Cooperative Phenomena in Two-Dimensional Active Rotator Systems

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(Received February 20, 1986)

Phase transitions of active rotator systems with short-range coupling are discussed. The constituents of the system which we call active rotators are represented by a phase model of a limit-cycle oscillator or an excitable element, i.e., \( \frac{d\phi}{dt} = \omega - b \sin\phi, \) \(|\omega/b| \geq 1\). The rotators are subject to noises, and ferromagnetic type coupling is assumed between them. The effect of infinitesimal noises on a perfectly ordered motion is examined for various spatial dimensions by using a linear approximation. As a result, a macroscopic in-phase oscillation throughout the system turned out impossible for 1 and 2 dimensions, but possible for 3 dimensions. An interesting feature expected for a two-dimensional system is that there is a finite parameter region where characteristic length scale is absent. In order to see the latter feature in more detail, we performed a Langevin simulation for a two-dimensional system, and confirmed the existence of a Kosterlitz-Thouless type parameter region.

§ 1. Introduction

By an active rotator we mean a phase model of either a limit-cycle oscillator or an excitable element. This is expressed by the equation of motion for phase \( \phi \)

\[
\frac{d\phi}{dt} = \omega - b \sin\phi .
\]  

(1.1)

Clearly, two different modes of motion are possible depending on the value of \(|b/\omega|\). If \(|b/\omega| < 1\), \(\phi(t)\) increases monotonically, and then we have an oscillator with frequency

\[
\bar{\omega} = 2\pi \left[ \int_{-\pi}^{\pi} \frac{d\phi}{\omega - b \sin\phi} \right]^{-1} = \omega \sqrt{1 - \left(\frac{b}{\omega}\right)^2} .
\]  

(1.2)

If \(|b/\omega| > 1\), we have a stable-unstable pair of fixed points \(\phi_s\) and \(\phi_u\), given by

\[
\phi_s = \arcsin(\omega/b) \quad \text{and} \quad b \cos\phi_s > 0 ,
\]

\[
\phi_u = \pi - \phi_s .
\]  

(1.3)

If \(|b/\omega|\) is slightly greater than 1, the rotator is sensitive to external perturbations. This is because the system is then easily kicked out of its stable equilibrium thus making a long tour before coming back to its original equilibrium. Such a feature is known to be basic to excitable elements.

Equation (1.1) may also be looked upon as an equation describing the heavily damped motion of a particle in a periodic potential under the action of a constant driving force. The population model of the elements like (1.1) has been employed in the study of a driven and heavily damped sine-Gordon chain\(^1\) and also of the surface roughening transition.\(^2\) In those cases, the elements are supposed to be under the effect of external noises, and nearest-neighbour mutual coupling is assumed. However, the coupling assumed is linear in phase difference, so that the transformation \(\phi \to \phi + 2\pi\) for individual elements does not
Recently, the present authors investigated the phase transitions in communities of active rotators with infinite-range coupling and external noises.\textsuperscript{3) The coupling assumed there is given by a 2\pi-periodic function in phase difference so that the model satisfies the required invariance and also allows for the phase slip. Then we found some interesting features, in particular, the existence of a dynamically ordered state where the system exhibits a coherent rhythmic motion over the entire space.

The present paper deals with the system of active rotators with short-range coupling. Otherwise the model is essentially the same as in a previous paper.\textsuperscript{3) Our main purpose is to know the relation between spatial dimension and possible types of phase transitions. After briefly introducing our model in § 2, we shall develop in § 3 a preliminary consideration based on a linear theory (similar to the spin wave theory) to study the effect of weak noises on a perfectly ordered behaviour. The analysis shows the following. In the region $|b/\omega|<1$, a macroscopic in-phase oscillation over the entire system persists for $d\geq3$ ($d$ for dimension). For $d=2$ and 1, the long-range coherence is destroyed immediately on introducing weak noises. In two-dimensional systems, however, there is an indication that the spatial correlation of some 2\pi-periodic function of $\phi$ obeys a power-law decay and consequently, the fluctuation in 'magnetization' $M=\sum_i e^{i\phi_i}$ or the quantity $\tilde{x}$ given by

$$\tilde{x} = \langle M - \langle M \rangle \rangle / N$$

diverges. For vanishing $b$, the system reduces to the equilibrium planar model as we shall see in § 2. In this special case, the system is known to exhibit the divergence of the susceptibility $\chi (=\tilde{x}/2T)$ for all temperatures below $T_c$, the Kosterlitz-Thouless transition temperature. The above results in two extreme cases (i.e., cases of weak noises and of vanishing $b$) show that in $T-b$ plane ($T$ for noise intensity) we have at least two special regions of divergent $\tilde{x}$, one for $|b/\omega|<1$ near the $b$-axis ($T\geq0$) and the other for $0<T\leq T_c$ on the $T$ axis ($b=0$). We shall call the regime in which $\tilde{x}$ diverges 'generalized Kosterlitz-Thouless regime'. An interesting question is whether the generalized KT regime can be extended away from band $T$ axes. In order to study this problem, a Langevin simulation on the two-dimensional system is carried out for sizes of $32 \times 32$ and $64 \times 64$ elements. The results are summarized in § 4, where the above question is answered positively.

§ 2. Model

Let our system of $N$ interacting active rotators be represented by a Langevin-equation form,

$$\frac{d\phi_i}{dt} = \omega - b\sin\phi_i - K \sum_{(ij)} \sin(\phi_i - \phi_j) + \eta_i(t),$$

where $\sum_{(ij)}$ means the summation over all the nearest-neighbours to the $i$-th site. Each element is placed on a lattice point of a cubic lattice of linear scale $L (L^d = N)$. $\eta_i(t)$ are the Gaussian white random forces with statistical averages

$$\langle \eta_i(t) \rangle = 0, \quad \langle \eta_i(t) \eta_j(t') \rangle = 2T \delta_{ij} \delta(t-t').$$

(2.2)

The parameters included in this model are $\omega$, $b$, $K$ and $T$, but we shall use a suitable time
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unit in which $K=1$; the choice $K=-1$ would imply antiferromagnetic coupling which we will not consider. Without loss of generality, we assume the other three parameters to be non-negative. There are known limiting cases corresponding to some peripheral regions in our parameter space:

(i) $\omega=0$

The dynamics is governed by a kinetic potential $H$ which is $2\pi$-periodic in each $\phi_i$:

$$\frac{d\phi_i}{dt} = -\frac{\partial H}{\partial \phi_i} + \eta_i(t), \quad (2.3)$$

$$H = -K\sum_{(ij)}\cos(\phi_i - \phi_j) - b\sum\cos\phi_i, \quad (2.4)$$

where $(ij)$-summation is taken over all nearest-neighbour pairs (each pair counted only once). Thus the system has an equilibrium probability distribution $\exp(-H/T)$ and becomes equivalent to the equilibrium planar model with external field.

(ii) $b=0$

The driving force term $\omega$ may be eliminated from Eq. (2.1) through simultaneous transformations $\phi_i \rightarrow \phi_i + \omega t, (i = 1, 2, \cdots, N)$. Thus the system again reduces to the planar model but without external field.

(iii) $T=0$

This corresponds to the absence of noises. The phases of the rotators then show a perfect synchrony, i.e., $\phi_i(t) = \phi_0(t)$ for all $i$, $\phi_0(t)$ obeying Eq. (1.1).

§ 3. Linear approximation — case of small $T$

Without mutual coupling, the phase $\phi(t)$ shows a monotone increase with time if $b<\omega$, but this is generally accompanied by a jerky oscillation. In some cases, it is more appropriate to work with a new phase variable $\phi(\phi(t))$ which shows a steady increase with the same average rate of increase as $\phi(t)$, i.e., $d\phi/dt = \bar{\omega}$. It is clear that

$$\phi(\phi) = \bar{\omega} \int^{\phi} \frac{d\phi'}{\omega - b\sin\phi'} = 2 \arctan\left(\frac{\omega \tan(\phi/2) - b}{\bar{\omega}}\right). \quad (3.1)$$

Equation (2.1) then takes the form

$$\frac{d\phi_i}{dt} = \bar{\omega} + \sum_{(ij)} \Gamma(\phi_i, \phi_j) + \xi_i(\phi_i, t), \quad (3.2)$$

where

$$\Gamma(\phi_i, \phi_j) = \frac{K\bar{\omega}}{\omega - b\sin\phi_i} \sin[\phi(\phi_i) - \phi(\phi_j)] \quad (3.3)$$

and

$$\xi_i(\phi_i, t) = \frac{\bar{\omega}}{\omega - b\sin\phi_i} \eta_i(t). \quad (3.4)$$

We now consider the case of sufficiently weak noises for which an averaging method as developed in Ref. 4) is available. Note that for weak noises the phase difference between any interacting pair will also be small. This means that the effect of the last two
terms in Eq. (3·2) is small as compared to $\omega$ term. In order to express this situation more clearly, we introduce a smallness indicator $\epsilon$ (finally set to 1), and rewrite Eq. (3·2) as

$$\frac{d\phi_i}{dt} = \omega + \epsilon \{ \sum_{ij} \Gamma'(\phi_i, \phi_j) + \xi_i(\phi_i, t) \}$$

or

$$\frac{d\theta_i}{dt} = \epsilon \{ \sum_{ij} \Gamma'(\theta_i + \omega t, \theta_j + \omega t) + \xi_i(\theta_i + \omega t, t) \}, \quad (3·5)$$

where $\theta_i = \phi_i - \omega t$. Since $\xi_i(\phi_i, t)$ are Gaussian noises with properties

$$\langle \xi_i(\phi_i, t) \rangle = 0,$$
$$\langle \xi_i(\phi_i, t) \xi_j(\phi_j, t') \rangle = 2TD(\phi_i)\delta_{ij}\delta(t-t'),$$

where

$$D(\phi_i) = \left[ \frac{\omega}{\omega - b\sin(\phi_i)} \right]^2,$$

Eq. (3·5) is now cast into the form of a Fokker-Planck equation for the probability distribution $P(\{\theta_i\}, t)$ or

$$\frac{\partial P}{\partial t} = \epsilon \{ -\sum_i \frac{\partial}{\partial \theta_i} \sum_{ij} \Gamma'(\theta_i + \omega t, \theta_j + \omega t) P + \sum_i \frac{\partial^2}{\partial \theta_i^2} TD(\theta_i + \omega t) P \}. \quad (3·6)$$

Note that the evolution of $P$ is slow because of the smallness parameter on the r.h.s., whereas the drift and diffusion terms involve rapidly oscillating quantities $\Gamma'(\theta_i + \omega t, \theta_j + \omega t)$ and $D(\theta_i + \omega t)$. Such quantities may therefore be time-averaged over the period, and we get after putting $\epsilon = 1$

$$\frac{\partial P}{\partial t} = -\sum_i \frac{\partial}{\partial \theta_i} \sum_{ij} \bar{\Gamma}(\theta_i - \theta_j) P + \sum_i \frac{\partial^2}{\partial \theta_i^2} \bar{T} \bar{D} P, \quad (3·7)$$

where

$$\bar{\Gamma}(\theta_i - \theta_j) = \frac{\omega}{2\pi} \int_0^{2\pi/\omega} dt \Gamma'(\theta_i + \omega t, \theta_j + \omega t) \quad (3·8)$$

and

$$\bar{D} = \frac{\omega}{2\pi} \int_0^{2\pi/\omega} dt D(\theta_i + \omega t). \quad (3·9)$$

Equation (3·7) is equivalent to the simple Langevin equation

$$\frac{d\theta_i}{dt} = \sum_{ij} \bar{\Gamma}(\theta_i - \theta_j) + \sqrt{\bar{D}} \eta_i(t).$$

The linearization of the above equation may be permitted for sufficiently small phase difference, and we get

$$\frac{d\theta_i}{dt} = -\bar{K} \sum_{ij} (\theta_i - \theta_j) + \sqrt{\bar{D}} \eta_i(t), \quad (3·10)$$

where $\bar{K} = \bar{\Gamma}''(0)$. The above argument implies that the effect of non-vanishing $b$ is
irrelevant (as far as \( |b/\omega| < 1 \)) to the persistence or collapse of ordered motion on introducing weak noises.

The linear Langevin equation (3·10) or the corresponding Fokker-Planck equation gives stationary distribution \( P(\{\theta_i\}) \) in the Gaussian form

\[
P(\{\theta_i\}) = C \exp\left[-\tilde{K} \sum_{(ij)} (\theta_i - \theta_j)^2 / (\tilde{D}T)\right].
\]

Thus we have

\[
\langle (\phi_j - \phi_{j+l})^2 \rangle = \langle (\theta_j - \theta_{j+l})^2 \rangle \propto \int d\mathbf{k} \frac{1}{k^2} [1 - \cos(\mathbf{k} \cdot \mathbf{l})].
\]

(3·11)

The above quantity diverges as \( |l| \to \infty \) if \( d \leq 2 \) and remain finite if \( d > 2 \). Thus the long range phase order or in-phase oscillation is absent for \( d = 1 \) and \( d = 2 \) and present for \( d = 3 \).

The correlation function

\[
c(l) = \langle e^{i\phi_j} e^{-i\phi_{j+l}} \rangle
\]

(3·12)

is found to be

\[
c(l) \sim \begin{cases} 
e^{-l^d}, & d = 1, \\ l^{-a}, & d = 2, \\ \text{const.}, & d \geq 3, \end{cases}
\]

(3·13)

The above asymptotic forms of \( c(l) \) are expected to be preserved when we consider the correlation function of fluctuation of more general 2\( \pi \)-periodic function of \( \phi \).

In the case \( \omega < b \), \( \phi_0(t) \) becomes time-independent as \( t \to \infty \). If we linearize Eq. (2·1) directly in the deviation \( \delta \phi_i \) from \( \phi_0 \), each deviation \( \delta \phi_i \) experiences its harmonic kinetic force, and the mean square deviation \( \langle |\delta \phi_i|^2 \rangle \) remains finite for all dimensions.

We summarize here the results of our linear approximation together with some known equilibrium (i.e., \( b = 0 \)) results:

\((d \geq 3)\)

A macroscopic order persists on introducing sufficiently weak noises. Thus a clear phase boundary between stationary regime \( S \) and periodic regime \( P \) is expected on the \( T-b \) plane. For \( b = 0 \), in particular, existing theories tell that an order-disorder transition at some critical temperature \( T_c \) occurs. Thus the basic feature of the phase diagram seems qualitatively the same as in the case of infinite-range interaction.3)

\((d = 2)\)

In the region \( \omega > b \) at sufficiently low temperature, the spatial correlation function of \( e^{i\phi} \) is expected to show a power-law decay \( \Delta c(l) \sim l^{-a} \), where \( \Delta c(l) = c(l) - c(\infty) \) and \( a \) depends linearly on \( T \). The fluctuation of magnetization is then given by

\[
\bar{\chi} \sim \int_0^L dl \; l \Delta c(l) \sim L^{2-a} = N^{1-a/2},
\]

(3·14)

which diverges as \( N \to \infty \) if \( a < 2 \). No singularity is expected in the region \( \omega \leq b \). In the case \( b = 0 \), existing theories tell \( \bar{\chi} \) diverges at all temperatures below \( T_c \), the Kosteritz-Thouless transition temperature.

\((d = 1)\)

There is no singularity in \( \bar{\chi} \) as far as the temperature \( T \) remains finite.
§ 4. Langevin simulation

In order to see if the parameter region of divergent $\tilde{\chi}$ is extended away from the vicinity of $T$ and $b$ axes in two-dimensional system, we carried out a Langevin simulation for various $T$ and $b$. The parameter $\omega$ is fixed to unity throughout. Some details on the method of numerical simulations are the following. The elements are placed on the lattice points of an $L \times L = N$ square lattice where we have chosen $L=32$ ($N=1024$) and $L=64$ ($N=4096$). For technical reasons we employed the boundary conditions $(i+L,j)=(i,j+1)$ and $(i,j+L)=(i,j)$ which is expected to produce no essential difference from the usual periodic boundary condition. As a method of the time-discretization of the Langevin equation, we adopted the one proposed by Ukawa and Fukugita, which is similar to the Heun method applicable to usual differential equations. The time-increment by discretization is chosen to be $0.025$. For given $b$ and $T$, the system runs the first $4 \times 10^3 (2 \times 10^3)$ steps for 'idling' or 'equilibration' and the subsequent $2 \times 10^4 (1 \times 10^4)$ steps for calculating physical quantities at every $40$ ($20$) steps. Here, the numbers with and without the brackets are for the systems of $L=32$ and $L=64$, respectively. We carried out the simulation for parameter values $b=0.0, 0.4, 0.8, 0.9, 1.0$ and $1.1$. For each value of $b$, the temperature $T$ is cooled stepwise from $1.5$ down to $0.1$ at intervals $0.1$. The calculation for $L=32$ is used to check the size effect and hysteretic behaviour on cooling and warming. No evidence for hysteresis is found. In what follows, we shall show the results mainly for the system of $L=64$.

(i) order parameter $\sigma$

As discussed in the preceding section, the order parameter defined by $\sigma(t)=M(t)/N$ is expected to lose time-dependence in the limit $N \to \infty$. In finite systems, however, there may be a sizable fluctuation in $M(t)/N$, so that it would be more appropriate to redefine $\sigma$ by

$$\sigma = \bar{M}/N \, ,$$

where the bar denotes the long-time average. The obtained $\sigma$-values are plotted in Fig. 1, without apparent singularity.

(ii) fluctuation of magnetization, $\tilde{\chi}$

No systematic time-dependence of $\tilde{\chi}$ is expected in two-dimensional systems. The ensemble average in defining $\tilde{\chi}$ may therefore be replaced by a long-time average. Thus we adopt the definition

$$\tilde{\chi} = |M-\bar{M}|^2/N \, .$$

The results are shown in Fig. 2. In the equilibrium limit $b=0$, the fittings of the data with a theoretical curve $\tilde{\chi} \sim e^{bt \cdot \omega}(t=(T-\bar{T})/\bar{T})$ is used for obtaining $\bar{T}$. In our more general case, however, such a method is not applicable. We therefore determined the phase boundary $T_c(b)$ simply by looking for where an anomalously large value of $|\partial \tilde{\chi}/\partial T|$ is attained. More specifically, $T_c(b)$ is determined as the temperature such that the product of the gradient of $\tilde{\chi}$ on both sides of that temperature has the maximum value (see Table I). Here, we mention that the size-dependence of $\tilde{\chi}$ becomes prominent below
$T_c(b)$ determined in this way.

(iii) vortex density $n_v$

The Kosterlitz-Thouless transition of a planar model is related to the dissociation of a vortex pair. A similar mechanism is expected to work in our generalized model with $b \neq 0$. Vorticity is defined as a net change of the phases as we move along a closed loop on the lattice. This change must be an integer multiple of $2\pi$. We calculated the vorticity in each plaquette. The mean density of the vortex pairs per plaquette is given by

$$n_v = \frac{\bar{N}_v}{N},$$

(4.3)

Fig. 1. Magnitude of the order parameter $\sigma (L=64)$.
Fig. 2. Generalized susceptibility $\chi(L=64)$.

Fig. 3. Logarithm of the vortex density $n_v$ versus $1/T (L=64)$.
Fig. 4. Mean angular velocity $v (L=64)$. 
Table I. Critical temperature $T_c(b)$ for some values of $b(\omega=1)$.

<table>
<thead>
<tr>
<th>$b$</th>
<th>0.0</th>
<th>0.4</th>
<th>0.8</th>
<th>0.9</th>
</tr>
</thead>
<tbody>
<tr>
<td>$T_c(b), L=32$</td>
<td>$1.0\pm0.1$</td>
<td>$1.0\pm0.1$</td>
<td>$0.6\pm0.1$</td>
<td>$0.5\pm0.1$</td>
</tr>
<tr>
<td>$L=64$</td>
<td>$1.0\pm0.1$</td>
<td>$0.9\pm0.1$</td>
<td>$0.5\pm0.1$</td>
<td>$0.5\pm0.1$</td>
</tr>
</tbody>
</table>

Table II. Creation energy of a vortex pair ($L=64$).

<table>
<thead>
<tr>
<th>$b$</th>
<th>0.0</th>
<th>0.4</th>
<th>0.8</th>
<th>0.9</th>
</tr>
</thead>
<tbody>
<tr>
<td>$E_p(b)$</td>
<td>$7.8\pm0.5$</td>
<td>$6.6\pm0.4$</td>
<td>$3.3\pm0.3$</td>
<td>$2.2\pm0.3$</td>
</tr>
</tbody>
</table>

where $N_v$ is the total number of plus or minus vortices. Note that the total vorticity is a conserved quantity which we have set to zero initially. If the vortex density is approximated by an activation type, then

$$\ln n_v = -\frac{E_p}{T} + \text{const.},$$

(4.4)

where $E_p$ represents the creation energy of a vortex pair. We show the $T$-dependence of $n_v$ in Fig. 3, where the linearity of $\ln(n_v)$ in $1/T$ seems to hold well in our generalized KT regime. The values of the $E_p(b)$'s are summarized in Table II.

(iv) mean angular velocity $v$

As a quantity of essentially dynamic nature, we define the mean angular velocity $v$ by

$$v = \frac{1}{N} \sum_{i=1}^{N} (\phi_i(t) - \phi_i(0))/t.$$  

(4.5)

From Fig. 4, we see that $v$ shows no singular behaviour.

§ 5. Summary and relevance to other fields

In two-dimensional active-rotator systems we have shown, first by a simple linear approximation and next by a Langevin simulation, some evidence for the existence of the generalized KT regime in which $\bar{\omega}$ diverges.

Two-dimensional networks of the oscillators or excitable elements are commonly found in living organisms. In those systems there seem to be as many situations where the uniformly synchronized motion would be unfavorable (such as wave propagation in some nervelike tissues), as those where it is favorable (as in circadian rhythms). The present study may also have some relevance to such problems.

Acknowledgements

We are greatly benefited from the comments of Professor M. Fukugita on Langevin simulation. We are also grateful for informative comments to Mr. H. Sakaguchi, Dr. H. Kawamura, Dr. Y. Okabe and Dr. M. Kikuchi. One of the authors (S.S.) is indebted to Yukawa foundation for financial support. The present work was also supported by the Grant-in Aid for Fusion Research of Ministry of Education, Culture and Science of Japan.
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