It is shown on the basis of BRS algebra and renormalization group that gluons are confined when the number of flavors does not exceed nine. We exploit the analysis of the gluon propagator by Oehme and Zimmermann, but we give an interpretation of the results different from theirs.

§ 1. Introduction

In a preceding paper, to be referred to as I, we have studied the condition for color confinement based on BRS algebra without reference to the asymptotic conditions for the quark and gluon fields. In that paper we have postulated the asymptotic completeness only for BRS singlet operators in conformity with the unitarity condition of the $S$ matrix expressed in terms of BRS singlet states alone.

In a series of papers Oehme and Zimmermann have analyzed the structure of the gluon propagator in detail by making use of asymptotic freedom and analyticity. Furthermore, they have introduced an ingenuous idea of analyzing the asymptotic behavior of the projected transverse gluon propagator to find a restriction imposed on the number of flavors.

It is the purpose of the present paper to reexamine their consistency condition in the light of the confinement condition obtained in I and deduce the condition for gluon confinement. It turns out that their consistency condition is a necessary condition for unconfined gluons, so that gluons are confined when their condition is violated. Thus we are led to the conclusion that gluons are confined when the number of quark flavors does not exceed nine.

The present paper is organized as follows: In our approach the concept of confinement is closely related to formation of bound states of a pair of Faddeev-Popov, ghosts, so that color confinement is never realized in perturbation theory. On the other hand, when fundamental particles, such as quarks and gluons, are not confined and belong to the BRS singlet representation, they are represented by the asymptotic fundamental fields that can be realized in perturbation theory. Thus, crudely speaking, unconfined fundamental particles are realized in the perturbative phase, whereas confined fundamental particles are realized only in the non-perturbative phase. On the basis of this distinction we shall formulate the condition for unconfined gluons in § 2.

Then we introduce the renormalization group equation in § 3 in order to exploit the structure analysis of the gluon propagator by Oehme and Zimmermann. The projected gluon propagator plays an essential rôle in their theory, so that we shall explicitly show in § 4 that it satisfies the same renormalization group equation as the ordinary gluon propagator when the projection is defined with reference to the BRS singlet subspace of positive-definite metric.
Finally in § 5 an argument is presented to show that gluons are confined when the number of flavors does not exceed nine.

§ 2. The condition for unconfined gluons

In I we have postulated the asymptotic completeness in the sense that BRS singlet Heisenberg operators can be expressed as sums of normal products of BRS singlet asymptotic fields. This does not exclude the possibility of defining asymptotic fields of certain BRS doublet operators. Indeed, we can find such examples as is clear from the following identities:

\[
\langle A_{\mu}^a(x), B^b(y) \rangle = -\delta_{ab}\partial_{\mu}D_F(x-y),
\]

\[
\langle (D_{\mu}c)^a(x), \bar{c}^b(y) \rangle = i\delta_{ab}\partial_{\mu}D_F(x-y),
\]

where we have adopted the same notation as in I. These identities show that various Heisenberg operators generate massless single particle states when they are applied to the vacuum state. Correspondingly, we have introduced massless scalar asymptotic fields as:

\[
A_\mu \rightarrow \partial_\mu \chi, \quad B \rightarrow \beta, \quad D_{\mu}c \rightarrow \partial_\mu \gamma, \quad \bar{c} \rightarrow \bar{\gamma},
\]

without specifying what happens to the vector part of the vector fields \(A_\mu\) and \(D_{\mu}c\). Similarly, we may introduce the following asymptotic fields:

\[
\bar{B} \rightarrow \bar{\beta}, \quad D_{\mu}\bar{c} \rightarrow \partial_\mu \bar{\Gamma}, \quad c \rightarrow \Gamma,
\]

where \(\bar{B}\) is defined by

\[
B + \bar{B} - ig(c \times \bar{c}) = 0.
\]

It should be recalled that all these asymptotic fields are BRS doublet operators as seen from their BRS transformation:

\[
\delta \chi = \gamma, \quad \delta \bar{\gamma} = i\beta, \quad \delta \gamma = 0, \quad \delta \beta = 0,
\]

\[
\delta \bar{\chi} = \bar{\Gamma}, \quad \delta \Gamma = i\bar{\beta}, \quad \delta \bar{\Gamma} = 0, \quad \delta \beta = 0.
\]

Then the condition for color confinement is given by

\[
\delta \delta \chi = 0.
\]

This is a sufficient condition for all the colored particles to be confined.

In discussing gluon confinement it is convenient to decompose the gluon field \(A_\mu\) into BRS singlet and doublet parts,

\[
A_\mu = A_{\mu s} + A_{\mu d}.
\]

Subject to later refinements, we expect from the condition (2.7)

\[
A_{\mu s} = 0 \quad \text{or} \quad A_{\mu}|\psi \in U_d,
\]

and gluons are confined. Thus the condition for unconfined gluons is given by

\[
A_{\mu s} \neq 0
\]
(2.11)

\[ \langle 0 | A_\mu^a(x) P_s A_\nu^a(y) | 0 \rangle \neq 0, \]

where \( P_s \) is an abbreviation of \( P(\mathcal{U}_s) \), the projection operator to the BRS singlet subspace \( \mathcal{U}_s \) of positive-definite metric. Because of the asymptotic completeness postulated for BRS singlet operators, \( A_{\mu s} \) can be expressed as a sum of normal products of BRS singlet asymptotic fields, and the first linear term of this series is given by the asymptotic field of the unconfined gluon. Since we are interested only in the transverse part of the unconfined gluon, we shall consider, instead of \( A_\mu \) itself, the following operator \( \text{à la Oehme and Zimmermann:}^{2,3} \)

\[ A_\mu^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a. \tag{2.12} \]

Then we introduce two kinds of propagators defined by

\[ \langle 0 | T[A_\mu^a(x), A_\nu^b(y)] | 0 \rangle = \delta_{ab} G_{\mu\nu\sigma}(x-y), \]
\[ \langle 0 | T[A_\mu^a(x) P_s A_\nu^b(y)] | 0 \rangle = \delta_{ab} G_{s\mu\nu\sigma}(x-y), \tag{2.13} \]

where we have assumed the unbroken color symmetry.

We also introduce their Fourier transforms:

\[ G_{\mu\nu\sigma}(x) = \frac{-i}{(2\pi)^4} \int d^4 k e^{i k \cdot x} G_{\mu\nu\sigma}(k), \tag{2.14} \]

and a similar expression for \( G_{s\mu\nu\sigma}(x) \).

When the gluon remains massless, however, we should be aware of the fact that the BRS singlet subspace \( \mathcal{U}_s \) cannot be defined in a Lorentz-invariant manner. It depends, for instance, on the choice of a special direction for the time-axis. The corresponding unit vector will be denoted by \( n \), so that we have \( n^2 = -1 \).

The transverse propagator \( D(k) \) is defined by

\[ G_{\mu\nu\sigma}(k) = (k_\mu k_\nu \delta_{\sigma\rho} - k_\mu k_\rho \delta_{\nu\sigma} - k_\rho k_\nu \delta_{\mu\sigma} + k_\rho k_\sigma \delta_{\nu\mu}) D(k), \tag{2.15} \]

and it depends on \( k \) only through the Lorentz scalar \( k^2 \). On the other hand, \( G_{s\mu\nu\sigma}^+ \) depends not only on \( k \) but also on \( n \), so that it is decomposed as

\[ G_{s\mu\nu\sigma}^+(k) = (k_\mu k_\nu \delta_{\sigma\rho} - k_\mu k_\rho \delta_{\nu\sigma} - k_\rho k_\nu \delta_{\mu\sigma} + k_\rho k_\sigma \delta_{\nu\mu}) D^+(k) \]
\[ + (k_\rho n_\nu - k_\nu n_\rho)(k_\sigma n_\mu - k_\mu n_\sigma) D_1^+(k). \tag{2.16} \]

Both \( D^+(k) \) and \( D_1^+(k) \) are functions of \( k^2 \) and \( k^2 = k^2 + (n \cdot k)^2 \).

The function \( D(k) \) is normalized subject to the condition

\[ k^2 D(k) = 1 \tag{2.17} \]

at the space-like point

\[ k^2 = k^2 - k_0^2 = k^2 - (n \cdot k)^2 = \mu^2 > 0, \tag{2.18} \]

where we have adopted the Pauli metric. Then \( D \) depends on \( k^2, \mu^2 \) and \( g \), whereas \( D^+ \) depends on \( k^2, \mu^2 \) and \( g \). We shall now introduce dimensionless expressions \( R \) and \( R^+ \) defined by
\[ \begin{align*}
R(k, \mu, g) &= k^2 D(k, \mu, g) = R\left( \frac{k^2}{\mu^2}, g \right), \\
R^+(k, \mu, g) &= k^2 D^+(k, \mu, g) = R^+\left( \frac{k^2}{\mu^2}, \frac{k^2}{\mu^2}, g \right).
\end{align*} \tag{2.19-20} \]

Then the normalization condition (2.17) reduces to
\[ R(1, g) = 1. \tag{2.21} \]

Due to infrared singularities, a branch cut may start from \( k^2 = 0 \) in \( R \), and this is the reason why we have chosen the off-shell point \( k^2 = \mu^2 \) as the normalization point. In fact, the electron propagator under a similar condition is represented, in the neighborhood of the mass shell, by\(^6\)
\[ S_F(p) \sim \frac{1}{ip \cdot \gamma + m} \left( 1 + \frac{p^2}{m^2} \right)^{-\left( 3 - \sigma \right)\pi^2/8}. \tag{2.22} \]

The function \( R^+ \) survives only when gluons are not confined as is clear from the definition (2.13). In the lowest order perturbation theory, we have
\[ R^+\left( \frac{k^2}{\mu^2}, \frac{k^2}{\mu^2}, 0 \right) = 1. \tag{2.23} \]

By generalizing this result to the unconfirmed gluon phase, we assume
\[ \lim_{g \to 0} R^+(1, 0, g) = c > 0, \tag{2.24} \]
where we have avoided the mass shell by choosing \( k^2 = \mu^2 \), and have reduced the number of independent variables by putting \( k^2 = 0 \). The positivity of \( c \) follows from Eqs. (4.13) and (4.14) as we see later.

On the basis of this assumption we shall study the condition for gluon confinement in the following sections. We are aware of the fact that this condition is essentially equivalent to the consistency condition postulated by Oehme and Zimmermann in their normalization of the projected transverse gluon propagator.

\section*{§ 3. Renormalization group}

In their analysis of the asymptotic behavior of the gluon propagator Oehme and Zimmermann\(^2,3\) fully exploited the method of renormalization group (RG),\(^6-9\) so that we shall recapitulate the essence of their arguments.

The generator of the RG is given by the following differential operator in the Landau gauge:
\[ \mathcal{D} = \mu \frac{\partial}{\partial \mu} + \beta(g) \frac{\partial}{\partial g}. \tag{3.1} \]

The invariant charge \( Q(k, \mu, g) \) is dimensionless and is defined by the RG equation,
\[ \mathcal{D} Q = 0 \tag{3.2} \]
with the initial condition
In quantum chromodynamics (QCD) based on the color $SU(3)$ group the $\beta$ function can be expressed as a power series in the coupling constant $g$ as

$$\beta(g) = g^3(\beta_0 + \beta_1 g^2 + \cdots).$$

The first two coefficients are given in a covariant gauge by

$$\beta_0 = -\frac{1}{16\pi^2} \left(11 - \frac{2}{3} N_f\right)$$

and

$$\beta_1 = \frac{1}{2} \frac{1}{(8\pi^2)^2} \left(\frac{19}{3} N_f - 51\right),$$

respectively. $N_f$ denotes the number of quark flavors.

Since $Q$ is dimensionless we may express it as

$$Q(k, \mu, g) = Q\left(\frac{k^2}{\mu^2}, g\right),$$

and asymptotic freedom leads to

$$\lim_{|k^2| \to \infty} \ln \frac{k^2}{\mu^2} \cdot Q^2\left(\frac{k^2}{\mu^2}, g\right) = |\beta_0|^{-1}.\tag{3.8}$$

The $D$ function in (2.15) satisfies the RG equation,

$$[\mathcal{D} + 2\gamma_V(g)]D(k, \mu, g) = 0,$$\tag{3.9}

where the anomalous dimension of the gluon field $\gamma_V$ can be expanded in powers of $g^2$ as

$$\gamma_V(g) = g^2(\gamma_0 + \gamma_1 g^2 + \cdots),$$\tag{3.10}

where $\gamma_0$ is given in the Landau gauge by

$$\gamma_0 = -\frac{1}{16\pi^2} \left(13 - \frac{4}{3} N_f\right).$$\tag{3.11}

The $R$ function satisfies the same equation as $D$,

$$[\mathcal{D} + 2\gamma_V(g)]R\left(\frac{k^2}{\mu^2}, g\right) = 0.$$

Oehme and Zimmermann have expressed the solution of this equation in the following form:\textsuperscript{23}

$$R\left(\frac{k^2}{\mu^2}, g\right) = R(1, Q)\exp\left[2\int_0^{g'} dg' \gamma_V(g')\right].$$\tag{3.13}

This form is valid for sufficiently large $k^2$ and small $Q$ and $g$ provided that the theory is asymptotically free, namely, $\beta_0 < 0$. The renormalization condition (2.24) implies that the first factor $R(1, Q)$ is equal to unity.

In order to find the asymptotic behavior of $R$ for $k^2 \to \infty$, we have to be aware of the singularity at $g' = 0$ of the integrand in the exponent of Eq. (3.13). For small values of
\( g', \gamma g/\beta \) is inversely proportional to \( g' \), and we shall separate this singularity in \( \gamma g/\beta \) as

\[
\gamma g'(g')/\beta(g') = \gamma_0/\beta_0 g' + \tau(g'),
\]

(3.14)

Then we find

\[
\exp \left[ 2 \int_0^q dg' \frac{\gamma g'}{\beta(g')} \right] = \left( \frac{Q^2}{g^2} \right)^{\gamma_0/\beta_0} \exp \left[ 2 \int_0^q dg' \tau(g') \right],
\]

(3.15)

where the exponential on the r.h.s. converges to a finite limit for \( k^2 \to \infty \) and hence for \( Q \to 0 \).

Thus the asymptotic form of \( R \) is given by

\[
R \left( \frac{k^2}{\mu^2}, g \right) \sim C_{\gamma} \left( \ln \frac{k^2}{\mu^2} \right)^{\gamma_0/\beta_0} \text{ for } k^2 \to \infty ,
\]

(3.16)

where

\[
C_{\gamma} = (g^2|\beta_0|)^{-\gamma_0/\beta_0} \exp \left[ 2 \int_0^q dg' \tau(g') \right] > 0 .
\]

(3.17)

We also have

\[
D(k, \mu, g) \sim C_{\gamma} k^{-2} \left( \ln \frac{k^2}{\mu^2} \right)^{\gamma_0/\beta_0} \text{ for } k^2 \to \infty .
\]

(3.18)

Hence we may write down the unsubtracted Källén-Lehmann representation for \( D \),

\[
D(k, \mu, g) = \int_0^\infty dm^2 \rho(m^2) \frac{k^2 + m^2 - i\varepsilon}{k^2 + m^2 - i\varepsilon} .
\]

(3.19)

Then Oehme and Zimmermann argue that Eqs. (3.16) and (3.18) are valid generally for \( |k^2| \to \infty \) in the cut \( k^2 \) plane by analytic continuation. The spectral function \( \rho \) is obtained as the absorptive part of \( D \) for time-like \( k \),

\[
\pi \rho = \im D = k^{-2} \im R ,
\]

(3.20)

We quote the asymptotic form of \( \rho \) obtained by them,

\[
\rho \sim -\frac{\gamma_0}{\beta_0} \cdot C_{\gamma} |k^2|^{-1} \left( \ln \frac{|k^2|}{\mu^2} \right)^{-\gamma_0/\beta_0 - 1}
\]

for \( k^2 \to -\infty \). This is a very significant result. An important observation is concerned with the sign of \( -\gamma_0/\beta_0 \). When it is negative \( (N_f \leq 9) \), not only \( D \) but also \( R \) satisfies the unsubtracted Källén-Lehmann representation, and consequently we obtain the supercovergence relation for \( \rho \),

\[
\int_0^\infty dm^2 \rho(m^2) = 0 . \quad (N_f \leq 9)
\]

(3.21)

(3.22)

Since the gauge fields in a covariant gauge is quantized with indefinite metric, \( \rho \) is not positive-definite, and the relation above is satisfied by non-vanishing \( \rho \).

The next subject is concerned with the extention of the above arguments to \( D^+ \) and \( R^+ \).
§ 4. The projected gluon propagator

In this section we shall study the RG equation for the projected transverse gluon propagators $D^+$ and $R^+$. First, we consider the following expression:

\begin{equation}
\langle 0 | A_\mu^a(x) P_5 A_\nu^b(y) | 0 \rangle \ .
\end{equation}

(4·1)

Because of the asymptotic completeness of the BRS singlet subspace postulated in the present paper, any state in this subspace can be constructed by applying BRS singlet asymptotic fields to the vacuum state. For instance, we consider a state

\begin{equation}
\Phi^{a_1,\ldots,a_k}_{a_1,\ldots,a_k} \in U_5,
\end{equation}

(4·2)

then we have, with the help of the reduction formula, the relation,

\begin{equation}
\langle 0 | A_\mu^a(x) | a_1,\ldots,a_k, \text{in} \rangle = (-i)^k \int d^4 u_1 \cdots d^4 u_k f_{a_1}(u_1) \cdots f_{a_k}(u_k) \times \bigl[K_{u_1} \cdots K_{u_k} \langle 0 | T[A_\mu^a(x) \phi(u_1) \cdots \phi(u_k)] | 0 \rangle \bigr] ,
\end{equation}

(4·3)

in a familiar notation. For an illustrative purpose, we have assumed that this state is formed by scalar particles alone. Then, taking the complex conjugate of this relation we find a similar relation,

\begin{equation}
\langle a_1,\ldots,a_k, \text{in} | A_\nu^b(y) | 0 \rangle = i^k \int d^4 v_1 \cdots d^4 v_k f_{a_1}^*(v_1) \cdots f_{a_k}^*(v_k) \times \bigl[K_{v_1} \cdots K_{v_k} \langle 0 | \tilde{T}[A_\nu^b(y) \phi^+(v_1) \cdots \phi^+(v_k)] | 0 \rangle \bigr] ,
\end{equation}

(4·4)

where \( \tilde{T} \) denotes the antichronological time-ordering operator. Thus, summing up over all the states of the BRS singlet subspace, we find

\begin{equation}
\langle 0 | A_\mu^a(x) P_5 A_\nu^b(y) | 0 \rangle = \sum_{k=1}^{\infty} \frac{1}{k!} \int d^4 u_1 \cdots d^4 u_k d^4 v_1 \cdots d^4 v_k \times K_{u_1} \cdots K_{u_k} \langle 0 | T[A_\mu^a(x) \phi(u_1) \cdots \phi(u_k)] | 0 \rangle \times i \Delta^{(+)}(u_1 - v_1) \cdots i \Delta^{(+)}(u_k - v_k) \times K_{v_1} \cdots K_{v_k} \langle 0 | \tilde{T}[A_\nu^b(y) \phi^+(v_1) \cdots \phi^+(v_k)] | 0 \rangle ,
\end{equation}

(4·5)

where

\begin{equation}
i \Delta^{(+)}(u - v) = \sum_a f_a(u) f_a^*(v) .
\end{equation}

(4·6)

Now we are ready to write down the RG equation for (4·5). The RG equations for the time-ordered Green's functions are given by

\begin{align}
[\mathcal{D} + \gamma \phi(g) + k \gamma \phi(g)] \langle 0 | T[A_\mu^a(x) \phi(u_1) \cdots \phi(u_k)] | 0 \rangle &= 0 , \\
[\mathcal{D} + \gamma \phi(g) + k \gamma \phi(g)] \langle 0 | \tilde{T}[A_\nu^b(y) \phi^+(v_1) \cdots \phi^+(v_k)] | 0 \rangle &= 0 , \\
[\mathcal{D} - 2 \gamma \phi(g)] \Delta^{(+)}(u - v) &= 0 ,
\end{align}

(4·7)
where we have suppressed all the indices of the $\phi$ fields expressing their quantum numbers. Furthermore, the Klein-Gordon operators and their generalizations involve only space-time derivatives and RG invariant masses, so that they commute with $\mathcal{D}$. Hence, we arrive at the RG equation for the expression (4·1):

$$[\mathcal{D} + 2\gamma_v(g)]0|A^a(x)P_\alpha A^\alpha(y)|0> = 0,$$

and consequently we also have

$$[\mathcal{D} + 2\gamma_v(g)]D^+(k, \mu, g) = 0,$$

$$[\mathcal{D} + 2\gamma_v(g)]R^+(k, \mu, g) = 0.$$

Thus we have established that both $D^+$ and $R^+$ satisfy the same RG equation as $D$ and $R$.

In order to obtain the solution of Eq. (4·10) corresponding to (3·13), however, we have to reduce the number of independent variables in $R^+$. Instead of considering $R^+$ as a function of $k^2/\mu^2, k^2/\mu^2$ and $g$ in Eq. (2·20), we shall confine ourselves to $R^+$ in the configuration $k=0$. Then,

$$R^+(k^2/\mu^2; 0, g) = R^+(1, 0, Q) \exp \left[ 2 \int_0^\infty dg' \gamma_v(g') \beta(g') \right].$$

For unconfined gluons we may assume (2·24), and then we find the asymptotic form of $R^+$ for large $|k^2|$ as

$$R^+(k^2/\mu^2, 0, g) \sim CC_v \left( \ln \frac{k^2}{\mu^2} \right)^{-\gamma_0/\beta_0}.$$

The function $D^+$ satisfies a dispersion relation of the form

$$D^+(k, \mu, g) = \int_0^\infty \frac{dm^2 \rho^+(m^2, k^2)}{k^2 + m^2 - i\epsilon},$$

which is an analogue of the Källén-Lehmann representation (3·19) for $D$. As one can easily check, the non-invariant spectral function $\rho^+$ is positive-definite,

$$\rho^+(m^2, k^2) \geq 0.$$

Then the asymptotic forms of $D^+$ and $\rho^+$ for $k=0$ are given, respectively, by

$$D^+(k, \mu, g) \sim CC_v k^2 \left( \ln \frac{k^2}{\mu^2} \right)^{-\gamma_0/\beta_0}$$

for $k=0$ and $|k_0| \to \infty$, and

$$\rho^+(k_0^2; 0) \sim -\left( \frac{\gamma_0}{\beta_0} \right) CC_v |k_0^2|^{-1} \left( \ln \frac{|k_0^2|}{\mu^2} \right)^{-\gamma_0/\beta_0 - 1}$$

for $k=0$ and $k_0^2 \to \infty$.

§ 5. Gluon confinement and the number of flavors

In the preceding section we have obtained Eq. (4·16), and we will discuss its implications. The postulated asymptotic freedom implies
\[ N_f \leq 16 \quad (5.1) \]

as a result of \( \beta_0 < 0 \) and Eq. (3.5). The superconvergence relation (3.22) for \( N_f \leq 9 \) is also valid for \( \rho^+(m^2, 0) \),

\[ \int_0^\infty dm^2 \rho^+(m^2, 0) = 0 \quad (5.2) \]

This relation along with Eq. (4.14) leads to

\[ \rho^+(m^2, 0) = 0 \quad (N_f \leq 9) \quad (5.3) \]

When this is the case, we also have

\[ R^+(\frac{k^2}{\mu^2}, 0, g) = 0, \quad (N_f \leq 9) \quad (5.4) \]

violating the condition for unconfined gluons (2.24), since (5.4) means \( c = 0 \). We interpret this as a sign of gluon confinement. Indeed, Eq. (5.4) leads, as is clear from (2.13), to

\[ A_{\mu \nu}(k)|0\rangle \in U_D \quad \text{for} \quad k = 0 \quad (5.5) \]

Confinement of BRS doublet stems from the unitarity condition,

\[ \langle \beta | \alpha \rangle = \langle \beta | S^\dagger P(U_S) S | \alpha \rangle, \quad (5.6) \]

where the states \( |\alpha\rangle \) and \( |\beta\rangle \) satisfy the subsidiary condition,

\[ Q_B |\alpha\rangle = Q_B |\beta\rangle = 0 \quad (5.7) \]

The choice of \( U_S \), as has been discussed so far, depends on the choice of the frame of reference. However, the unitarity condition (5.6) is valid for any choice of the reference frame, so that the concept of confinement is Lorentz-invariant.

Equation (5.5) indicates that the state \( A_{\mu \nu}^+(k)|0\rangle \) is confined when \( k = 0 \). The Lorentz-invariance of confinement implies that such a state is confined not only in its rest frame but also in any moving frame, although Eq. (5.5) is valid only in a special frame of reference and for a special choice of \( U_D \).

In this connection it is important to recognize that the residue of \( D^+ \) at \( k^2 = 0 \) does not depend on the direction of the time axis, so that Eq. (5.3) implies confinement of single gluon states.

Thus we may conclude that gluons are confined when the number of quark flavors does not exceed nine. We may also put it in a different manner: Gluons are confined when the number of generations does not exceed four. Confinement is closely related to low energy properties of gauge fields, and one might wonder why the confinement condition emerged from the high energy asymptotic behavior of the projected gluon propagators. When we contemplate this equation, we find that what really matters is the relative sign between low and high energies of the projected gluon spectral function, with the former sign fixed by the condition (2.25). Thus the low energy properties are implicitly taken into account in this analysis.

Finally, we may raise the question of what happens to quarks and other colored particles. The condition (2.7) is a sufficient condition for color confinement, but is it also
a necessary condition? We have no definite answer to this question, but a preliminary analysis in a simple approximation\textsuperscript{12) indicates that the condition (2·7) is not only sufficient but also necessary for color confinement. If we take this statement for granted, gluon confinement implies the condition (2·7), which in turn implies quark confinement or more generally color confinement. That is, when the number of generations does not exceed four, all the colored particles are confined.

It still remains to be seen whether or not color confinement ceases to be realized when the number of generations exceeds four, although it seems likely that colored particles are no longer confined in this case.

References

7) M. Gell-Mann and F. Low, Phys. Rev. 95 (1954), 1300.
10) See Ref. 2) for original papers.
12) K. Nishijima and Y. Okada, Prog. Theor. Phys. 72 (1984), 294. See, in particular, the last section.