphi_j \rangle e^{-\nu r^2} L | \tilde{\varphi}_R \varphi_j \rangle \right. \]

\[ \left. - \{ 3 u(^3O) - u(^3E) \} \sum_{j=1}^{A_T/4} \langle \tilde{\varphi}_R \varphi_j \rangle e^{-\nu r^2} L | \varphi_j \tilde{\varphi}_R \rangle \right], \]

\[ \tilde{\varphi}_R \equiv \langle \tilde{\varphi}_R | \tilde{\varphi}_R \rangle, \]

where \( \eta_{\sigma} \) is a spin function with \( z \)-component \( \sigma \). Third, we have
Of course, when \( P \) is a nucleon, this term is absent in the GCM kernel. The appearance of only \( u^{(3}E) \) is due to the fact that \( \hat{\phi}_{R_0}(P) \) is totally symmetric in spatial coordinates.

Altogether, the component \( M_{\Sigma S}^{\text{GCM}}(\sigma, \sigma'; R; R') \) of the GCM kernel that comes from \( \nu^{LS} \) is written as

\[
M_{\Sigma S}^{\text{GCM}}(\sigma, \sigma'; R; R') = \langle \Gamma' (r - R) \phi_0 (P) \phi (T) | \sum_{i,j}^{A \rho + A \tau} v_{\Sigma}^{i j} | \mathcal{A} \{ \Gamma' (r - R') \phi_0 (P) \phi (T) \rangle \rangle
\]

\[
= \langle \eta_{0} | S | \eta_{0} \rangle \exp \left\{ \frac{A_{\rho}^{2}}{2(2A_{\rho} + 2A_{\tau})} \nu (r - \mathbf{R}')^{2} \right\}
\times \left[ \hat{\phi}_{R_0}^{A \rho - 1} \left\{ (3 u^{(3)E} + u^{(3)E}) \right\} \sum_{j=1}^{A \tau/4} \langle \hat{\phi}_{R_0} \phi_{j} | e^{-\xi r^{2}} L | \phi_{j} \phi_{R} \rangle \right.
\]

\[
- (3 u^{(3)O} - u^{(3)E}) \sum_{j=1}^{A \tau/4} \langle \hat{\phi}_{R_0} \phi_{j} | e^{-\xi r^{2}} L | \phi_{j} \phi_{R} \rangle \right) + \hat{\phi}_{R_0}^{A \rho - 2} u^{(3)E} \sum_{j=1}^{A \tau/4} \langle \hat{\phi}_{R_0} \phi_{j} | e^{-\xi r^{2}} L | \phi_{j} \phi_{R} \rangle \right].
\]  \tag{2.7}

2.2. Important consequences due to the characteristic properties of \( \nu^{LS} \)

Two-nucleon spin-orbit force \( \nu^{LS} \) has the following two characteristic properties: The first is that the force range of \( \nu^{LS} \) is extremely short; namely, the range parameter \( \chi \) is very large. The second is that the \( ^{3}E \) component of \( \nu^{LS} \) is zero or weakly repulsive; namely, \( u^{(3)E} \approx 0 \) and \( u^{(3)O} > 0 \).

The above-mentioned two properties of \( \nu^{LS} \) give us

\[
\langle \hat{\phi}_{R_0} (P) | \sum_{i,j}^{A \rho} v_{\Sigma}^{i j} | \phi_{R_0} \rangle = \text{negligibly small}.
\]  \tag{2.8}

The first reason of this result is, of course, due to \( u^{(3)E} \approx 0 \). There is another strong reason for Eq. (2.8), which comes from the first property of \( \nu^{LS} \), the extremely short range of \( \nu^{LS} \). It is obtained from the following important relation due to Scheerbaum,\(^7\)

\[
\langle f_{i} f_{j} | g (r) L | f_{i} f_{j} \rangle \approx - \langle f_{i} f_{j} | g (r) L | f_{i} f_{j} \rangle
\]

for any \( g (r) \) which is very short-ranged. \( \tag{2.9} \)

We call this relation Scheerbaum’s relation, whose proof we give in Appendix I for the sake of self-containedness. From Scheerbaum’s relation, we immediately obtain \( \langle \phi_{R} \phi_{R} | e^{-\xi r^{2}} L | \phi_{R} \phi_{R} \rangle \approx 0 \), which constitutes another strong reason of Eq. (2.8).

The result of Eq. (2.8) is important. The term of Eq. (2.8) is absent when \( P \) is a nucleon and therefore if it were not small it would constitute an important factor which makes the \( l \cdot s \) force of the composite projectile different from that of a nucleon. In the case of the two-nucleon central force \( \nu^{c} \), the corresponding term \( \langle \hat{\phi}_{R_0} (P) | \sum_{i,j}^{A \rho} v_{\Sigma}^{i j} | \hat{\phi}_{R_0} (P) \rangle \) has been found not to be small and to cause an important projectile-mass-number-dependence of the inter-nucleus central potential.\(^8\)

Next we consider the matrix element of \( U_{\Sigma S}^{LS} \). According to Scheerbaum’s relation, we have
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\[ \langle \Phi_R \Phi_i | e^{-\epsilon R^2} L | \Phi_R \Phi_i \rangle \approx -\langle \Phi_R \Phi_i | e^{-\epsilon R^2} L | \Phi_i \Phi_R \rangle, \]  \hspace{1cm} (2.10)  

which gives us

\[ (\text{Fock term}) \approx \frac{3u(\sigma O) - u(\sigma E)}{3u(\sigma O) + u(\sigma E)} \times (\text{Hartree term}). \]  \hspace{1cm} (2.11)  

From this relation and Eq. (2.8), we can say that the \( l \cdot s \) potential of a \( 3N \) particle as well as that of a nucleon is built up of the Hartree-Fock terms in which the Fock term is nearly equal to the Hartree term.

§ 3. Calculational procedure of the \( l \cdot s \) potential

3.1. Transformation of the GCM kernel to the RGM kernel

The GCM kernel of \( v_{LS} \), \( M_{LS}^{\text{GCM}}(\sigma; R; \sigma', R') \) of Eq. (2.7), can be written as

\[ M_{LS}^{\text{GCM}}(\sigma; R; \sigma', R') = \langle \eta_\sigma | S | \eta_{\sigma'} \rangle (-i) (R \times R') K_{LS}^{\text{GCM}}(R \cdot R'), \]  \hspace{1cm} (3.1)  

where \( K_{LS}^{\text{GCM}}(R, R') \) is a function of \( R^2, R'^2 \) and \( (R - R') \) and is expressed as

\[ K_{LS}^{\text{GCM}}(R, R') = \sum_j \sum_{l,m,n} C_{jmn} R^{2j} R'^{2m} (R \cdot R')^{n} \exp \left[ -X_j R^2 - Y_j R'^2 - Z_j (R \cdot R') \right]. \]  \hspace{1cm} (3.2)  

As is well-known, the GCM kernel of the central two-nucleon force can be written in the form of \( K_{LS}^{\text{GCM}}(R, R'). \)  \hspace{1cm} (3.1)  

Equation (3.1) can be proved by actually calculating the matrix elements in Eq. (2.7) for \( P = N, 3N \) and for \( T = ^{16}O \) and \( ^{40}Ca \). A formal proof of Eq. (3.1) is given in Appendix II.

Now we transform \( M_{LS}^{\text{GCM}}(\sigma; R; \sigma', R') \) to the corresponding RGM kernel \( M_{LS}^{\text{RGM}}(\sigma; a; \sigma', a') \). According to the single-Fourier-transformation formula,\(^{11)} \) we have

\[ M_{LS}^{\text{RGM}}(\sigma; a; \sigma', a') \]

\[ \begin{align*} 
= & \langle \delta (r - a') \phi_\sigma (P) \phi (T) | \sum_{l>l} v_{LS}^{l \sigma} | \delta (r - a') \phi_\sigma (P) \phi (T) \rangle \\
= & \left( \frac{1}{2\pi} \right)^6 \left( \frac{\pi}{2\gamma} \right)^{3/2} e^{\gamma (a^2 + a'^2)} \int d\mathbf{k} d\mathbf{k}' \\
& \times \exp \left[ -\frac{1}{4\gamma} (k^2 + k'^2) - ik \cdot a - ik' \cdot a' \right] M_{LS}^{\text{GCM}}(\sigma; \frac{i}{2\gamma} \mathbf{k}; \frac{i}{2\gamma} \mathbf{k}') \\
= & \langle \eta_\sigma | S | \eta_{\sigma'} \rangle \left( \frac{1}{2\pi} \right)^6 \left( \frac{\pi}{2\gamma} \right)^{3/2} e^{\gamma (a^2 + a'^2)} \frac{1}{4\gamma} (-i) \left( \frac{\partial}{\partial a} \times \frac{\partial}{\partial a'} \right) \\
& \times \int d\mathbf{k} d\mathbf{k}' \exp \left[ -\frac{1}{4\gamma} (k^2 + k'^2) - ik \cdot a - ik' \cdot a' \right] K_{LS}^{\text{GCM}} \left( \frac{i}{2\gamma} \mathbf{k}, \frac{i}{2\gamma} \mathbf{k}' \right) \\
= & \langle \eta_\sigma | S | \eta_{\sigma'} \rangle e^{\gamma (a^2 + a'^2)} \frac{1}{4\gamma} (-i) \left( \frac{\partial}{\partial a} \times \frac{\partial}{\partial a'} \right) e^{-\gamma (a^2 + a'^2)} K_{LS} (a, a'), \]  \hspace{1cm} (3.3)  

where \( K_{LS} (a, a') \) is the RGM kernel corresponding to the GCM kernel \( K_{LS}^{\text{GCM}} (R, R') \). Since \( K_{LS}^{\text{GCM}} (R, R') \) is of the form of the central-force GCM kernel, the calculation of \( K_{LS} (a, a') \) is well-known and \( K_{LS} (a, a') \) is given by\(^{10,12} \)
\[ K_{LS}(a, a') = \left( \frac{\gamma}{2\pi} \right)^{3/2} \sum_{j,m,n} \sum_{\ell} \mathcal{C}_{jmn}(\ell+m+n) \left( \frac{\partial}{\partial X_j} \right)^{\ell} \left( \frac{\partial}{\partial Y_j} \right)^{m} \left( \frac{\partial}{\partial Z_j} \right)^{n} \left( \frac{4\gamma^2}{4X_{o,j}Y_{o,j}-Z_{j}^2} \right)^{3/2} \times \exp[-A_{j}a^2 - B_{j}a'^2 - C_{j}a \cdot a'], \]

\[ X_{o,j} = X_j - \gamma, \quad Y_{o,j} = Y_j - \gamma, \]

\[ A_{j} = -4\gamma^2 \frac{X_{o,j}}{4X_{o,j}Y_{o,j}-Z_{j}^2} - \gamma, \quad B_{j} = -4\gamma^2 \frac{X_{o,j}}{4X_{o,j}Y_{o,j}-Z_{j}^2} - \gamma, \]

\[ C_{j} = 4\gamma^2 \frac{Z_{j}}{4X_{o,j}Y_{o,j}-Z_{j}^2}. \]

By using

\[ e^{\gamma(a^2+a'^2)} \left( \frac{\partial}{\partial a} \times \frac{\partial}{\partial a'} \right) e^{-\gamma(a^2+a'^2)} e^{-\gamma a^2 - B_{j}a'^2 - C_{j}a \cdot a'} \]

\[ = 4(A_{j}+\gamma)(B_{j}+\gamma) - C_{j}^2 \right)(a \times a') e^{-\gamma a^2 - B_{j}a'^2 - C_{j}a \cdot a'} \]

\[ = \frac{16\gamma^4}{4X_{o,j}Y_{o,j}-Z_{j}^2} (a \times a') e^{-\gamma a^2 - B_{j}a'^2 - C_{j}a \cdot a'}, \]

we obtain

\[ M_{LS}^{RGM}(\sigma, a; \sigma', a') = \langle \eta_{\sigma} | S | \eta_{\sigma'} \rangle (-i)(a \times a') F_{LS}(a, a'), \]

\[ F_{LS}(a, a') = \left( \frac{\gamma}{2\pi} \right)^{3/2} \sum_{j,m,n} \sum_{\ell} \mathcal{C}_{jmn}(\ell+m+n) \left( \frac{\partial}{\partial X_j} \right)^{\ell} \left( \frac{\partial}{\partial Y_j} \right)^{m} \left( \frac{\partial}{\partial Z_j} \right)^{n} \left( \frac{4\gamma^2}{4X_{o,j}Y_{o,j}-Z_{j}^2} \right)^{3/2} \times \exp[-A_{j}a^2 - B_{j}a'^2 - C_{j}a \cdot a'], \]

It is to be noticed that \( F_{LS}(a, a') \) is obtained from \( K_{LS}(a, a') \) simply by replacing \((4\gamma^2/(4X_{o,j}Y_{o,j}-Z_{j}^2))^{3/2}\) by \((4\gamma^2/(4X_{o,j}Y_{o,j}-Z_{j}^2))^{3/2}\).

When we express \( M_{LS}^{RGM}(\sigma, a; \sigma', a') \) in the form of the equivalent differential operator, we have

\[ M_{LS}^{RGM}(\sigma, a; \sigma', a') = \langle \eta_{\sigma} | S | \eta_{\sigma'} \rangle \cdot J(a, \partial/\partial a) \delta(a - a'), \]

\[ J(a, \partial/\partial a) \] is a scalar operator composed of \( a^2, (a \cdot \partial/\partial a) \) and \((\partial/\partial a \cdot \partial/\partial a)\). A proof of Eq. (3.7) is given in Appendix III. Equation (3.7) shows that \( M_{LS}^{RGM}(\sigma, a; \sigma', a') \) is a spin-orbit interaction operator with a non-local form factor \( J(a, \partial/\partial a) \).

### 3.2. Formula to calculate the l·s potential

Our procedure to transform the \( l\cdot s \) potential with non-local form factor, \( M_{LS}^{RGM}(\sigma, a; \sigma', a') \), into the equivalent \( l\cdot s \) potential with local form factor is the same as the procedure to get the equivalent local central potential from the non-local RGM central potential,\(^3\) which has been extensively used with many interesting results.\(^9\)

The RGM equation of motion in our present problem is written as

\[ \int [H(a, a') - EN(a, a')] \chi(a') da' = 0; \]
\[
H(a, a') = T(a, a') + G_c(a, a') + G_{LS}(a, a', S) ,
\]
\[
G_{LS}(a, a', S) = S \cdot (-i)(a \times a') F_{LS}(a, a') ,
\]
where \(N(a, a')\) is the norm kernel, \(T(a, a')\) the kinetic kernel and \(G_c(a, a')\) the kernel coming from the two-nucleon central force. The definition of the relative wave function \(\chi(r)\) is as follows: When we express the original RGM many-body wave function as
\[
\sum_\sigma M_\sigma \{ \chi_\sigma(r) \phi_\sigma(P) \phi(T) \} ,
\]
\(\chi(r)\) is given by
\[
\chi(r) = \sum_\sigma \chi_\sigma(r) \eta_\sigma .
\]

In order to get the local momentum from which we calculate the equivalent local potential, we need to evaluate the Wigner transforms of the non-local operators in Eq. (3·8). When a non-local operator is of the usual type, namely, when the non-locality range in \(|a - a'|\) is shorter than that in \(|a + a'|\), we call such a non-local operator to be of the Wigner type or to be of the A-type. For an A-type non-local operator \(O_A(a, a')\), the Wigner transform we need is the usual one defined by
\[
O^w_A(a, p) = \int dt e^{i(n) t \cdot p} O_A\left(a - \frac{1}{2} t, a + \frac{1}{2} t \right) .
\]

On the other hand, when the non-locality range in \(|a - a'|\) is longer than that in \(|a + a'|\), we call such a non-local operator to be of the Majorana type or to be of the B-type. For a B-type non-local operator \(O_B(a, a')\), the Wigner transform we need is given by \(\pm O^w_B(a, p)\) where the plus and minus signs are to be used for the even and odd partial waves, respectively, and \(O^w_B(a, p)\) is defined by
\[
O^w_B(a, p) = \int dt e^{i(n) t \cdot p} O_B\left(a - \frac{1}{2} t, -(a + \frac{1}{2} t) \right) .
\]

In general, a non-local operator \(O(a, a')\) is a sum of the A-type component \(O_A(a, a')\) and the B-type component \(O_B(a, a')\); \(O(a, a') = O_A(a, a') + O_B(a, a')\). For such \(O(a, a')\) the Wigner transform we need is expressed as
\[
O^w_x(a, p) = O^w_A(a, p) \pm O^w_B(a, p) .
\]

It should be mentioned here that, except in special systems with \(A_F \approx A_T\), the contribution from the Majorana-type kernels \(O_B(a, a')\) are fairly small and so we can use simply \(O^w_A(a, p)\) instead of \(O^w_x(a, p)\) in a good approximation.

Now by the use of the notation of Eq. (3·13), the equation to determine the local momentum \(p(a)\) is given by
\[
H^w_x(a, p(a)) - EN^w_x(a, p(a)) = 0 .
\]

The local momentum \(p(a)\) with a fixed orbital angular momentum \(l\) is written as
\[
p(a) = p_r(a) \frac{a}{a} + \frac{\hbar \sqrt{l(l+1)}}{a} \cdot n ,
\]
\[
n \perp a ,
\]
and so what we determine by Eq. (3-13) is the radial component $p_r(a)$ of $p(a)$.

In the case of the spin-orbit RGM kernel $G_{LS}(a, a', S)$, by decomposing $F_{LS}(a, a')$ into $A$ and $B$ types, $F_{LS}(a, a') = F_{LS,A}(a, a') + F_{LS,B}(a, a')$, we have

$$G_{LS,\pm}(a, p, S) = \int dt \mathcal{e}^{i(tS)\cdot S} \cdot (-i) \left( a - \frac{t}{2} \right) \times \left( a + \frac{t}{2} \right) F_{LS,\pm}(a - \frac{t}{2}, a + \frac{t}{2})$$

$$= - \hbar S \cdot \left( a \times \frac{\partial}{\partial p} \right) [F_{LS,A}(a, p) \mp F_{LS,B}(a, p)]$$

$$= (S \cdot l) \left( -2 \hbar^2 \frac{\partial}{\partial p} \right) F_{LS,\pm}(a, p), \tag{3-16}$$

where of course $\hbar l = a \times p$. In deriving the final equality in Eq. (3-16), we have used the fact that both $F_{LS,A}(a, p)$ and $F_{LS,B}(a, p)$ are functions of three arguments $p^2$, $(p \cdot a)^2$ and $a^2$. By the use of Eq. (3-16) we can write $H_{\pm \gamma}(a, p)$ as

$$H_{\pm \gamma}(a, p) = T_{\pm \gamma}(a, p) + G_{\gamma,\pm}(a, p)$$

$$+ \left\{ -2 \hbar^2 \frac{\partial}{\partial p} F_{LS,\pm}(a, p) \right\} \frac{1}{2} \left\{ j(j+1) - l(l+1) - \frac{3}{4} \right\}, \tag{3-17}$$

where $j$ is the magnitude of $j = l + S$ and $j = l + 1/2$ or $l - 1/2$.

We denote the local momenta for $j = l \pm 1/2$ as $p_{l \pm 1/2}(a)$. The equivalent local potentials $V_{l \pm 1/2}(a)$ for $j = l \pm 1/2$ are defined by

$$V_{l \pm 1/2}(a) = E - \frac{1}{2\mu} (p_{l \pm 1/2}(a))^2, \tag{3-18}$$

where $\mu$ is the reduced mass of $P$ and $T$. When we denote the equivalent local central potential as $V_c^{eq}(a)$ and the equivalent local form factor of the $l \cdot s$ potential as $V_{ls}^{eq}(a)$, there should hold

$$V_c^{eq}(a) = V_c^{eq}(a) + V_{ls}^{eq}(a) \frac{1}{2} \left\{ j(j+1) - l(l+1) - \frac{3}{4} \right\}, \tag{3-19}$$

from which we obtain the desired formulas to calculate $V_c^{eq}(a)$ and $V_{ls}^{eq}(a)$ as below

$$V_c^{eq}(a) = \frac{1}{2l+1} [V_{l \pm 1/2}(a) + lV_{l \mp 1/2}(a)],$$

$$V_{ls}^{eq}(a) = -\frac{1}{l + \frac{1}{2}} [V_{l \mp 1/2}(a) - V_{l \pm 1/2}(a)]. \tag{3-20}$$

3.3. Calculation of $(-2\hbar^2 \partial/\partial p^2) F_{LS,\pm}(a, p)$

As seen in Eq. (3-6), $F_{LS}(a, a')$ is a linear combination of the terms

$$f_{lnm}(a, a') = \left( \frac{\gamma}{2\pi} \right)^{3/2} \left( \frac{\partial}{\partial X_j} \right)^l \left( \frac{\partial}{\partial Y_j} \right)^m \left( \frac{\partial}{\partial Z_j} \right)^n \left( \frac{4\gamma^2}{4X_{ij}Y_{ij}Z_j^2} \right)^{5/2}$$

$$\times \exp[-A_j a^2 - B_j d^2 - C_j a \cdot a']. \tag{3-21}$$
According to Ref. 5), the non-local kernel of Eq. (3·21) is of the Wigner type when \( C_j < 0 \) and of the Majorana type when \( C_j > 0 \). Therefore the Wigner transform we need for \( E_j < 0 \) is that of \( f_{lmn}(a, \alpha') \) itself while for \( C_j > 0 \) we need to calculate the Wigner transform of \( f_{lmn}(a, -\alpha') \). By noticing that \( C_j \geq 0 \) is equivalent to \( Z_j \geq 0 \), \( f_{lmn}(a, \alpha') \) for \( C_j < 0 \) and \( f_{lmn}(a, -\alpha') \) for \( C_j > 0 \) can be expressed unifiedly by \( \tilde{f}_{lmn}(a, \alpha') \) defined by

\[
\tilde{f}_{lmn}(a, \alpha') = \left( \frac{\gamma}{2\pi} \right)^{3/2} \left( \frac{Z_j}{|Z_j|} \right)^n \left( \frac{\partial}{\partial X_j} \right)^l \left( \frac{\partial}{\partial Y_j} \right)^m \left( \frac{\partial}{\partial Z_j} \right)^n \left( \frac{4\gamma^2}{4X_{0j}Y_{0j} - Z_j^2} \right)^{5/2} \times \exp \left[ -A_j a^2 - B_j a'^2 + |C_j| a \cdot a' \right] 
\]

\[
= \begin{cases} 
  f_{lmn}(a, \alpha') & \text{for } C_j < 0, \\
  f_{lmn}(a, -\alpha') & \text{for } C_j > 0.
\end{cases}
\] (3·22)

The Wigner transform of \( \tilde{f}_{lmn}(a, \alpha') \) is calculated to be\(^{10} \)

\[
\tilde{f}_{lmn}^w(a, p) = \left( \frac{Z_j}{|Z_j|} \right)^n \left( \frac{\partial}{\partial X_j} \right)^l \left( \frac{\partial}{\partial Y_j} \right)^m \left( \frac{\partial}{\partial Z_j} \right)^n \left( \frac{4\gamma^2}{4X_{0j}Y_{0j} - Z_j^2} \right)^{3/2} \times \left( \frac{4\gamma^2}{4X_{0j}Y_{0j} - Z_j^2} \right) \exp \left[ -p_j a^2 - q_j \left( \frac{p}{\hbar} \right)^2 + i r_j \left( \frac{p \cdot a}{\hbar} \right) \right],
\]

\[
X_{wj} = X_j - \frac{\gamma}{2} = X_{oj} + \frac{\gamma}{2}, \quad Y_{wj} = Y_j - \frac{\gamma}{2} = Y_{oj} + \frac{\gamma}{2},
\]

\[
Z_{wj} = |Z_j| + \gamma,
\]

\[
p_j = 4\gamma^2 X_{wj} + Y_{wj} + Z_{wj} - 2\gamma,
\]

\[
q_j = \frac{Z_{wj} - X_{wj} - Y_{vj}}{Z_{wj} - 4X_{wj}Y_{wj}} - \frac{1}{2\gamma} = \frac{1}{2\gamma} \frac{4X_{0j}Y_{0j} - Z_j^2}{Z_{wj} - 4X_{wj}Y_{wj}},
\]

\[
r_j = 4\gamma^2 X_{wj} - Y_{wj} - \frac{Z_{wj} - Y_{wj}}{Z_{wj} - 4X_{wj}Y_{wj}},
\] (3·23)

from which we obtain

\[
\left( -2\hbar^2 \frac{\partial}{\partial p^2} \right) \tilde{f}_{lmn}^w(a, p) = \frac{1}{\gamma} \left( \frac{Z_j}{|Z_j|} \right)^n \left( \frac{\partial}{\partial X_j} \right)^l \left( \frac{\partial}{\partial Y_j} \right)^m \left( \frac{\partial}{\partial Z_j} \right)^n \left( \frac{4\gamma^2}{4X_{0j}Y_{0j} - Z_j^2} \right)^{5/2} \times \exp \left[ -p_j a^2 - q_j \left( \frac{p}{\hbar} \right)^2 + i r_j \left( \frac{p \cdot a}{\hbar} \right) \right].
\] (3·24)

The execution of the differentiation \( \left( \partial / \partial X_j \right)^l \left( \partial / \partial Y_j \right)^m \left( \partial / \partial Z_j \right)^n \) in Eq. (3·24) can be made by the technique explained in detail in Ref. 10). When \( \left( -2\hbar^2 \partial / \partial p^2 \right) \tilde{f}_{lmn}^w(a, p) \) are calculated, the desired quantity \( \left( -2\hbar^2 \partial / \partial p^2 \right) F_{lmn}^{w}(a, p) \) is obtained by linearly combining them.

§ 4. Results of numerical calculations

We report in this paper the results of the numerical calculations for the two systems, \(^3\text{He}^+^{16}\text{O}\) and \(^3\text{He}^+^{40}\text{Ca}\). The Coulomb force is included by approximating the RGM Coulomb kernel as \( \sqrt{N} V_{dc} \sqrt{N} \) where \( V_{dc}(a) \) is the direct (or double folding) potential of
the Coulomb interaction. The harmonic oscillator parameter $\nu$ of the wave functions of $P$ and $T$ is 0.16 fm$^{-2}$ for $^3\text{He}+^{16}\text{O}$ and is 0.14 fm$^{-2}$ for $^3\text{He}+^{40}\text{Ca}$. The effective central nuclear force adopted is the Volkov No. 1 force$^{13}$ with the Majorana exchange mixture $m$ being $m=0.60$ for $^3\text{He}+^{16}\text{O}$ and $m=0.658$ for $^3\text{He}+^{40}\text{Ca}$. The parameter values given above for $^3\text{He}+^{16}\text{O}$ and $^3\text{He}+^{40}\text{Ca}$ are the same as those which have been used in the studies of the inter-nucleus central potentials for respective systems. As for the effective two-nucleon spin-orbit force $v^{ls}$, we adopt the following one:

$$v^{ls} = \left\{ \sum_{i=1}^{2} u_i(3O) P(3O) e^{-kx^2} \right\} L \cdot (S_1 + S_2),$$

$$u_i(3O) = 900 \text{ MeV}, \quad x_1 = 5.0 \text{ fm}^{-2},$$

$$u_2(3O) = -900 \text{ MeV}, \quad x_2 = 2.778 \text{ fm}^{-2},$$

(4.1).

which is taken from Ref. 14 and is due to Ref. 15. Since this $v^{ls}$ has no $^3E$ component, the spin-orbit RGM kernel consists only of Hartree and Fock terms, as is discussed in § 2.

In Fig. 1, we show the calculated $l \cdot s$ potential form factor $V^{ls}_0(a)$ of the $^3\text{He}+^{16}\text{O}$ system at the incident energies per nucleon $E=5, 10, 15$ and 20 MeV/u. For the sake of comparison, we also show the direct $l \cdot s$ potential $V^0_0(a)$ and the ratio $V^{ls}_0(a)/V^0_0(a)$. The direct (or double folding) $l \cdot s$ potential $V^0_0(a)$ is the non-exchange part of $G_{ls}(a, a', S)$ and satisfies the relation

$$V^0_0(a) \langle \eta_\sigma | l \cdot S | \eta_\sigma \rangle = \langle \phi_\sigma(P) \phi(T) | \sum_{j \in \Gamma} v^{ls}_j | \phi_\sigma(P) \phi(T) \rangle.$$  

(4.2)

Figure 2 shows the same quantities, $V^{ls}_0(a), V^0_0(a), V^{ls}_0(a)/V^0_0(a)$, for the $^3\text{He}+^{40}\text{Ca}$ system at the same incident energies per nucleon, $E=5, 10, 15$ and 20 MeV/u.

The $l \cdot s$ potentials $V^{ls}_0(a)$ shown in Figs. 1 and 2 are those for the orbital angular momentum $l=2$. Like the equivalent local central potential $V^{eq}_c(a)$, the equivalent local form factor $V^{ls}_c(a)$ of the spin-orbit potential is dependent on the orbital angular momentum $l$ in general. But the actual calculations show that this $l$-dependence is small at least for $E \gtrsim 5$ MeV/u. We show in Figs. 3 and 4 the comparison of $V^{ls}_0(a)$ with $l=2$ to that with $l=1$ for $^3\text{He}+^{16}\text{O}$ and $^3\text{He}+^{40}\text{Ca}$ systems, respectively, at $E=5, 10$ and 15 MeV/u. We see surely that except in the inner spatial region $V^{ls}_0(a)$ is independent of $l$ and that even in the inner region the $l$-dependence is not so large and it disappears rapidly as $E$ gets higher. Therefore in this section we report the numerical results in the case of only one orbital angular momentum which we choose $l=2$.

We here indicate three points which we notice commonly in Figs. 1 and 2. The first is that $V^{ls}_0(a)$ are of the volume type. The second is that, around the contact distance region, the ratio $V^{ls}_0(a)/V^0_0(a)$ exceeds two and approaches three at low incident energy. The third point is that the energy-dependence of $V^{ls}_0(a)$ is such that $V^{ls}_0(a)$ becomes deeper in the inner spatial region as $E$ gets higher while it becomes shallower in the tail region.

Next we decompose $V^{ls}_0(a)$ into two components; one coming from the $L \cdot S$-force kernel $G_{ls}(a, a', S)$ and the other from the renormalization effects of other kernels. For this purpose, we express $V^{eq}_0(a)$ of Eq. (3.15) as follows:

$$V^{eq}_0(a) = E - \frac{1}{2\mu} (p_0(a))^2$$

where $p_0(a)$ is the vanishing momentum $p_0$ for $l=2$. The energy dependence of $V^{ls}_0(a)$ is such that $V^{eq}_0(a)$ becomes deeper in the inner spatial region as $E$ gets higher while it becomes shallower in the tail region.
Fig. 1. Form factors $V^{os}_f(r)$ of the spin-orbit potential of $^3\text{He}+^{16}\text{O}$ for $l=2$ at $E=5, 10, 15$ and 20 MeV/u are shown in the lower part of the figure by the solid line. In the upper part of the figure, the ratio $V^{os}_f(r)/V^{D}_f(r)$ is displayed where $V^{D}_f(r)$ is the double folding spin-orbit potential and is shown in the lower part of the figure by the dotted line. The ordinate of the upper part of the figure is given at the right-hand side.

$$V^{os}_f(r) = \frac{H^{w}\left(a, p_j(a)\right)}{N^{w}\left(a, p_j(a)\right)} \frac{1}{2\mu} (p_j(a))^2$$

$$= t^{\text{ex}}_j(a) + V^{c}_j(a) + V^{LS}_j(a),$$

$$t^{\text{ex}}_j(a) = \frac{T^{w}\left(a, p_j(a)\right)}{N^{w}\left(a, p_j(a)\right)} \frac{1}{2\mu} (p_j(a))^2,$$

$$V^{c}_j(a) = \frac{G^{w}\left(a, p_j(a)\right)}{N^{w}\left(a, p_j(a)\right)},$$

$$= V_0(a) + V_{dc}(a) + v^{\text{ex}}_j(a),$$

$$V^{LS}_j(a) = \frac{-2\hbar^2}{\frac{\partial}{\partial p^2}} F^{w}_{LS,\gamma}\left(a, p_j(a)\right) \times \frac{1}{2} \left[j(j+1) - l(l+1) - \frac{3}{4}\right].$$

$$V^{LS}_j(a) = \left(4 \cdot 3\right) \text{(4-3)}$$
Fig. 2. The same quantities as in Fig. 1 in the case $^3\text{He}+^{40}\text{Ca}$.

Fig. 3. Comparison of $V_{ls}^D(r)$ with $l=2$ (solid line) to that with $l=1$ (dotted line) in $^3\text{He}+^{16}\text{O}$ system at $E=5, 10$ and $15\text{MeV}/u$.

Fig. 4. The same quantities as in Fig. 3 in the case $^3\text{He}+^{40}\text{Ca}$. 

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where \( V_D(a) \) is the direct (or double folding) potential due to the central nuclear force. By using Eq. (4.3) we have

\[
V_{ls}^{eq}(a) = \frac{1}{l+1/2} \left[ V_{ls}^{eq}(a) - V_{ls}^{eq}(a) \right]
\]

\[
= V_{ls,ls}(a) + V_{ls,v}(a) + V_{ls,v}(a)
\]

\[
V_{ls,ls}(a) = \frac{1}{l+1/2} \left[ V_{ls}^{eq}(a) - V_{ls}^{eq}(a) \right],
\]

\[
V_{ls,v}(a) = \frac{1}{l+1/2} \left[ V_{ls}^{eq}(a) - V_{ls}^{eq}(a) \right].
\]

Figures 5 and 6 display \( V_{ls,ls}^{eq}(a) \) and \( V_{ls,ren}(a) = V_{ls,ls}(a) + V_{ls,v}(a) \) for \( ^3\text{He} + ^{16}\text{O} \) and \( ^3\text{He} + ^{40}\text{Ca} \) systems, respectively, at \( E = 5, 10 \) and \( 15 \text{ MeV/}u \).

We see that the renormalization potential \( V_{ls,ren}(a) \) is fairly large and is attractive in the tail region. The energy-dependence of \( V_{ls,ren}(a) \) is such that the amplitude of \( V_{ls,ren}(a) \) decreases in the whole spatial region as \( E \) gets higher. Since the energy-dependence of \( V_{ls,ren}(a) \) is larger than that of \( V_{ls,ls}(a) \) in the inner spatial region, the energy-dependence of \( V_{ls}^{eq}(a) \) in the inner region is due to that of \( V_{ls,ren}(a) \). In the tail region, the energy-dependence of \( V_{ls,ren}(a) \) is comparable with that of \( V_{ls,ls}(a) \) and these two add up coherently.

Figures 7 and 8 show the decomposition of \( V_{ls,ren}(a) \) into \( V_{ls,v}(a) \) and \( V_{ls,v}(a) \) for \( ^3\text{He} + ^{16}\text{O} \) and \( ^3\text{He} + ^{40}\text{Ca} \) systems, respectively, at \( E = 5, 10 \) and \( 15 \text{ MeV/}u \). We see that the energy-dependence of \( V_{ls,v}(a) \) is larger than that of \( V_{ls,ls}(a) \) and so the energy-dependence of \( V_{ls,ren}(a) \) shows the same behaviour as that of \( V_{ls,v}(a) \). This result is reasonable since we know already from the previous studies \(^{10}\) that the energy-dependence of \( t^{eq}(a) \) is larger than that of \( v^{eq}(a) \).

Finally we show in Figs. 9 and 10 the decomposition of \( V_{ls,ls}(a) \) into two components, \( V_{ls,ls}^{eq}(a) = V_{ls,h}(a) + V_{ls,f}(a) \), where \( V_{ls,h}(a) \) comes from the Hartree potential of the valence nucleons (nucleons of \( ^3\text{He} \) accommodated in the valence orbits around the target nucleus) and \( V_{ls,f}(a) \) comes from the Fock potential of the valence nucleons (see § 2). For the sake of comparison we also show the direct \( l-s \) potential \( V_{ls}^{eq}(a) \). We see in these figures that \( V_{ls,h}(a) \) is slightly deeper than \( V_{ls}^{eq}(a) \) in the outer spatial region while it is slightly shallower than \( V_{ls}^{eq}(a) \) in the inner region. As \( E \) gets higher, \( V_{ls,h}(a) \) approaches \( V_{ls}^{eq}(a) \) in both spatial regions. As for \( V_{ls,f}(a) \), it is shallower than \( V_{ls,h}(a) \) and in the tail region its depth decreases as \( E \) gets higher like \( V_{ls,h}(a) \). According to Scheerbaum’s relation which we have discussed in § 2, \( V_{ls,f}(a) \) becomes identical to \( V_{ls,h}(a) \) in the zero-range limit of the two-nucleon spin-orbit force \( v^{ls} \) when \( u(3E) = 0 \). Therefore the difference between the calculated \( V_{ls,f}(a) \) and \( V_{ls,h}(a) \) is attributed to the finiteness of the ranges of \( v^{ls} \) adopted here although these ranges are very short compared to the ranges of the two-nucleon central force.
Fig. 5. Decomposition of $V^{eq}_{\text{LS}}(r)(l=2)$ into $V^{eq}_{\text{LS,Ren}}(r)$ (solid line) and $V^{eq}_{\text{LS,Sen}}(r)$ (dotted line) in $^3\text{He}+^{16}\text{O}$ system at $E=5, 10$ and $15\text{MeV}/u$. $V^{eq}_{\text{LS,S}}(a)$ is the $l$-$s$ potential coming directly from the two-nucleon spin-orbit force and $V^{eq}_{\text{LS,Sen}}(r)$ is the $l$-$s$ potential coming from the renormalization of the kinetic exchange potential and the central-force exchange potential.

Fig. 6. The same quantities as in Fig. 5 in the case of $^3\text{He}+^{40}\text{Ca}$.

Fig. 7. Decomposition of $V^{eq}_{\text{Sen}}(r)(l=2)$ into $V^{eq}_{\text{Sen,t}}(r)$ (solid line) and $V^{eq}_{\text{Sen,v}}(r)$ (dotted line) in $^3\text{He}+^{16}\text{O}$ system at $E=5, 10$ and $15\text{MeV}/u$. $V^{eq}_{\text{Sen,t}}(r)$ is the $l$-$s$ potential coming from the renormalization of the kinetic exchange potential and $V^{eq}_{\text{Sen,v}}(r)$ is that of the central-force exchange potential.

Fig. 8. The same quantities as in Fig. 7 in the case of $^3\text{He}+^{40}\text{Ca}$. 
§ 5. Summary

We have given the formulation how to derive the $l\cdot s$ potential of $3N$ particle ($^3\text{He}$ or $^t$) from the two-nucleon spin-orbit force $v^{ls}$ microscopically on the basis of the resonating group method, and have analysed the calculated $l\cdot s$ potentials of $^3\text{He}$ for the two target nuclei, $^{16}\text{O}$ and $^{40}\text{Ca}$.

The $3N$ $l\cdot s$ potential is generated through two mechanisms: One is the piling of the direct and exchange matrix elements of $v^{ls}$. The $l\cdot s$ potential due to this mechanism is denoted as $V_1^{ls,LS}$. The other is due to the renormalization of $v^e$ and the kinetic energy. The $l\cdot s$ potential due to this mechanism is denoted as $V_1^{ls,Ren}$ and has been shown to be by no means small by the numerical calculations.

$V_1^{ls,LS}$ consists of three components which come from the three kinds of energy of nucleons of $3N$ particle accommodated in the valence orbits around the target nucleus; the Hartree potential energy, the Fock potential energy and the mutual interaction energy of valence nucleons. As for the contribution from the mutual interaction energy among valence nucleons, we have shown that it is negligibly small in general. According to the numerical calculations, the component of $V_1^{ls,LS}$ coming from the Hartree term, denoted as $V_1^{ls,H}$, is quite near the direct $3N$ $l\cdot s$ potential $V^D_l$ in magnitude. If the force range of $v^{ls}$ is of the zero-range limit, we can show that the component of $V_1^{ls,LS}$ coming from the Fock term, denoted as $V_1^{ls,F}$, is identical to $V_1^{ls,H}$ when $v^{ls}$ has no $^3E$ component. However the
actual calculations have shown that, although the adopted $\psi^{lS}$ is quite short-ranged, $V_{ls,\rho}^{es}$ is shallower than $V_{ls,\rho}^{es,H}$ by about $1/4 \sim 1/3$ times of $V_{ls,\rho}^{es,H}$, which means that the finite range effect of $\psi^{lS}$ is important.

The origin of the renormalization $l\cdot s$ potential, $V_{ls,\rho}^{es,\text{Ren}}$, is the difference of the local momentum for $j = l+1/2$ from that for $j = l-1/2$. Since the kinetic exchange potential $t^{ex}$ and the central-interaction exchange potential $v^{ex}$ take the different values for different local momentum, the difference of $(t^{ex} + v^{ex})$ between $j = l+1/2$ and $j = l-1/2$ is renormalized effectively into the $l\cdot s$ potential. We have seen that $V_{ls,\rho}^{es,\text{Ren}}$ is attractive in the tail region and repulsive in the inner region. Due to the large attractive contribution of $V_{ls,\rho}^{es,\text{Ren}}$ in the tail region, the total $l\cdot s$ potential $V_{ls,\rho}^{es}$ becomes even three times larger than the direct $l\cdot s$ potential $V_{ls,\rho}$ in the tail region for low incident energy.

In this paper we have reported the numerical calculations only for $^3\text{He}$ case. In subsequent papers we will discuss the comparison of the $3\text{N}$ $l\cdot s$ potential with the proton $l\cdot s$ potential and also the comparison of $^3\text{He}$ $l\cdot s$ potential with triton $l\cdot s$ potential.16)

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Appendix I

--- Proof of Eq. (2.9) ---

By expressing the single-particle wave function $f_j(r)$ as

$$f_j(r) = \int dk e^{i\mathbf{k} \cdot \mathbf{r}} \phi_j(k),$$

we have

$$\langle f_i f_\lambda | g(r) \mathbf{L} | f_\lambda f_i \rangle = \int dk_1 dk_2 dk_3 dk_4 \phi_1^*(k_1) \phi_2^*(k_2) \phi_3(k_3) \phi_4(k_4)$$

$$\times \langle e^{i\mathbf{k}_1 \cdot \mathbf{r} + i\mathbf{k}_3 \cdot \mathbf{r}} g(r) \mathbf{L} | e^{i\mathbf{k}_2 \cdot \mathbf{r} + i\mathbf{k}_4 \cdot \mathbf{r}} \rangle.$$  \hfill (I.1)

The matrix element in the integrand is easily calculated to be

$$M \equiv \langle e^{i\mathbf{k}_1 \cdot \mathbf{r} + i\mathbf{k}_3 \cdot \mathbf{r}} g(r) \mathbf{L} | e^{i\mathbf{k}_2 \cdot \mathbf{r} + i\mathbf{k}_4 \cdot \mathbf{r}} \rangle$$

$$= (2\pi)^3 \delta (\mathbf{k}_1 + \mathbf{k}_2 - \mathbf{k}_3 - \mathbf{k}_4) \times \frac{1}{2} \int d\mathbf{r} \times (\mathbf{k}_3 - \mathbf{k}_4) e^{-i\mathbf{q} \cdot (\mathbf{k}_3 - \mathbf{k}_4)} g(r).$$ \hfill (I.2)

Now we use the following relation

$$\int d\mathbf{r} \times A e^{-i\mathbf{q} \cdot \mathbf{r}} g(r) = \left( i \frac{\partial}{\partial \mathbf{q}} \times A \right) \int d\mathbf{r} e^{-i\mathbf{q} \cdot \mathbf{r}} g(r)$$

$$= \left( i \frac{\partial}{\partial \mathbf{q}} \times A \right) 4\pi \int_0^\infty dr \ r^2 g(r) j_0(qr).$$
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\[ -i \left( \frac{q}{q} \times A \right) 4\pi \int_0^\infty dr \ r^3 g(r) j_1(qr) . \]  

(1.4)

Then we have

\[ M = (2\pi)^3 \delta(k_1 + k_2 - k_3 - k_4) \]

\[ \times (-2i\pi) \left( \frac{(k_1 - k_2) \times (k_3 - k_4)}{|k_1 - k_2 - k_3 + k_4|} \right) \int_0^\infty dr \ r^3 g(r) j_1 \left( \frac{|k_1 - k_2 - k_3 + k_4|}{2} , r \right) . \]

(1.5)

By the same procedure we have for the exchange matrix element

\[ \langle f_1 f_2 | g(r) L | f_3 f_4 \rangle = \int dk_1 dk_2 dk_3 dk_4 \phi_1^*(k_1) \phi_2^*(k_2) \phi_3(k_3) \phi_4(k_4) \cdot M , \]

\[ \tilde{M} = M(k_3 \leftrightarrow k_4) \]

\[ = (2\pi)^3 \delta(k_1 + k_2 - k_3 - k_4) \]

\[ \times (2i\pi) \left( \frac{(k_1 - k_2) \times (k_3 - k_4)}{|k_1 - k_2 + k_3 - k_4|} \right) \int_0^\infty dr \ r^3 g(r) j_1 \left( \frac{|k_1 - k_2 + k_3 - k_4|}{2} , r \right) . \]

(1.6)

Since \( g(r) \) is assumed to be very short-ranged, we can adopt the following approximation for \( j_1(qr) \) in the integrands of \( M \) and \( \tilde{M} \);

\[ j_1(qr) \approx \frac{1}{3} qr . \]

(1.7)

Under the approximation of Eq. (1.7) we have

\[ M \approx -\tilde{M} , \]

(1.8)

which gives us the desired relation

\[ \langle f_1 f_2 | g(r) L | f_3 f_4 \rangle \approx -\langle f_1 f_2 | g(r) L | f_3 f_4 \rangle \]

(1.9)

when \( g(r) \) is very short-ranged.

Appendix II
—— Formal Proof of Eq.(3·1) ——

By introducing two spinors \( \xi_\sigma \) and \( \zeta_\sigma \), we have

\[ \sum_{\sigma'} \xi_\sigma (-)^{1/2 - \sigma'_-} \zeta_\sigma \langle \eta_{\sigma'} S_{\mu} | \eta_{\sigma} \rangle = \frac{1}{\sqrt{2}} \sum_{\sigma'} \left( \frac{1}{2} \sigma \cdot \frac{1}{2} - \sigma'_- | \mu \right) \xi_\sigma \zeta_\sigma \]

\[ = \frac{1}{\sqrt{2}} \xi_\sigma \zeta_\sigma \]

(II·1)

Therefore, the relation of Eq. (3·1) which we are going to prove is rewritten as

\[ \sum_{\sigma'} \xi_\sigma (-)^{1/2 - \sigma'_-} \zeta_\sigma M_{E^3}^{pGCM}(\sigma, R; \sigma'_-, R') \]

\[ = \frac{4\pi}{\sqrt{3}} \left[ [\xi \cdot \zeta]_0 [y_1(R) \cdot y_1(R')]_0 K_{E^3}^{pGCM}(R, R') . \]

(II·2)

Here we have used the relation
\[(R \times R')_\mu = -i \frac{8\pi}{3\sqrt{2}} [y_i(R) \cdot y_i(R')]_{1,\mu}\]  
(II·3)

with \(y_{lm}(R)\) standing for
\[y_{lm}(R) = R^l Y_{lm}(\hat{R}).\]  
(II·4)

Now by using the partial wave expansion of \(\Gamma (r-R)\)
\[\Gamma (r-R) = \sum_{lm} H_i(r, R^2) Y_{lm}(\hat{r}) y_{lm}(R),\]
\[H_i(r, R^2) = 4\pi \left(\frac{2\gamma}{\pi}\right)^{\frac{3}{4}} e^{-\gamma(R^2+r^2)} i_l(2\gamma Rr) R^l,\]  
(II·5)

where \(i_l(x)\) is modified spherical Bessel function, we express \(M_{\text{LS}}^{\text{GCM}}(\sigma, R; \sigma', R')\) as follows:
\[M_{\text{LS}}^{\text{GCM}}(\sigma, R; \sigma', R') = \sum_{lm, l'm'} \langle H_i(r, R^2) \Phi_\sigma(P) \Phi(T) | \sum_{p \neq q} v_{l,m}^p \rangle \times \lambda \{ H_i(r, R'^2) \Phi_\sigma(P) \Phi(T) \} y_{lm}(R) y_{l'm'}(R') \]
\[= \sum_{l, l'} Q_{l,l'}(R^2, R'^2) \left( \left| m + \frac{1}{2} \right| \sigma | jM \right) \left( \left| m' + \frac{1}{2} \right| \sigma | jM \right) y_{lm}(R) y_{l'm'}(R'),\]  
(II·6)

Then we obtain
\[\sum_{\sigma, \sigma'} \xi_\sigma(-)^{1/2-\sigma'} \xi_\sigma M_{\text{LS}}^{\text{GCM}}(\sigma, R; \sigma', R') \]
\[= \sum_{l, l'} Q_{l,l'}(R^2, R'^2) y_{l}(R) \xi l = \sum_{l, l'} Q_{l,l'}(R^2, R'^2) \sqrt{2j+1} (-)^l \xi [y_{l}(R) \xi]_{l, l'} \]
\[= \sum_{l, l'} Q_{l,l'}(R^2, R'^2) \sqrt{2j+1} (-)^l \xi \cdot 3 \cdot (2j+1) \left\{ \begin{array}{ccc} l & 1/2 & j \\ 1 & 1 & 0 \end{array} \right\} \]
\[\times [y_{l}(R) y_{l}(R')]_{1, \xi l} \xi ,\]  
(II·7)

where in getting the final equality we have used the fact that \(\xi\) and \(\xi\) can appear only in the form of \([\xi \cdot \xi]_l\), since we know that \(M_{\text{GCM}}^{\text{GCM}}(\sigma, R; \sigma', R')\) contains \(\sigma\) and \(\sigma'\) only in the form of \([\eta|\eta\sigma]\).

Thus, what we need to prove is now reduced to the following relation:
\[[y_{l}(R) y_{l}(R')]_{1, \mu} = F(R^2, R'^2, R \cdot R') [y_{l}(R) y_{l}(R')]_{1, \mu}.\]  
(II·8)

For the proof of Eq. (II·8) it is enough for us to note the relation
\[[y_{l+1}(R) y_{l+1}(R')]_{1, \mu} = \alpha_{l+1} [y_{l+1}(R) y_{l+1}(R')]_{1, \mu} + \beta_{l-1} (RR')^2 [y_{l-1}(R) y_{l-1}(R')]_{1, \mu},\]  
(II·9)
which is easy to show. From Eq. (II·9) we obtain

$$\left[ y_i(R)y_i(R') \right]_{j\mu} = \left[ y_i(R)y_i(R') \right]_{j\mu} \left\{ \frac{1}{a_i} \left[ y_{i-1}(R)y_{i-1}(R') \right]_0 - \frac{\beta_{i-2}}{a_i a_{i-2}} (RR')_2 \left[ y_{i-3}(R)y_{i-3}(R') \right]_0 + \cdots \right\},$$

(II·10)

which is just of the form of Eq. (II·8).

**Appendix III**

--- Proof of Eq. (3·7) ---

By using the general formula

$$m(a, a') = \int dt e^{i/2t \cdot R} m(a-t/2, a+t/2) e^{i/2t \cdot P} \delta(a-a'),$$

$$F \equiv \frac{\partial}{\partial a},$$

we obtain

$$(a \times a') F_{LS}(a, a') = \int dt e^{i/2t \cdot P} (a \times t) F_{LS}(a-t/2, a+t/2) e^{i/2t \cdot P} \delta(a-a')$$

$$= \left(a \times \frac{\partial}{\partial F} \right) \tilde{F}_{LS}(F, a) \delta(a-a'),$$

$$\tilde{F}_{LS}(F, a) \equiv \int dt e^{i/2t \cdot P} F_{LS}(a-t/2, a+t/2) e^{i/2t \cdot P}.$$  (III·2)

Since $F_{LS}(a, a')$ is a rotationally invariant function of $a^2$, $a'^2$ and $(a, a')$, $\tilde{F}_{LS}(F, a)$ is also a rotationally invariant function of $F^2$, $a^2$ and $(a \cdot F)$. Hence we can express $\tilde{F}_{LS}(F, a)$ as follows:

$$\tilde{F}_{LS}(F, a) = \sum_{i, m} (F^2)_i^m (a \cdot F)^m B_{im}(a^2).$$  (III·3)

From this expression of $\tilde{F}_{LS}(F, a)$ we obtain

$$\left(a \times \frac{\partial}{\partial F} \right) \tilde{F}_{LS}(F, a)$$

$$= \sum_{i, m} \left\{ 2l(a \times F) (F^2)_{i-l}^m(a \cdot F)^m + a \times (F^2)_i^m \sum_{p=0}^{m-1} (a \cdot F)^p (a \cdot F)^{m-p-1} \right\} B_{im}(a^2)$$

$$= (a \times F) \sum_{i, m} 2l(F^2)_{i-l}^m (a \cdot F)^m + \sum_{p=0}^{m-1} (1+a \cdot F)^p (a \cdot F)^{m-p-1} \right\} B_{im}(a^2),$$  (III·4)

which proves Eq. (3·7).

**References**
