On oscillations of the Earth's fluid core

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Summary. In the conventional spheroidal and toroidal representation, the differential equations governing long-period free oscillations of the Earth's fluid core form an infinitely coupled system. Because of the mathematical complexity, solutions of this system are customarily attempted by numerical integration. The approach, however, necessitates a severe truncation of the coupling chain and this, in turn, renders it difficult to interpret the results. An alternative approach is sought in this work. By invoking the 'solenoidal flow' approximation which neglects the dilatation, and the 'subseismic' approximation which neglects the flow pressure respectively, we have succeeded in developing analytic solutions for the inertial and gravitational oscillations. This results in a significant simplification of the eigenvalue problem because the infinitely coupled system of differential equations is now reduced to an algebraic one. More importantly, the analytic solutions reveal immediately the roles played by the rotation and density stratification in core dynamics. We find that the inertial oscillations, which arise from the Earth's rotation, are essentially independent of the core's density stratification. Thus the fluid core can be approximated by a homogeneous, incompressible and non-gravitating inviscid Newtonian fluid with only minor modifications. The gravitational oscillations, on the other hand, are governed by both the rotation and density stratification. In a non-rotating earth, gravitational oscillations are not possible for neutrally or unstably stratified cores. Because of rotation, however, gravitational oscillations become possible for any density stratification in the core, and the eigenfrequencies are divided into alternating allowed and forbidden zones. We show that in each allowed zone, there exist an infinite number of gravitational oscillations with their eigenfrequencies approaching asymptotically the eigenfrequency of a corresponding inertial oscillation.

Both the solenoidal flow and subseismic approximations ignore the fluid flow arising from the variation in gravitational potential. To correct for this neglect, as well as taking into account the deformation in the solid part of the Earth, an asymptotic formulation is developed. It is shown that the eigenfrequencies can be determined without explicitly solving for the 'correction'
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terms. An attempt has also been made to consider the effects of a solid inner core on core oscillations. We find that for those inertial oscillations which exist in a core bounded by only the solid mantle, the presence of the interior boundary leads to modifications of the eigenfunctions but not the eigenfrequencies.

1 Introduction

In theoretical studies of normal modes of the Earth, it is customary to describe the eigenfunctions in terms of vector spherical harmonics. When a non-rotating spherical earth model is used (Alterman, Jarosch & Pekeris 1959), each spheroidal and toroidal component in this representation can be separated and the solution of the dynamic equation of equilibrium presents no major problems. Once the rotation and ellipticity are taken into account, however, the spheroidal and toroidal fields become infinitely coupled (Smith 1974; Shen & Mansinha 1976). The natural coordinates for a rotating fluid are the oblate spheroidal system which yields solutions in closed form for inertial oscillations of a rotating, incompressible, homogeneous, inviscid, Newtonian fluid bounded by a spherical rigid shell (Greenspan 1964). However, the fluid outer core of the Earth is bounded by a deformable solid mantle as well as an inner core. Moreover, the effects of self-gravitation are not entirely negligible. Since the variation in self-gravitation and the deformation in the solid Earth are most readily expressed in the spheroidal and toroidal representation, it is desirable to express solutions for the fluid core in the same form. To circumvent the problem of infinite coupling, we can consider the effects of rotation and ellipticity as perturbations to a spherical, non-rotating earth. This is feasible when the frequencies of the normal modes are much greater than the diurnal frequency. Indeed, for elastic oscillations with periods of less than a sidereal hour, adequate results can be obtained with a first-order perturbation approximation (e.g. Backus & Gilbert 1961; Pekeris, Alterman & Jarosch 1961; Dahlen 1968; Luh 1974). The approach fails, however, when long-period oscillations such as those confined mainly to the liquid outer core are to be considered. Because the eigenperiods for these oscillations are comparable or greater than the diurnal period, the full effects of rotation and in some cases, of ellipticity, must be taken into account. We are then faced with an infinitely coupled system of ordinary differential equations. Traditionally, solutions of this system are attempted by straightforward numerical integration (e.g. Smylie 1974; Crossley 1975a; Shen & Mansinha 1976; Johnson & Smylie 1977; Crossley & Rochester 1980). This approach, however, runs into the unavoidable problem of having to truncate the coupling chain severely and it is also difficult to interpret the numerical results. We therefore seek an alternative approach. It is shown that by invoking appropriate approximations, analytic solutions can be developed for free oscillation of the Earth's fluid outer core.

According to the nature of the fluid motion and the primary driving force, free oscillations of the Earth's core may be divided into two classes: the gravitational oscillations and the inertial oscillations. The gravitational oscillations are those which arise from density stratification in the core and are characterized by fluid flows in which the pressure variation due to compression is approximately balanced by that due to transport (Shen 1978; Smylie & Rochester 1981). The inertial oscillations, on the other hand, are characterized by a predominantly solenoidal flow in the fluid and result from the rotation of the Earth (Greenspan 1964). The capability of the Earth's core to support solenoidal flows was recently questioned by Smylie & Rochester (1981). Earlier, Batchelor (1967, pp. 167-171) has shown that, in general, a fluid flow may be considered solenoidal if: (a) for steady flow the flow speed is much less than the speed of sound; (b) for oscillatory flow the frequency is much less than the acoustic frequency; and (c) for stratified fluid the scale-height for
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density variation is much greater than the vertical scale of the fluid motion. Smylie & Rochester (1981) remarked that in core hydromagnetics, (c) is usually violated and for free oscillations, neither (b) nor (c) is satisfied. However, (a), (b) and (c) are the sufficient but not the necessary conditions for the validity of the solenoidal flow approximation. Fluid flows exist for which a non-vanishing pressure variation is sustained by the action of gravity so that conditions (b) and (c) are vitiated. This is the case for all inertial oscillations for which the fluid flow is essentially solenoidal. A well-known example is the tilt-over mode which leads to the ‘nearly diurnal wobble’ (Toomre 1974; Rochester, Jensen & Smylie 1974) and resonant amplification of the diurnal earth tides and nutations (Jeffreys 1948). Theories (e.g. Poincaré 1910; Jefferys & Vicente 1957a, b; Molodensky 1961; Shen 1976; Smith 1977; Sasao, Okubo & Saito 1981; Wahr 1981) have shown that the tilt-over mode is essentially a $T^1_l$ toroidal oscillation of the core relative to the mantle for which both the compressibility and self-gravitation can be treated as small perturbations. In fact, the equation of motion for inertial oscillations of the Earth’s core is identical in form to that of a non-gravitating, homogeneous and incompressible fluid. The density stratification comes into play only when the change in gravitational potential is taken into account. Thus the existence of inertial oscillations is independent of the density stratification in the liquid core, a conclusion in disagreement with those of Olson (1977) and Smylie & Rochester (1981). Results of the latter, however, have been derived under the ‘subseismic approximation’ (named by Smylie & Rochester 1981) which neglects the flow pressure altogether. This approximation is not appropriate for the inertial oscillations.

It is interesting to point out that inertial oscillations of the fluid core have no equivalent under the Boussinesq approximation which is commonly assumed to hold in ocean/atmosphere fluid dynamics. In the Boussinesq approximation, the density fluctuation is considered to arise from temperature effects, and since the coefficient of thermal expansion for most fluid is small, density perturbation is ignored in the equation of motion in all but the buoyancy term. The buoyancy term, then, is proportional to the component of velocity in the direction of the gravitational field (e.g. Friedlander & Siegmann 1982a, eqn 2.17), and provided that this velocity component does not vanish, plays a part in the free oscillation. The Earth’s fluid core, on the other hand, is self-gravitating and compressible, and is believed to have a nearly adiabatic temperature gradient. Because the pressure and temperature scale heights are of comparable magnitude, the pressure effects on the variation in density, in particular that due to the transport of fluid across the ambient pressure gradient, cannot be neglected. The equation of motion thus takes on a somewhat different form (eqn 2.1). Instead of being associated with the velocity in the direction of gravity, the buoyancy term is now proportional to the dilatation. Consequently, for inertial oscillations which have vanishing dilatation, the buoyancy force plays no part and the fluid core behaves like a rotating, incompressible and homogeneous fluid.

Detailed and analytic studies of inertial oscillations for an incompressible and non-gravitating fluid, bounded by a rigid shell, have already been given by Greenspan (1964, 1965, 1968) and Kudlick (1966). As remarked earlier, these solutions are usually given in oblate spheroidal coordinates. In Section 4, we develop them in the spheroidal and toroidal representation so as to include the corrections due to the variation in self-gravitation and the deformation of the bounding surfaces. It will be shown that in this representation, the solutions remain in closed form. The correction terms are dealt with in Section 6, using a formulation similar to those of Molodensky (1961) and Shen (1976). The spheroidal and toroidal representation reveals an interesting property of the inertial oscillations which has previously not been observed. This is the identical vanishing of the $S^m_k$ spheroidal displacement field for those inertial oscillations, which are designated $Y^l_{km}$ (Section 4.1). An
attempt is also made to develop solutions for a fluid outer core which is bounded by both a solid mantle and a solid inner core. Stewartson & Rickard (1969) handled this problem by expanding the equation of motion in powers of $h/r_b$, where $h$ is the thickness of the outer core and $r_b$ the radius of the core–mantle boundary. They found the equations for $h > 0$ singular at the 'inertial latitudes' and attributed the result to the presence of the interior boundary. This implies that the solutions with a solid inner core are significantly different from those without an inner core. The singularities, however, were later shown by Miles (1974) to be consequences of the limiting process invoked in deriving Laplace's tidal equations. We show that for those inertial oscillations which exist without a solid inner core, the presence of an interior solid surface will modify only the eigenfunctions but not the eigenfrequencies.

The argument of Smylie & Rochester (1981) for non-solenoidal flow is valid for gravitational oscillations of the outer core in which density stratification is a principal governing factor. For these gravity waves or core undertones, the subseismic approximation can be applied and the resulting equation of motion, called the subseismic equation (Smylie & Rochester 1981), is a second-order partial differential equation in a single scalar variable. This reduction from the original fourth-order partial differential, vector equation permits solutions to be given in analytical form, as we shall demonstrate. Similar approximations have been made in meteorological studies and are called the 'anelastic' (Ogura & Phillips 1962; Gough 1969) or the 'deep convection' (Dutton 1976, p. 494) approximation. In contrast, the solenoidal flow approximation is called the 'shallow convection' (Dutton 1976, p. 494) approximation. The argument of Smylie & Rochester (1981) for non-solenoidal flow is valid for gravitational oscillations of the outer core in which density stratification is a principal governing factor. For these gravity waves or core undertones, the subseismic approximation can be applied and the resulting equation of motion, called the subseismic equation (Smylie & Rochester 1981), is a second-order partial differential equation in a single scalar variable. This reduction from the original fourth-order partial differential, vector equation permits solutions to be given in analytical form, as we shall demonstrate. Similar approximations have been made in meteorological studies and are called the 'anelastic' (Ogura & Phillips 1962; Gough 1969) or the 'deep convection' (Dutton 1976, p. 494) approximation. In contrast, the solenoidal flow approximation is called the 'shallow convection' (Dutton 1976, p. 494) approximation. The validity of the subseismic approximation was first demonstrated by Pekeris & Accad (1972) for gravitational oscillations of the fluid core in a non-rotating earth, using explicit solutions, and by Smylie & Rochester (1981) for a rotating earth, using scaling analysis. In an 'asymptotic theory', Pekeris & Accad deduced analytic solutions for the subseismic equation. A 'correction' term was then added to account for the modifications arising from the change in gravitational potential, which has been neglected in developing the subseismic solution. Unfortunately, in subsequent studies of core undertones with rotation, this analytic approach is usually abandoned. We will demonstrate in Section 5 that, despite the Coriolis coupling, the subseismic equation can still be solved analytically for some core models. It is not necessary to neglect the horizontal component of the Earth's angular velocity (Olson 1977), known in meteorology as the 'traditional approximation' (Eckhart 1960). Also, in Section 6, it will be shown that the correction term can be incorporated through a formulation identical to that for the inertial oscillations.

Some interesting effects of the rotation on gravity waves are immediately observable from the analytic solutions. In a non-rotating earth, gravitational oscillations are possible only for a stably stratified fluid core (Pekeris & Accad 1972). With rotation, however, gravitational oscillations become possible for any density stratification in the outer core, and the eigenfrequencies are divided into alternating allowed and forbidden zones (Section 5). Furthermore, in each allowed zone, there exist an infinite number of such normal modes with their eigenfrequencies approaching asymptotically the eigenfrequency of a corresponding inertial oscillation. The analytic solutions also allow us to explain why the 'transverse inertia' (Smylie 1974; Crossley 1975a) can be used to determine the distribution of allowed and forbidden zones.

We point out that any gravitational (or any buoyancy related) oscillation of the Earth's core must be non-solenoidal (and consequently has a non-vanishing displacement in the direction of gravity), because dilatation provides the only link to the buoyancy force. This would appear to suggest that such oscillations may have characteristics significantly different from those obtained under the Boussinesq approximation. However, with the subseismic
approximation, the equation of motion for the fluid core (eqn 2.11) becomes formally identical to that for a Boussinesq fluid. Consequently, some general properties of the eigenvalues and eigenfunctions given for a Boussinesq fluid (Friedlander & Seigmann 1982a, b) may be applicable to gravitational oscillations of the fluid core. For example, the critical surface across which the subseismic equation changes type (Smylie & Rochester 1981, eqn 68), is described by the same equation for the critical surface across which the pressure equations for a Boussinesq fluid changes type (Friedlander & Seigmann 1982b, eqn 2.9). The general properties of eigenvalues and eigenfunctions, obtained under the subseismic approximation, will be taken up in a separate study. The present work is confined to the development of analytic solutions which will facilitate the computation of eigenvalues.

2 The equations governing core oscillations

We consider here an earth model that is everywhere in hydrostatic equilibrium, has an isotropic, linearly elastic constitutive relation, and rotates about a fixed axis at a constant angular velocity \( \Omega \). If we choose a coordinate system that is rotating with the earth so that \( \mathbf{B} = \mathbf{Q} \mathbf{i} \) where \( \mathbf{i} \) is the unit vector in the \( z \)-direction, the linearized Lagrangian equation governing infinitesimal oscillations of the inviscid fluid outer core is (Shen 1976)

\[
\frac{\partial^2 \mathbf{u}}{\partial t^2} + 2 \mathbf{Q} \times \frac{\partial \mathbf{u}}{\partial t} = -\nabla \Psi - \beta \Delta g
\]  

where

\[
\Psi = -W_a - \frac{\lambda}{\rho} \Delta - g \cdot \mathbf{u}.
\]  

\( \rho, \lambda \) and \( g \) are the undisturbed density, incompressibility and gravity respectively, \( \mathbf{u} \) is the displacement, \( \Delta = \nabla \cdot \mathbf{u} \) the dilatation, and \( W_a \) the change in gravitational potential.

\( W_a \) satisfies the Poisson equation:

\[
\nabla^2 W_a = 4\pi G \nabla \cdot (\rho \mathbf{u})
\]  

and the function \( \Psi \) can be written as

\[
\Psi = -W_a + \frac{1}{\rho} P_e
\]  

where \( P_e \) is the Eulerian pressure variation given by

\[
P_e = -\lambda \Delta - \rho g \cdot \mathbf{u}.
\]  

The first term in the right hand side of (2.5) is the pressure variation due to compression/dilatation and the second due to fluid transport along the unperturbed pressure gradient. For convenience, we shall write

\[
\eta = g \cdot \mathbf{u}.
\]  

Notice that \( -\eta/g \) is the displacement normal to the equipotential surface. \( \beta(r) \) is a stability parameter introduced by Pekeris & Accad (1972) to describe the density profile:

\[
\nabla \rho = (1 - \beta) \frac{\rho^2}{\lambda} g
\]  

and is related to the Brunt-Väisälä frequency $N(r)$ by

$$N^2(r) = -\beta \frac{\rho g^2}{\lambda}.$$  \hfill (2.8)

It is important to point out here that both the stability factor $\beta$ and the gravity $g$, which bring in the effects of ellipticity, appear in equation (2.1) only as multipliers of the dilatation $\Delta$. Thus under the solenoidal flow approximation:

$$\Delta = 0,$$  \hfill (2.9)

the fluid would behave as if the outer core were spherically symmetric with uniform density. Whether the fluid is actually stratified or not, then, has little importance. Equation (2.1) in this case, is separable from (2.3) and is identical in form to that for an incompressible and non-gravitating inviscid fluid. Exact solutions for inertial solutions of such a fluid, bounded by a rigid, spherical shell, were given by Greenspan (1964). For a real Earth, however, the inertial oscillations are not exactly solenoidal because the change in self-gravitation $\mathbf{W}$, as well as the deformation in the solid part of the Earth, must be considered. An attempt to take these modifications into account was made by Shen (1976). The theory, however, is valid only for a particular group of the inertial oscillations, called the $\mathbf{S}_q + 1 \mathbf{T}_q$ toroidal oscillations. In Section 4, we will develop eigenfunctions for inertial oscillations of a fluid bounded not only by an exterior surface but also by an interior one. The modifications due to self-gravitation and solid earth elasticity are considered in Section 6.

The inertial oscillations of the outer core are characterized by fluid flows in which the pressure variation due to dilatation/compression is negligible in comparison with that due to the action of gravity. For gravitational oscillations, neither the pressure variation due to compression/dilatation nor that due to fluid transport in the gravity field is negligible. However, the two components balance each other so that the subseismic approximation:

$$\frac{\lambda}{\rho} \frac{\partial P}{\partial \rho} = -\eta$$  \hfill (2.10)

is valid and the dynamic pressure variation $P_d$ can be considered negligibly small in the determination of density variation (Smylie & Rochester 1981). By using (2.8) and (2.10), the equation of motion (2.1) now becomes

$$\frac{\partial^2 \mathbf{u}}{\partial t^2} + 2\mathbf{\Omega} \times \frac{\partial \mathbf{u}}{\partial t} = -\nabla \Psi - N^2(\hat{\mathbf{g}} \cdot \mathbf{u}) \hat{\mathbf{g}}$$  \hfill (2.11)

where $\hat{\mathbf{g}}$ is the unit vector in the direction of $\mathbf{g}$. This is identical in form to the conservation of momentum equation (Friedlander & Siegmann 1982b, eqn 2.2) for a rotating, stratified, Boussinesq fluid, as we have already pointed out earlier. However, it should be noted again that the second fluid equation, namely the conservation of mass for a Boussinesq fluid, gives zero divergence of the velocity vector; this result is not reproduced for the subseismic approximation of core dynamics.

Equation (2.11), with the help of (2.10), can be solved without using the Poisson equation (2.3). However, because the dilatation is non-vanishing, both the density stratification and ellipticity appear explicitly through the term $N^2(\hat{\mathbf{g}} \cdot \mathbf{u}) \hat{\mathbf{g}}$. Even when the core is everywhere neutrally stratified so that $N^2(r) = 0$, the ellipticity will enter the solution through (2.10). Fortunately, the gravitational oscillations are principally governed by the density stratification and rotation (Crossley & Rochester 1980). The ellipticity is likely to contribute only a small perturbing effect. Thus spherical symmetry will be assumed when we
develop analytical solutions in Section 5 for the subseismic equation. Under this approximation, the gravity $g$ becomes $-\hat{g} \hat{r}$ where $\hat{r}$ is the unit vector in the radial direction. Also, $X'$, which is the derivative of any function $X(r, \theta, \phi)$ along the normal to the equipotential surface, becomes $\partial X/\partial r$ and will be abbreviated $\dot{X}$. The convention of using $\dot{X}$ to express $\partial X(r, \theta, \phi)/\partial r$ or $dX(r)/dr$ will be adopted throughout this work.

Both the solenoidal flow and subseismic approximation permit the reduction of equation (2.1) to a second-order partial differential equation in a single scalar variable. This is the key simplification which makes possible the development of analytic solutions. The customary procedure is to eliminate the displacement $u$ and express the reduced equation in terms of the function $\Psi$ (Greenspan 1964; Smylie & Rochester 1981; Friedlander & Siegmann 1982a, b). In this work, however, our main interest is in the computation of eigenvalues. For this purpose, it proves more convenient to choose, $u_r$, the radial component of $u$, as the scalar variable because the solution can then be immediately used in the boundary conditions. We will not derive the reduced second-order partial differential equation here. Instead, a system of second-order ordinary differential equations, which is equivalent to the single partial differential equation, will be derived in Section 3, following the conversion of (2.1) and (2.3) into scalar equations through the spherical harmonic expansion.

3 The scalar equations

In the spheroidal and toroidal representation, the scalar equivalent of equations (2.1) for a small oscillation with a purely real angular frequency $\omega$ are (Shen 1976)

$$\Delta_n = \dot{U}_n + \frac{2}{r} U_n - \frac{n(n+1)}{r} V_n,$$

$$\Psi_n - \beta g \Delta_n = \frac{(n-1)(n-m)}{2n-1} 2\Omega \omega T_{n-1} + \omega^2 U_n - 2m \Omega \omega V_n - \frac{(n+2)(n+1+m)}{2n+3} 2\Omega \omega T_{n+1},$$

$$\frac{1}{r} \Psi_n = \frac{(n-1)(n-m)}{n(2n-1)} 2\Omega \omega T_{n-1} - \frac{2m \Omega \omega}{n(n+1)} U_n + \left(\frac{\omega^2 - \frac{2m \Omega \omega}{n(n+1)}}{2\Omega \omega} \right) V_n + \frac{(n+2)(n+1+m)}{(n+1)(2n+3)} 2\Omega \omega T_{n+1},$$

$$0 = \frac{n-m}{n(2n-1)} \left[ -U_{n-1} + (n-1) V_{n-1} \right] + \left[ \frac{\omega}{2\Omega} - \frac{m}{(n+1)} \right] T_n + \frac{n+1+m}{(n+1)(2n+3)} \left[ U_{n+1} + (n+2) V_{n+1} \right],$$

and the equivalent of (2.3) is

$$\dot{H}_n + \frac{2}{r} H_n - \frac{n(n+1)}{r^2} H_n = 4\pi G \left[ \rho \Delta_n - \frac{\rho'}{g} \eta_n \right].$$

Here $U_n(r), V_n(r)$ and $T_n(r)$ are the radial coefficients of the radial, transverse and toroidal displacements respectively. $\Delta_n(r)$ is the radial coefficient of the dilatation $\Delta$ and $H_n(r)$ that of the change in gravitational potential $W_\eta$. In the $y$-notation of Alterman et al. (1959), $U_n = y_1, \lambda \Delta_n = y_2, V_n = y_3, H_n = y_5$ and $T_n = y_7$. $\eta_n(r)$ is the radial coefficient of $\eta$ and is
given by
\[ \eta_n = -g U_n - e g \left[ \frac{2m(n-m)}{2n-1} T_{n-1} + \frac{2m(n+1+m)}{2n+3} T_{n+1} \right]. \]  

Equation (2.2) is not needed for the solution of (2.1) under either the solenoidal flow or the subseismic approximation. However, it is needed in the boundary conditions. The scalar equivalent of this equation is
\[ \Psi_n = -H_n - \frac{\lambda}{\rho} \Delta_n. \]  

We note that small terms of the order \( e^2 g \Delta_n \) and \( e r (d/dr)(\lambda/\rho) \Delta_n \), where \( e \) is the ellipticity, have been omitted from equations (3.2)-(3.4) and (3.7). These terms are neglected in the present treatment because for inertial oscillations, they involve the product of two small quantities \( e \) and \( \Delta \), while for gravitational oscillations, we are considering only spherical earth models. We choose to retain the ellipticity terms in (3.6). For some inertial oscillations, such as the tilt-over mode, they lead to first-order modifications of the eigenfrequencies (e.g. Shen 1976).

The radial functions \( U_n \) and \( V_n \) uniquely define a spheroidal vector displacement field \( \mathbf{S}_n \), while \( T_n \) a toroidal field \( \mathbf{T}_n \). The superscript \( m \), which specifies the order of a spherical harmonic, is not given explicitly for the radial coefficients. This is not likely to cause ambiguity because for the earth model under consideration, no coupling exists between displacement fields of different order (Smith 1974). In general, the displacement \( u \) for any core oscillation is an infinite coupling chain of spheroidal and toroidal terms, either of the form
\[ u = S^m_m + T^m_{m+1} + S^m_{m+2} + \ldots \]  

or
\[ u = T^m_m + S^m_{m+1} + T^m_{m+2} + \ldots \]  

The infinite coupling is, of course, the result of rotation which destroys the spherical symmetry. However, for inertial oscillations, the coupling chain is finite provided that the fluid core is not bounded from within (Section 4.1).

We have pointed out earlier that both the solenoidal flow and subseismic approximations allow the reduction of (2.1) to a second-order partial differential equation in a single scalar variable. We shall now derive the system of ordinary differential equations which are equivalent to this reduced equation. To do this, we observe that the quantities which are involved in these approximations are the dilatation \( \Delta \) and the displacement normal to the equipotential surface, which, for a spherical earth, is the radial displacement. We therefore eliminate \( V_n \), \( T_n \), and \( \Psi_n \) from (3.1) to (3.4). The resulting infinitely coupled, second-order ordinary differential equations are
\[
\frac{(A_n + B_n)}{n+1} \left( \bar{U}_n + \frac{4}{r} \frac{d}{dr} \bar{U}_n - \frac{n^2 + n - 2}{r^2} U_n \right) - \frac{2n+1}{r^2} \frac{\rho g}{\rho} \Delta_n \\
\frac{A_n}{n+1} \frac{r^{-n-3}}{dr} \left( r^{n+3} \Delta_n \right) - \frac{B_n}{n} \frac{r^{-n-2}}{dr} \left( r^{-n+2} \Delta_n \right) \\
\frac{C_n}{n+2} \frac{r^{-n-3}}{dr} \left[ \frac{1}{r} \frac{d}{dr} \left( r^{n+4} U_{n+2} \right) - r^{n+3} \Delta_{n+2} \right] \\
+ \frac{D_n}{n-1} \frac{r^{-n-2}}{dr} \left[ \frac{1}{r} \frac{d}{dr} \left( r^{n+3} U_{n-2} \right) - r^{-n+2} \Delta_{n-2} \right],
\]  

\( 3.10 \)
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where the degree $n$ either follows the sequence $m, m + 2, m + 4, \ldots$, or the sequence $m + 1, m + 3, m + 5, \ldots$, and the parameters $A_n, B_n, C_n$ and $D_n$ are given by

\begin{align*}
A_n &= \omega^2 + \frac{2m\omega}{n+1} - \frac{(n+2)(n+1-m)(n+1+m)}{(n+1)^2(2n+3)} \frac{4\Omega^2\omega}{\omega - 2m\Omega/[(n+1)(n+2)]}, \\
B_n &= \omega^2 - \frac{2m\omega}{n} - \frac{(n-1)(n-m)(n+m)}{n^2(2n-1)} \frac{4\Omega^2\omega}{\omega - 2m\Omega/[(n+1)(n+2)]}, \\
C_n &= \frac{(2n+1)(n+1+m)(n+2+m)}{(n+1)(2n+3)(2n+5)} \frac{4\Omega^2\omega}{\omega - 2m\Omega/[(n+1)(n+2)]}, \\
D_n &= \frac{(2n+1)(n-1-m)(n-m)}{n(2n-3)(2n-1)} \frac{4\Omega^2\omega}{\omega - 2m\Omega/[(n-1)n]},
\end{align*}

In the following two sections, the system (3.10) will be solved under the approximations (2.9) and (2.10). Once $U_n$ is determined, the other radial functions $V_n, T_n$ and $\Psi_n$ can be obtained from (3.1)-(3.4) by straightforward substitution. Notice, however, that the Poisson equation (3.5) has not been used in deriving this solution. To take into account the change in gravitational potential, and the corresponding modification in the displacement, an asymptotic formulation is presented in Section 6.

4 The solenoidal flow approximation

Under the solenoidal flow approximation $\Delta = 0$, equation (3.10) becomes

\begin{align*}
\left( \frac{A_n}{n+1} + B_n \right) \left( \dot{U}_n + \frac{4}{r} \frac{n^2+n-2}{r^2} U_n \right) &= \frac{C_n}{n+2} r^{-n-3} \frac{d}{dr} \left[ \frac{1}{r} \frac{d}{dr} (r^{n+4} U_{n+2}) \right] \\
&\quad + \frac{D_n}{n-1} r^{-n-2} \frac{d}{dr} \left[ \frac{1}{r} \frac{d}{dr} (r^{-n+3} U_{n-2}) \right].
\end{align*}

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The solution of equation (2.1) for this core model was given by Greenspan (1964). For each pair of integers $(k, m)$ where $k > m$, the pressure variation $\Psi$ is

\begin{equation}
\Psi = \Psi_{km} = P_k^m(\xi/c) P_k^m(\xi) \cos(\omega t + m\phi)
\end{equation}

where $P_k^m$ is an associate Legendre function of degree $k$ and order $m$, $|\omega| < 2\Omega$ and $c^2 = 4\Omega^2/(4\Omega^2 - \omega^2)$. $\xi$ and $\xi$ are the modified ‘oblate-spheroidal’ coordinates defined by

\begin{equation}
\xi = \frac{c |\omega|}{2\Omega} - \frac{r}{r_0} \cos \theta
\end{equation}
and
\[(c^2 - \xi^2) (1 - \xi^2) = \frac{r^2}{r_b^2} \sin^2 \theta \tag{4.4}\]
with \(r_b\) the radius of the core-mantle boundary. The eigenfrequency \(\omega = 2\Omega \chi\) is determined by
\[mP^m_k(\chi) - (1 - \chi^2) \frac{d}{d\chi} P^m_k(\chi) = 0. \tag{4.5}\]

In this work, an eigensolution of the system (2.1) and (2.3) will be written as \(Y = (\omega, u, W, \Psi)\). Recall that \(\omega\) is the eigenfrequency, \(u\) the eigendisplacement and \(W\) the change in gravitational potential, while \(\Psi\) is defined by (2.2). With this convention, the above solution can be designated \(Y^1_{km}\), where the superscript I indicates that it is an inertial mode. The subscripts \(k\) and \(m\) reflect two interesting properties of the \(Y^1_{km}\) inertial oscillations.

Property 1: on the rigid core-mantle boundary, the function \(\xi\), which is the Eulerian pressure variation when we neglect self-gravitation, behaves exactly like a solid spherical harmonic of degree \(k\) and order \(m\).

Property 2: the \(S^m_k\) spheroidal component of the displacement vanishes identically throughout the fluid core.

The validity of property 1 was demonstrated by Kudlick (1966). Property 2 will become self-evident when the eigenfunctions are developed in the spheroidal and toroidal representation. It is sufficient to give only the expression for \(U_n\) because the other radial coefficients \(V_n, T_n, \text{ and } \xi_n\) are readily obtained by substituting \(U_n\) into (3.1)-(3.4).

The solution of (4.1) for \(Y^1_{km}\) inertial oscillation is
\[r U_n = \sum_{j=0}^{(k-n)/2} \frac{(k-n)!}{j! n! (j-n)!} r^{n+2j} \frac{\Gamma(n+2j)}{\Gamma(n+2)} (j+n+1) (j+n+3) r^{n+2j} \tag{4.6}\]

For any other \(n\), \(U_n\) vanishes identically. A consequence of this is that for any \(n > k\), the functions \(V_n, T_n\) and \(\xi_n\) also vanish identically so that the eigenfunction has a closed form. In (4.6), the left subscript \('r'\) for \(U_n\) indicates that the solution is regular at the origin. The constants \(E^j_k\) are not all independent. For each \(j = k, k-2, \ldots, m\), they satisfy the recursion relation
\[\left( \frac{A_n}{n+1} + \frac{B_n}{n} \right) [j(j+1) - n(n+1)] E^j_k = C_n \frac{(j+n+1)(j+n+3)}{n+2} r E^j_{n+2} + \frac{D_n}{n-1} (j-n)(j-n+2) r E^j_{n-2}, \quad n = j-2, j-4, \ldots, m \tag{4.7}\]
so that exactly \(L = \text{INT}[(k-m)/2]\) free constants remain. These \(L\) free constants, together with the eigenfrequency, are determined by the boundary conditions at \(r = r_b\). For a fluid core bounded by a spherical and rigid mantle, the conditions are \(U_n(r_b) = 0\) for all \(n\). The validity of property 2 now becomes clear because for the \(Y^1_{km}\) inertial oscillation, \(r U_k = r E^k_k r^{k-1}\). For \(r U_k\) to vanish at \(r_b\), the constant \(r E^k_k\) must be set to zero. Since \(V_k = (1/k) U_k\), this means that the \(S^m_k\) spheroidal term in the displacement vanishes throughout the fluid core. A consequence of this property is that the eigenfrequencies of \(Y^1_{km}\) inertial oscillations can be determined by setting \(r E^k_k\) to zero, provided that \(r E^k_k\) is given in terms of \(r E^m_m\) (or \(r E^m_{m+1}\)) through the recursion relation (4.7). For non-trivial solutions to exist, \(r E^m_m\) (or \(r E^m_{m+1}\)) must not vanish.
It is important to point out that the validity of property 2 depends on the sphericity of the core–mantle boundary. If the boundary has an ellipticity, the boundary conditions would be \( \eta_n(r_b) = 0 \) and it is no longer necessary to set the constant \( r E_k^n \) to zero. Property 1, however, holds for a fluid core bounded by an ellipsoidal mantle (Kudlick 1966).

4.2 \( Y_{km}^1 \) INTERIAL OSCILLATIONS WITH A SOLID INNER CORE

When the fluid outer core is also bounded from within by a solid inner core with radius \( r_c > 0 \), the solution of (4.1) takes the form of

\[
U_n = r U_n + s U_n = \sum_{j=0}^{\infty} \sum_{j=0}^{(n-m)/2} s E_n^{n+2j} r^{n+2j-1} + \sum_{j=0}^{(n-m)/2} s E_n^{n-2j} r^{n-2j-2}.
\]

\( n = m, m + 2, \ldots, \infty \), or \( n = m + 1, m + 3, \ldots, \infty \). (4.8)

Here, the second summation is over terms which arise from the presence of an interior boundary, and the constants \( s E^j_n \) are related by:

\[
\left( \frac{A_n}{n+1} + \frac{B_n}{n} \right) [j(j+1) - n(n+1)] s E^j_n = \frac{C_n}{n+2} (j-n-2)(j-n) s E^{j+2}_n
\]

\[+ \frac{D_n}{n-1} (j+n-1)(j+n+1) s E^{j-2}_n, \quad n = j, j+2, \ldots, \infty. \] (4.9)

The eigenfrequencies can, in principle, be determined by substituting (4.8) into the boundary conditions at \( r = r_b \) and \( r = r_c \). However, because of the infinite coupling, a practical solution is impossible except for one special case which we will consider in the following.

We consider a special case of (4.8) in which the constants \( r E_n^{n+2j} \) are identically zero for \( n > k \) or \( 2j > (k-n) \), and the constants \( s E_n^{n-2j} \) are identically zero if \( n = k \). The regular part of the \( U_n \), then, becomes identical to \( \mu U_n \) given by eqn (4.6). Thus, we are, in effect, considering the solution which will reduce to that for the \( Y_{km}^1 \) inertial oscillation if the solid inner core is removed. The extra condition imposed on the constants \( s E^j_n \) will not render the solution trivial because the recursion relation (4.9) is an identity for \( n = j \). Notice that the solution is not in a closed form. However, the eigenfrequency can be determined by a finite number of boundary conditions: \( U_n(r_b) = U_n(r_c) = 0 \), for \( n-k, k-2, \ldots, m \). In fact, the eigenfrequency is again determined by setting the constant \( r E_k^n \) to zero because \( s E_k^{k-1} = 0 \) and the radial function \( U_k \) consists of the single regular term \( r E_k^n \). Thus, provided that the boundaries are spherical, the eigenfrequency is not affected by the presence of a solid inner core and the \( S_k^n \) spheroidal component of the displacement \( u \) vanishes identically (property 2). A well-known example is the \( Y_{2,1}^1 \) inertial oscillation, commonly referred to as the nearly diurnal wobble if the earth is ellipticai (Toomre 1974; Shen 1976). The eigenfrequency is modified by the ellipticity but not by the presence of a solid inner core.

4.3 THE EFFECTS OF TRUNCATION

The solutions given in the previous two sections show that, in general, the eigendisplacement for a \( Y_{km}^1 \) inertial oscillation consists of an infinite number of spheroidal and toroidal terms.
Even when the solid inner core is not present, the number of terms is \((k-m+1)\). In the customary numerical analysis, however, the coupling chain is usually arbitrarily truncated. We shall now examine the effects of truncation by considering the following frequently used three-term displacement:

\[
u = T_{k-1}^m + S_k^m + T_{k+1}^m.
\]

This representation is often interpreted as to require the other spheroidal and toroidal terms to vanish identically. An examination of the system of equations (3.1)–(3.4), however, shows that the truncation, in effect, requires only the following two conditions to be satisfied:

\[
0 = \left[ \frac{\omega}{2\Omega} - \frac{m}{(k-1)k} \right] T_{k-1} + \frac{k+m}{k(2k+1)} [U_k + (k+1) V_k]
\]

and

\[
0 = \left[ \frac{\omega}{2\Omega} - \frac{m}{(k+1)(k+2)} \right] T_{k+1} + \frac{k+1-m}{(k+1)(2k+1)} (-U_k + k V_k).
\]

These two conditions allow equations (3.1)–(3.4) for \(n = k\) to form a closed system so that explicit solutions become possible. The expression for \(U_k\) is

\[
U_k = -r B_k^k \frac{k^2}{k+m} \frac{\omega - 2m\Omega/[k(k-1)k]}{2\Omega} r^{k-1} + r B_k^k \frac{(k+1)^2}{k+1-m} \frac{\omega - 2m\Omega/[(k+1)(k+2)]}{2\Omega} r^{-k-2}.
\]

If we assume the mantle–outer core boundary and the inner core–outer core boundary to be rigid and spherical, two eigenfrequencies result:

\[
\omega_1 = \frac{2m\Omega}{[k(k-1)]}
\]

and

\[
\omega_2 = \frac{2m\Omega}{[(k+1)(k+2)]}.
\]

These are, of course, the eigenfrequencies of toroidal oscillations discussed by Shen (1976). Numerical values for \(k = 5\) and \(m = 1\) have also been found by Smith (1977). When \(k = m + 1\), the eigenvalue \(\omega_1\) is identical to that of the \(Y_{km}\) inertial oscillation discussed in Sections 4.1 and 4.2, and, if a solid inner core is not present, the 3-term representation of the eigen-displacement becomes a correct one. The eigenvalue \(\omega_2\), however, is the artefact of the truncation.

5 The subseismic approximation

Under the subseismic approximation which, for a spherical earth, is \(\lambda / \rho \Delta_n = g U_n\), equation (3.10) becomes

\[
\left( \frac{A_n}{n+1} + \frac{B_n}{n} \right) \left\{ \dot{z}_n + \frac{1}{r} \dot{r}_n + \left[ \alpha_n^2 - \frac{(n+\frac{3}{2})^2}{r^2} \right] z_n + \delta z_n \right\} = \left( \frac{C_n}{n+2} - n - \gamma_2 \right) \left\{ \frac{d}{dr} \left[ \frac{1}{r} \frac{d}{dr} \left( r^{n+5/2} z_{n+2} \right) \right] + \delta z_{n+2} \right\} + \left( \frac{D_n}{n-1} - n - \gamma_2 \right) \left\{ \frac{d}{dr} \left[ \frac{1}{r} \frac{d}{dr} \left( r^{-n+3/2} z_{n-2} \right) \right] + \delta z_{n-2} \right\}.
\]
Here, we have replaced the radial function $U_n(r)$ with $Z_n(r)$ defined by

$$z_n(r) = r^{\frac{3}{2}} \exp \left[ -\rho(r) \right] U_n(r)$$  \hspace{1cm} (5.2)

where

$$\rho(r) = \frac{1}{2} \int_{r_c}^{r} \frac{\rho(r') g(r')}{\lambda(r')} \, dr'.$$

The parameters $\alpha_n^2$ and $\delta$ are given by

$$\alpha_n^2(r) = \frac{2n + 1}{A_n/(n + 1) + B_n/n} \frac{N^2(r)}{r^2} - \frac{(n + \frac{3}{2}) A_n/(n + 1) - (n - \frac{3}{2}) B_n/n}{A_n/(n + 1) + B_n/n} \frac{\rho g}{\lambda r},$$

and

$$\delta(r) = \frac{1}{2} \frac{\rho g}{\lambda r} - \frac{1}{2} \frac{d}{dr} \left( \frac{\rho g}{\lambda} \right) - \frac{1}{4} \left( \frac{\rho g}{\lambda} \right)^2.$$

For a realistic outer core of the Earth, $\delta \ll r^{-1}$. Thus the terms involving $\delta$ can be neglected. $\alpha_n^2$, however, is usually a function of $r$ so that equation (5.1) must, in general, be solved by numerical integration. To avoid this difficulty, we consider a simple core model for which both $N$ and $\rho g/\lambda$ are linear in $r$. The parameter $\alpha_n^2$ is then independent of $r$ and the solution of (5.1) is given by

$$z_n = \sum_{j=1}^{\infty} r E^j_n J_{n+\frac{3}{2}}(q_j r) + \sum_{j=1}^{\infty} s E^j_n J_{n-\frac{3}{2}}(q_j r),$$

where $J_{n+\frac{3}{2}}$ and $J_{n-\frac{3}{2}}$ are Bessel functions. Notice that $J_{n-\frac{3}{2}}(q_j r)$ is singular at the origin and must be omitted from the solution if a solid inner core does not exist. The constants $r E^j_n$ and $s E^j_n$ both satisfy the same recursion relation:

$$\left( \frac{A_n}{n+1} + \frac{B_n}{n} \right) (\alpha_n^2 - q_j^2) E^j_n = \frac{C_n}{n+2} q_j^2 E^{j+2}_n + \frac{D_n}{n-1} q_j^2 E^{j-2}_n, \quad j = 1, 2, \ldots, \infty.$$  \hspace{1cm} (5.5)

For $n = m, m+2, \ldots$, or $n = m+1, m+3, \ldots$, equation (5.5) leads to an infinite system of homogeneous linear equations to be solved for $r E^j_n$ or $s E^j_n$. If non-trivial solutions are to exist, the determinant of the coefficient matrix must vanish and $q_j^2$ are the roots. The problem of solving the infinitely coupled ordinary differential equations (5.1) is now reduced to one of solving an infinite order algebraic equation. The practical procedure for handling this problem is to truncate the infinite coupling chain, say at $n = k$, and determine the eigenfrequencies from the boundary conditions. Then the coupling chain is extended successively to $n = k+2, k+4, \ldots$, until the eigenfrequencies converge to stable values.

To examine the properties of such solutions, let us consider the following truncated coupling chain

$$u = (T^m_{m-1}) + S^m_M + T^m_{M+1} + \ldots + S^m_I + T^m_{I+1}.$$  \hspace{1cm} (5.6)
Here, \( M = m \), or \( m + 1 \) and \( S_m^m \) is the spheroidal term with the lowest possible degree. The solutions obtained under this truncation will be designated the \( Y_{lm}^G \) gravitational oscillations. But it must be borne in mind that the true solutions are those for which \( l = \infty \).

Define \( L \) as the largest integer smaller or equal to \((l-M)/2 + 1\). Then the coupling chain (5.6) contains \( L \) spheroidal terms and the corresponding system of linear equations (5.5) has a coefficient matrix whose determinant is an \( L \) degree polynomial in \( q^2 \). We shall write the determinant as

\[
\det(q^2) = \sum_{l=0}^{L} (-1)^l f_l^l q^2(L - l)
\]

where each coefficient \( f_l^l \) is a function of \( l, m \) and \( \omega/\Omega \) alone. It can be shown by deduction that the \( L \) roots of \( \det(q^2) = 0 \) are real and distinct. Each \( q_j \) is, therefore, either purely real or purely imaginary. When \( q_j \) is real, the Bessel function \( J_{n+\frac{1}{2}}(q_j r) \) and \( J_{n-\frac{1}{2}}(q_j r) \) oscillate between positive and negative values and each possesses an infinite number of zeros. On the other hand, when \( q_j \) is imaginary, \( J_{n+\frac{1}{2}}(q_j r) \) and \( J_{n-\frac{1}{2}}(q_j r) \) are monotonic functions of \( r \). Clearly, gravitational oscillations are possible only when at least one of the \( L \) roots \( q_j^2 \) is positive. Thus by studying the zeros and poles of \( q_j^2 \), we can determine the frequency ranges for which the gravitational oscillations are allowed. At present, no simple formula has been found for the determination of the zeros of \( q_j^2 \). The poles, however, can be determined easily. This follows from the fact that the condition

\[
f_l^l\left(l, m, \frac{\omega}{\Omega}\right) = 0
\]

is identical to equation (4.5) with \( k = l + 2 \). In other words, the poles of the roots \( q_j^2 \) for \( Y_{lm}^G \) gravitational oscillations are the eigenfrequencies for \( Y_{l+2,m}^I \) inertial oscillations.

5.1 THE \( Y_{2,2}^G \) GRAVITATIONAL OSCILLATIONS

To exemplify the above results, we consider the \( Y_{2,2}^G \) gravitational oscillations, obtained under the truncated coupling chain \( u = S_2^2 + T_3^3 \). The equation \( \det(q^2) = 0 \), in this case, has only one root, given by

\[
q^2 = \alpha_2^2 = \frac{14(3\omega - \Omega)}{\omega(7\omega^2 - 7\Omega\omega - 2\Omega^2)} - \frac{35}{2} \frac{\omega^2 + 7\Omega\omega - 6\Omega^2}{\omega^2 - 7\Omega\omega - 2\Omega^2} \frac{\rho g}{\lambda r}.
\]

The reason for choosing this particular truncation is that it has been used extensively (e.g. Crossley 1975a, b; Shen & Mansinha 1976; Crossley & Rochester 1980) in numerical studies of core oscillations. Let us now determine the allowed and forbidden zones for \( Y_{2,2}^G \) gravitational oscillations by examining the zeros and poles of \( q^2 \). The necessary condition for these oscillations to exist is \( \alpha_2^2 > 0 \). The roots of \( 7\omega^2 - 7\Omega\omega - 2\Omega^2 = 0 \) are \( \omega/\Omega = 1.2319 \) and \(-0.2319\). The roots of \( \omega^2 + 7\Omega\omega - 6\Omega^2 = 0 \) are \( \omega/\Omega = 0.7720 \) and \(-7.7720 \). Thus for a neutrally stratified core with \( N^2 = 0 \), \( Y_{2,2}^G \) gravitational oscillations are allowed if \( 1.2319 > \omega/\Omega > 0.7720 \) or \(-0.2319 > \omega/\Omega > -7.7720 \), and forbidden otherwise. When \( N^2 \neq 0 \), the sign of \( \alpha_2^2 \) depends on the relative magnitude of the two terms in the right hand side of (5.9). For simplicity, let us consider a strongly stratified core such that \( |N^2/\Omega^2| \gg \).
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The core oscillations with frequencies $|\omega| < 2\Omega$, have generally been considered as inertial waves modified by buoyancy (Crossley 1975a; Olson 1977; Smylie & Rochester 1981). According to the present theory, however, it would seem more appropriate to identify any solutions, obtained under the subseismic approximation, as gravitational oscillations.

It is interesting to point out that for $Y_{2,2}^{G}$ gravitational oscillations, the 'transverse inertia' of a fluid element (Smylie 1974; Crossley 1975a) is proportional to

$$
\frac{\omega(7\omega^2 - 7\Omega\omega - 2\Omega^2)}{14(3\omega - \Omega)} \rho r V_2.
$$

(5.10)

The role of transverse inertia in determining the spectra of core oscillations has been the subject of some controversy (Smith 1976; Crossley 1976). But when we compare (5.10) with (5.9), we see that the coefficient of $\rho r V_2$ in (5.10) is exactly the inverse of the coefficient of $N^2/\rho^2$ in the right hand side of (5.9). This is why the sign of the transverse inertia can be used to determine the disturbance of allowed and forbidden zones.

Another interesting result can also be obtained from an inspection of (5.9) and (5.10). $\omega = 0$ is the eigenfrequency of the geostrophic mode (e.g. Greenspan 1965), and the roots of $7\omega^2 - 7\Omega\omega - 2\Omega^2 = 0$ are the eigenfrequencies of $Y_{4,2}$ inertial oscillations. As $\omega$ approaches one of these frequencies, the transverse inertia tends to zero, the parameter $\alpha^2$ tends to infinity, and the Bessel functions $J_{5/2}(\alpha r)$ and $J_{-5/2}(\alpha r)$ pass through an infinite number of zeros. This means that there exists, in each allowed zone, an infinite number of $Y_{4,2}^{G}$ gravitational oscillations with the eigenfrequencies approaching asymptotically that of a $Y_{4,2}$ inertial oscillation.

Crossley & Rochester (1980) called the gravitational oscillations, found in the allowed zone $\omega/\Omega > 1.2319$ when the core is stably stratified, the 'regular' undertones. On the other hand, the gravitational oscillations, which exist in the allowed zone $1.2319 > \omega/\Omega > 1/3$ for an unstably stratified core, were called 'irregular' undertones. The eigenfrequencies of the regular undertones decrease with the undertone number, while those of the irregular ones increase with the undertone number. If we follow this convention, then the gravitational oscillations, which exist in the allowed zones $1/3 > \omega/\Omega > 0$ and $-0.2319 > \omega/\Omega$ for a stable core, are all regular undertones. For an unstably stratified core, however, half of the gravitational oscillations in the allowed zone $0 > \omega/\Omega > -0.2319$ are regular undertones, while the other half are irregular. This is because the parameter $\alpha^2$ (given by eqn 5.9, but with the second term in the right hand side neglected) remains positive in this zone and approaches infinity asymptotically as $\omega/\Omega$ tends to either 0 or $-0.2319$.

The qualitative aspect of the above results for $Y_{2,2}^{G}$ gravitational oscillations can be applied to the coupling chain (5.6) of any length. In general, as $l$ goes to infinity, there will be an infinite number of alternating allowed and forbidden zones for $Y_{lm}^{G}$ gravitational oscillations. Crossley (1975a), interpreting the transverse inertia, has shown that as $l$ goes to infinity, the first zone, which is allowed for stably stratified cores and forbidden for unstably stratified cores, has a frequency lower bound of $\omega/\Omega = 2$. This frequency, of course, is the upper bound for all inertial oscillations (Greenspan 1964). In actual computation of the eigenfrequencies, $l$ cannot be set to infinity, and we must consider the question: For what value of $l$ is the truncated coupling chain (5.6) a satisfactory approximation? Numerical work, which is unavoidable for an answer to this question, will be attempted in the future.
5.2 GRAVITATIONAL OSCILLATIONS IN A NON-ROTATING EARTH

The solutions developed for the coupling chain (5.6) can also be applied to gravitational oscillations in a non-rotating earth. Because there is no Coriolis coupling, we now have purely spheroidal oscillations. When the displacement is an $S_k^m$ spheroidal field, the root $q^2$ is given by

$$q^2 = \alpha_k^2 = \frac{k(k+1)}{\omega^2} \frac{N^2}{r^2} - \frac{1}{2} \frac{\rho g}{\lambda r},$$

and the radial eigenfunction $U_k(r)$ by

$$U_k = r^{-3/2} e^p \left[ rE_k J_{k+\nu} (qr) + s E_k J_{-k-\nu} (qr) \right].$$

Since the second term in the right hand side of (5.11) is always negative, $\alpha_k^2$ can be positive only when the core is stably stratified. This reflects the fact that, in a non-rotating earth, gravitational oscillations are not possible for neutrally or unstably stratified cores (Pekeris & Accad 1972). Let us now consider a stable core model without a solid inner core, and restrict ourselves to those gravitational oscillations with very low frequencies. As $\omega$ tends to zero, $q^2$ tends to infinity. Then, except near the origin, the asymptotic form of the Bessel functions can be used and we find

$$U_k = rE_k \left( \frac{2}{\pi q} \right)^{\nu^2} r^{-2} e^p \sin (qr - k\pi/2).$$

This is essentially the expression derived by Pekeris & Accad (1972). The boundary condition requires $U_k$ to vanish at the core–mantle boundary. This means that $qr_b - k\pi/2 = l\pi$, or that the $l$th ($l > 1$) eigenfrequency is given by

$$\omega_l = \frac{\sqrt{k(k+1)}}{l+k/2} \frac{N}{\pi}.$$

6 The asymptotic formulation

In Sections 4 and 5, we developed analytic solutions for the inertial and gravitational oscillations by invoking respectively the 'solenoidal flow' and the 'subseismic' approximation. In doing so, we neglected the change in gravitational potential and the secondary deformation induced by this change. Also, the boundary conditions have been treated under the assumption that the mantle and inner core are rigid. Clearly, a 'correction' term must be added to the analytic solution, if the effects of self-gravitation and solid earth elasticity are to be included. In what follows, the analytic solution will be right-superscripted by the character 'A' and the correction term by 'C'. A true eigensolution $Y = (\omega, u, W, \Psi)$ for the outer core, then, is given by $Y = Y^A + Y^C$. The purpose of this section is to develop an asymptotic theory which will allow us to determine the eigenfrequencies $\omega$ without explicitly solving for the correction term. This is accomplished by eliminating $Y^C$ from the continuity conditions at the outer core–mantle boundary and the inner core–outer core boundary. The procedure has previously been used by Molodensky (1961) and Shen (1976). We will right-superscript the eigensolution for the solid mantle or inner core by the character 'S'.

It is important to point out here that a solution of the system (3.1)–(3.5) for the outer core contains four free constants for each spheroidal term in the displacement. This is the
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number required by the boundary conditions. Because of the solenoidal flow approximation (2.9) or the subseismic approximation (2.10), however, the analytic solution contains only two free constants. The other two, and two only, must come from the 'correction' term \( Y^C \). But \( Y^C \), without any constraints, would have four free constants for each spheroidal term. Thus, two constraints for each spheroidal term must be imposed on \( Y^C \). This will be done when we formulate the continuity condition for the displacement normal to the boundary of the outer core.

6.1 CONTINUITY OF THE CHANGE IN GRAVITATIONAL POTENTIAL

The radial coefficient \( H_n \) for the change in gravitational potential \( W_\sigma \) can be written as

\[
H_n = H_n^A + H_n^C
\]

where \( H_n^A \) is the analytic term defined by (3.7):

\[
H_n^A = -\frac{\Psi_n^A}{\rho} - \frac{\lambda}{\rho} \Delta_n^A.
\]

\( H_n^A \), however, is not the solution of the Poisson equation (3.5). We define, instead, two quantities \( H_n^F \) and \( h_n \) such that

\[
H_n = H_n^F + h_n.
\]

Here \( H_n^F \) is the 'general solution' of

\[
\dot{H}_n^F + \frac{2}{r} H_n^F + \left[ -\frac{4\pi G \rho'}{g} - \frac{n(n+1)}{r^2} \right] H_n^F = \frac{4\pi G \rho'}{g} \Psi_n^A + 4\pi G \rho \beta \Delta_n^A,
\]

and \( h_n \) the 'particular solution' of

\[
\dot{h}_n + \frac{2}{r} h_n + \left[ -\frac{4\pi G \rho'}{g} - \frac{n(n+1)}{r^2} \right] h_n = \frac{4\pi G \rho'}{g} \Psi_n^C + 4\pi G \rho \beta \Delta_n^C.
\]

In the asymptotic formulation, we assume that \( \Psi_n^C \) and \( h_n \) are negligible as compared to \( \psi_n^A \) and \( H_n^F \) respectively. The continuity condition for \( H_n \) therefore reads

\[
H_n^F = H_n^S.
\]

Notice that \( H_n^F \) may deviate significantly from \( H_n^A \). This gives rise to a correction term \( u^C \) in the displacement. Customarily, \( u^C \) is neglected (e.g. Olson 1977). In the asymptotic theory, \( u^C \) is eliminated, but not neglected, from the boundary conditions.

6.2 CONTINUITY OF THE CHANGE IN NORMAL GRAVITATIONAL FLUX DENSITY

The change in gravitational flux density normal to the boundary is \( \dot{H}_n + 4\pi G \rho \eta_n/g \). Here, \(- \eta_n/g\) is the displacement normal to the equipotential surface. Since \( \eta_n \) is continuous across the boundary, we can replace \( \eta_n \) for the outer core with that for the solid earth. The continuity condition, after \( h_n \) is neglected, reads

\[
\dot{H}_n^F = \dot{H}_n^S + 4\pi G (\rho_c - \rho_s) \frac{\eta_n^S}{g},
\]

where \( \rho_c \) is the density of the outer core at the boundary, and \( \rho_s \) that of the mantle or inner core.
6.3 Continuity of the Transverse Stress

The transverse stress is given by \( \mu \left[ \dot{V}_n - (1/r)V_n + (1/r)U_n \right] \) where \( \mu \) is the rigidity. This quantity must vanish in the outer core and the continuity condition reads

\[
0 = \dot{V}_n \frac{1}{r} - V_n \frac{1}{r} + \frac{1}{r} U_n .
\]  

\( \text{(6.5)} \)

6.4 Continuity of the Normal Stress

The normal stress is given by \( \lambda \Delta_n + 2 \mu \dot{U}_n \). In the outer core, the second term vanishes. But the dilatation \( \Delta_n \) is given by \( \Delta_n = \Delta_n^A + \Delta_n^C \). To eliminate the correction term \( \Delta_n^C \) from the boundary condition, we use equation (3.7). By replacing \( H_n \) and \( \eta_n \) with the corresponding quantities in the solid earth, the continuity condition reads

\[
- \rho_c \Psi_n^A = \rho_c (H_n^S + \eta_n^S) + (\lambda_s \Delta_n^S + 2 \mu_s \dot{U}_n^S).
\]  

\( \text{(6.6)} \)

where \( \lambda_s \) is the Lamé parameter and \( \mu_s \) the rigidity of the mantle or inner core.

6.5 Continuity of the Normal Displacement

This is the boundary condition in which the effects of self-gravitation and solid earth elasticity will be shown explicitly. The displacement normal to the boundary is \(-\eta_n/g \). In order to eliminate the correction term from this expression, we make use of equations (3.1) and (3.5) which can be rewritten as

\[
n \rho r^{n+1} [U_n + (n+1)V_n] = \frac{d}{dr} \left[ -r^{n+2} \frac{\rho \eta_n}{g} - \frac{1}{4\pi G} r^{2n+2} \frac{d}{dr} (r^{-n} H_n) \right]
\]  

\( \text{(6.7)} \)

and

\[
(n+1) \rho r^{-n} (-U_n + nV_n) = \frac{d}{dr} \left[ -r^{-n+1} \frac{\rho \eta_n}{g} - \frac{1}{4\pi G} r^{-2n} \frac{d}{dr} (r^{n+1} H_n) \right].
\]  

\( \text{(6.8)} \)

Integrating (6.7) and (6.8) over the fluid outer core and imposing the following two constraints on the ‘correction’ term \( \Psi^C \):

\[
\int_{r_c}^{r_b} \rho r^{n+1} [U_n^C + (n+1)V_n^C] \, dr = 0,
\]  

\( \text{(6.9)} \)

and

\[
\int_{r_c}^{r_b} \rho r^{-n} (-U_n^C + nV_n^C) \, dr = 0,
\]  

\( \text{(6.10)} \)

we get

\[
n \int_{r_c}^{r_b} \rho_c r^{n+1} [U_n^A + (n+1)V_n^A] \, dr
\]

\[
+ \frac{1}{4\pi G} \left[ r^{2n+2} \frac{d}{dr} (r^{-n} H_n^F) \right] \bigg|_{r_c}^{r_b} = - \left( r^{n+2} \frac{\rho_c \eta_n^S}{g} \right) \bigg|_{r_c}^{r_b},
\]  

\( \text{(6.11)} \)
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These are the desired continuity conditions for the normal displacement. The constraints (6.9) and (6.10), as we have mentioned earlier, are necessary for the solution to be unique. They are equivalent to re-defining the arbitrary constants contained in the analytic $Y^A$ solution.

Equations (6.3)-(6.6) applied at $r_b$ and $r_c$, together with (6.11) and (6.12), form exactly 10 boundary conditions for each spheroidal term in the displacement. Note that if $r_c = 0$, the continuity condition to be omitted for the normal displacement is (6.12).

Finally, equations (6.11) and (6.12) can be re-written, through simple manipulations, as

\[
(n + 1) \int_{r_c}^{r_b} \rho_c r^{-n} (-U_n^A + n V_n^A) dr + \frac{1}{4\pi G} \left[ \frac{r^{-2n}}{dr} \left( r^{n+1} H_n^F \right) \right]_{r_c}^{r_b} = - \left( r^{-n+1} \frac{\rho_c \eta_n^S}{g} \right) \left[ \right]_{r_c}^{r_b}.
\] (6.12)

where $H_n^D$ is the solution of the following homogeneous equation

\[
H_n^D + \frac{2}{r} H_n^D + \left[ -\frac{4\pi G\rho'}{g} - \frac{n(n + 1)}{r^2} \right] H_n^D = 0.
\]

The first terms in the right hand sides of (6.13) and (6.14) represent the deformation arising from the change in gravitational potential, while the second terms represent the deformation due to solid earth elasticity. In Olson’s (1977) treatment, the first terms are neglected. If we also neglect the second terms and assume the boundaries to be spherical, the two continuity conditions reduce to $U_n^A(r_b) = U_n^A(r_c) = 0$. These are precisely the boundary conditions which were used in Sections 4 and 5 to determine the eigenfrequencies. Notice that in those sections, the solution involves only algebraic operations. But if the effects of self-gravitation and elasticity of the solid earth are to be included, numerical integration would be required to evaluate $H_n^F$ for the outer core and the solution $Y^S$ for the solid earth. For each $n$, $H_n^F$ is determined from (6.1), a second-order ordinary differential equation. It is assumed that the effects of both the rotation and the ellipticity on $Y^S$ can be neglected (Shen & Mansinha 1976). Then, each spheroidal term in $Y^S$ is determined by a system of sixth-order ordinary differential equations.

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