Quantum-Classical Correspondence in Wave Functions of the Universe

Sumio WADA
Institute of Physics, University of Tokyo, Komaba, Tokyo 153

(Received November 25, 1985)

By using the classically solvable minimal massless scalar minisuperspace model we study two examples of wave functions of the universe in the semiclassical limit. The first one consists of a Lorentzian component and a Euclidean component and admits a clear semiclassical interpretation as a superposition of universes. The second one consists of a Lorentzian (or Euclidean) component and another oscillatory component which corresponds to (neither Lorentzian nor Euclidean) complex classical solutions. This example has some resemblance to Hawking's minimal massive scalar minisuperspace model. We suggest a possible way of recovering the classical interpretation in such a case.

§ 1. Introduction

In quantum cosmology, we describe the universe by a wave function which satisfies the Wheeler-DeWitt equation. This is a second-order linear partial differential equation. In the \( h \to 0 \) limit some of its solutions admit clear classical interpretation while others do not. In the present paper we study the semiclassical limit of wave functions in a classically solvable minisuperspace model and discuss the classical interpretation.

In § 2 we present the model and its classical solutions. The model is the minimal massless scalar theory coupled to the Einstein gravity with all the spatial degrees of freedom suppressed, i.e., in the minisuperspace approximation. By using a convenient set of variables we write down the most general complex classical solutions which include real Lorentzian and Euclidean solutions as special cases. (For another classically solvable model, see Ref. 6.) In § 3 we show the first set of examples for wave functions of the universe. In the semiclassical sense these wave functions correspond to a set of Lorentzian and Euclidean solutions which pass a certain fixed point in the minisuperspace (the configuration space), and therefore can be interpreted as a superposition of universes.

In § 4 we rewrite the above examples in the path integral representation, which includes slight refinement of the formula in previous works. We also study the consequence of Hartle and Hawking's initial condition which states that only the histories starting from the vanishing cosmic scale factor should be summed. In the present model this condition gives us a wave function which contains no universe.

In § 5 we present the second example of wave functions, in which appears a region that is neither Lorentzian nor Euclidean. We use a path integral representation written in terms of a new set of canonical variables. This is transformed to the wave function in the original variables by using the generating function of the canonical transformation. The result shows that the wave function in the saddle point approximation is given by (neither Lorentzian nor Euclidean) complex classical solutions in some region.

In § 6 we discuss the picture that the universe is created from nothing by the tunneling process. We first present the path integral representation of Vilenkin's pure gravity model and its classical interpretation. Next we discuss Hawking's minimal massive
scalar model and point out a similarity to the example of the massless model in § 5. Section 7 is for conclusion.

§ 2. The model and its classical solutions

In this paper we mainly deal with the minimal massless scalar theory in the minisuperspace approximation. The Lagrangian is

$$\mathcal{L} = R + \frac{1}{2} g^{\mu \nu} \partial_\mu \phi \partial_\nu \phi - \frac{1}{16 \pi G}.$$ 

In the minisuperspace approximation we assume that \( \phi \) depends only on time and that the metric is of the Robertson-Walker type with the closed spatial section,

$$ds^2 = N^2(t) dt^2 - a^2(t) d\Omega_3^2.\text{(closed)}.$$ 

Then the canonical form of the action is

$$S = \int \left( \dot{\pi}_a \dot{a} + \pi_\phi \dot{\phi} - NH \right) dt,$$

$$H = \frac{1}{2a} \left( -\frac{\pi_a^2}{12} + \frac{\pi_\phi^2}{a^2} - 12a^2 \right),$$

$$\pi_a = -\frac{12a\dot{a}}{N}, \quad \pi_\phi = \frac{a^2 \dot{\phi}}{N},$$

where \( \pi_a \) and \( \pi_\phi \) are canonical momenta and the overdot represents differentiation with respect to \( t \). The following set of the variables is convenient

$$X = a^2 e^{\phi/\sqrt{3}},$$

$$Y = a^2 e^{-\phi/\sqrt{3}},$$

in terms of which (1) becomes

$$S = \int \left( \pi_x \dot{X} + \pi_Y \dot{Y} - NH \right) dt,$$

$$H = -6(XY)^{1/4} \left( \frac{1}{9} \pi_x \pi_Y + 1 \right),$$

$$\pi_x = -\frac{3}{2N} (XY)^{-1/4} \dot{Y}, \quad \pi_Y = -\frac{3}{2N} (XY)^{-1/4} \dot{X}.\text{(3)}$$

Classical solutions of this model are obtained from

$$\frac{\dot{X}}{N^2} = -4(XY)^{1/2}, \text{(4a)}$$

$$\frac{1}{X} \frac{d}{dt} \left( \frac{X}{N} \right) - \frac{1}{Y} \frac{d}{dt} \left( \frac{Y}{N} \right) - \frac{1}{4N} \left( \frac{\dot{X}^2}{X^2} - \frac{\dot{Y}^2}{Y^2} \right) = 0. \text{(4b)}$$

The first equation is the (0, 0) component of Einstein's equations and corresponds to the constraint \( H = 0 \). The second one is the equation of motion of \( \phi \) and immediately gives
Quantum-Classical Correspondence in Wave Functions

\[ (XY)^{3/4} \frac{N}{N} \left( \frac{X}{X} - \frac{Y}{Y} \right) = c, \tag{5} \]

where \( c \) is an integration constant. Substituting this into (4a), we get

\[ c^2 \left( Y - X \frac{dY}{dX} \right)^2 = -4. \]

Its solution is

\[ Y = aX + \beta, \]

\[ a = -\frac{4\beta^2}{c^2}. \tag{6} \]

When \( Ndt \) is real, \( c \) in (5) is also real and \( a \) should be negative. This is an ordinary Lorentzian solution. Likewise \( a \) should be positive when \( Ndt \) is imaginary. This is nothing but a Euclidean solution. In general, \( a \) and \( \beta \) are complex.

In the following sections we need the value of the action for the classical solutions. From (3), (5) and (6) we get

\[ \pi_x = 3\sqrt{-a} \quad \text{and} \quad \pi_y = -3/\sqrt{-a}. \tag{7} \]

Then the action of the solution which goes from \((X_0, Y_0)\) to \((X, Y)\) is

\[ S(X, Y; X_0, Y_0) = \int \pi_x dx + \int \pi_y dy = 6\sqrt{-\left( X - X_0 \right) \left( Y - Y_0 \right)}, \tag{8} \]

where we used \( a = (Y - Y_0) / (X - X_0) \).

\[ \section{3. Wave function of the universe. I} \]

In the quantum theory the physical state \( \Psi \) is defined by the Wheeler-DeWitt equation \( \hat{H}\Psi = 0 \). In the model described in the previous section we get from (3)

\[ (XY)^{1/4} \left[ -\frac{\hbar^2}{9} \frac{\partial}{\partial X} \frac{\partial}{\partial Y} + 1 \right] \Psi(X, Y) = 0. \tag{9} \]

(We comment on ambiguities of the operator ordering below.) To study the classical limit \( (\hbar \to 0) \) we write the wave function as

\[ \Psi = A(X, Y) \exp(iW(X, Y)/\hbar), \tag{10} \]

where \( W \) satisfies the equation

\[ \frac{1}{9} \partial_x W \cdot \partial_y W + 1 = 0. \tag{11} \]

As is expected \( S \) in (8) is one of its solutions, namely

\[ W(X, Y)_{X_0, Y_0} = S(X, Y; X_0, Y_0). \tag{12} \]

\( X \) and \( Y \) are coordinates of the (mini) superspace, and \( X_0 \) and \( Y_0 \) are regarded as indices which distinguish the solutions.

In the region \( (X - X_0)(Y - Y_0) > 0 \), \( S \) becomes imaginary. This corresponds to the fact that there is no Lorentzian classical solution which connects the two points, since its
slope should be negative. In the standard language this is a classically forbidden region. \( S \) is real where \((X-X_0)(Y-Y_0) < 0\) and this is a classically allowed region. This situation is summarized in Fig. 1.

From the definition of \( S \) we can regard the wave functions (10) and (12) as a superposition of the universes which pass through the point \((X_0, Y_0)\). We make this interpretation more explicit by calculating the prefactor \( A \) in the WKB approximation. In the next-to-leading order in \( \hbar \), (9) gives

\[
\partial_t A \partial_y S + \partial_y A \partial_x S + A \partial_x \partial_y S = 0.
\]

Its solution is

\[
A = S^{-1/2} A_0 \left( \frac{Y-Y_0}{X-X_0} \right),
\]

where \( A_0 \) is an arbitrary function. When \( A_0(\xi) = \delta(\xi-\xi_0) \), for an instance, the wave function is non-zero only along the classical solution \( Y-Y_0 = \xi_0(X-X_0) \) and its classical meaning is clear. In general, \( A_0 \) is arbitrary and can be expressed as a superposition of the delta functions with an appropriate weight factor. In this sense (10) with (12) and (13) is a superposition of the universes. (When \( A_0 \) is a delta function, \( \Psi \) becomes an example of wave packets of a universe. But it is singular when \((X, Y) \rightarrow (X_0, Y_0)\) because \( S \rightarrow 0 \). A nonsingular wave packet, which is not of the \( \delta \)-function type, was given in Ref. 7.)

Two comments are in order. Firstly the Wheeler-DeWitt equation in the form of (9) is not unique but has ambiguities in higher order in \( \hbar \). Some of them (e.g., the operator ordering in the first term) cancels with the corresponding change of the measure which comes from the requirement that \( H \) is hermitian. In general, however, ambiguities in \( O(\hbar) \) have physical effects.

Secondly the plane wave is an exact solution of (9) if the above ambiguities are ignored. Classically this corresponds to the set of the parallel Lorentzian solutions, but its interpretation in terms of the action (and the path integral) needs a canonical transformation of the variables, \( X \rightarrow \pi_X \) and \( Y \rightarrow \pi_Y \). The technique of canonical transformations will be used in § 5 in a different example.

§ 4. Path integral and Hartle and Hawking's boundary conditions

In this section we discuss the wave function in the framework of the path integral scheme. The path integral representation of wave functions of the model described by (3) is

\[
\Psi(X, Y)_{X_0, Y_0} = \int \mathcal{D}X \mathcal{D}Y \mathcal{D}N \text{(gauge fixing)} \exp(i \int_0^T dt \mathcal{L}).
\]
The path integral is an expression for the transition amplitude between two classical configurations. In the above we mean that the initial configuration is \((X_0, Y_0)\) and the final configuration is \((X, Y)\). \(N\) is not a physical degree of freedom and \(N\) at \(t=0\) and \(T\) should be integrated. The origin of \(N\) and also of the gauge fixing is the time reparametrization invariance of the theory. In Ref. 7) we showed, by using the gauge \(\dot{N} = 0\), that \(\Psi\) in (16) satisfies the Wheeler-DeWitt equation in \((X, Y)\) and that \(\Psi\) does not depend on \(T\) except a trivial factor. In this sense \(\Psi\) in (16) is a physical state.

In the gauge \(\dot{N} = 0\) the integration \(\prod_t \delta N(t)\) reduces to the simple integral \(dN\). In the Euclidean formulation we should rotate the time as \(t \to i t\). But it is equivalent to the rotation of the integration contour for \(dN\) from the real axis to the imaginary one, since \(dt\) in the action (both for integration and for differentiation) is always multiplied by \(N\). Therefore we do not rotate the time in (16), but rotate the integration contour for \(dN\) so that it passes saddle points in the complex \(N\) plane.

We estimate (16) in the saddle point method (i.e., in the stationary phase approximation). The path which gives the stationary phase in (16) is the trajectory from \((X_0, Y_0)\) to \((X, Y)\) which satisfies

\[
\frac{\delta S}{\delta X(t)} = \frac{\delta S}{\delta Y(t)} = \frac{dS}{dN} = 0. \quad (S = \int_0^T dt \mathcal{L}(X, Y, N)) \tag{17}
\]

Though \(N\) is \(t\)-independent in the present gauge, the first two of (17) imply \(\delta S / \delta N(t)\) is a constant of motion, which should be zero because of the third of (17). Therefore (17) is equivalent to

\[
\frac{\delta S}{\delta X(t)} \left( \text{or} \frac{\delta S}{\delta Y(t)} \right) = \frac{\delta S}{\delta N(t)} = 0,
\]

and so is equivalent to (4). This implies that the saddle point is nothing but a classical solution and that, in the leading order of the saddle point approximation,

\[
\Psi \sim e^{iS}
\]

with \(S\) in (8). When \((X - X_0)(Y - Y_0) < 0\), the relevant solution is the one with \(a < 0\) in (6) and \(N dt\) is real. We call it Lorentzian region. When \((X - X_0)(Y - Y_0) > 0\), the relevant solution is the one with \(a > 0\) and \(N dt\) is imaginary. We call it Euclidean region. In this example the Lorentzian one and the Euclidean one coincide with the oscillatory regions (\(S\) real) and the nonoscillatory region (\(S\) imaginary), respectively. This correspondence breaks down in the example of the next section.

As is shown above the wave function depends on the initial condition \((X_0, Y_0)\). Hartle and Hawking proposed\(^3\) that, in the path integral expression, only the geometry with no boundary (i.e., \(a(t=0) = 0\)) should be summed. Provided that this condition is supplemented by the finiteness
of $\phi(t=0)$, we get a unique initial condition $X_0 = Y_0 = 0$ (see (2)). (Finiteness was not mentioned in Ref. 3), but it is implicitly assumed in later works. Note in passing that the finiteness does not uniquely determine the initial condition in a two dimensional model as was shown in Ref. 6.) If we take $X_0 = Y_0 = 0$ in (8) $S$ is imaginary in the whole physical region $XY > 0$. The whole physical region is classically forbidden, as is depicted in Fig. 2. In the language of Ref. 2), a universe is not created from nothing in this model (see § 6).

§ 5. Wave function of the universe. II

In this section we show another example of a wave function which, in some region, does not correspond to either Lorentzian or Euclidean solutions and does not admit ordinary classical interpretation. Our strategy is the following: Consider the set of the Lorentzian solutions which are tangential to the hyperbola $XY = A^2$, i.e.,

$$Y = aX + \beta \quad \text{with} \quad \beta^2 + 4aA^2 = 0.$$ (19)

More precisely we consider the solutions only for $0 < X < X_c$ where $X_c$ is the coordinate of the contact point. In terms of the original variables these solutions correspond to the universes which expand from $a = 0$ ($\phi = -\infty$) to the maximum value $a = \sqrt{A}$ at some $\phi$. This set of the solutions covers the region $XY < A^2$ as is shown in Fig. 3 and defines a wave function there in the semiclassical limit, as will be shown below. Our interest is what this wave function expresses when it is continued to the region $XY > A^2$.

Note that even the wave function for $XY < A^2$ is not trivial because the above set of the solutions does not have the initial points in common and the naive path integral expression fails. This difficulty is overcome by using a new variable

$$\tilde{Y} = Y \pi_Y,$$

instead of $Y$. In terms of $X$ and $\tilde{Y}$ (19) is written as

$$\tilde{Y} = 3\sqrt{-a} X - 6A,$$ (20)

and has an initial point $(X_0 = 0, \tilde{Y}_0 = -6A)$ in common. The generating function $F(Y, \tilde{Y})$ for the canonical transformation $Y \rightarrow \tilde{Y}$ is

$$F = \tilde{Y} \log Y,$$

since $\pi_Y = \partial F/\partial Y = \tilde{Y}/Y$. We also get that

$$\pi_{\tilde{Y}} = -\partial F/\partial \tilde{Y} = -\log Y,$$

$$= -\log (-\sqrt{-a}\, \tilde{Y}/3)$$

We write the Lagrangian and the wave function in terms of $X$ and $\tilde{Y}$ as $\mathcal{L}$ and $\Phi$, respectively. Then

$$\Phi(X, \tilde{Y}) = \int \mathcal{D}X \mathcal{D}Y dN e^{i\mathcal{L} \tilde{Y} dt},$$

$$\mathcal{L} = \dot{X} \pi_X + \dot{\tilde{Y}} \pi_{\tilde{Y}} - NH (X, Y = \exp(-\pi_Y)),$$
where $H$ is the same as that in (3).

The saddle point for this path integral is given again by a classical solution and in the leading order of the saddle point approximation we get

$$\bar{\Psi} \approx e^{iS},$$

$$S = \int \pi_x dX + \int \pi_y d\bar{Y}$$

$$= 2(\bar{Y} - \bar{Y}_0) - \bar{Y} \log\left(-\frac{\bar{Y}(\bar{Y} - \bar{Y}_0)}{9X}\right) + \bar{Y}_0 \log\left(-\frac{\bar{Y}_0(\bar{Y} - \bar{Y}_0)}{9X}\right).$$

$\bar{\Psi}$ is transformed to $\Psi$ by the generating function as

$$\Psi(X, Y) = \int d\bar{Y} e^{iY \log \bar{Y}} \bar{\Psi}(X, \bar{Y}).$$

The saddle point of this integral is given by

$$\bar{Y} = \frac{1}{2}(\bar{Y}_0 - \sqrt{\bar{Y}_0^2 - 36XY}),$$

and the wave function is $\Psi \approx e^{iS}$ with

$$S = -(\bar{Y}_0 + \sqrt{\bar{Y}_0^2 - 36XY}) + \bar{Y}_0 \log\left(\frac{\bar{Y}_0 + \sqrt{\bar{Y}_0^2 - 36XY}}{18X}\right).$$

It is easy to see that this satisfies the Wheeler-DeWitt equation (11) with $W = S$.

By construction this wave function at each point, $X = \bar{X}$ and $Y = \bar{Y}$, is given by one classical solution, which we write as

$$Y = a(\bar{X}, \bar{Y})X + \beta(\bar{X}, \bar{Y}).$$

(23)

$a$ and $\beta$ are obtained as follows: First from (20) we get

$$\sqrt{-a} = \frac{1}{3X}(\bar{Y} + 6A),$$

where $\bar{Y}$ is given by (21) with $X = \bar{X}$ and $Y = \bar{Y}$. Then

$$a(\bar{X}, \bar{Y}) = -\frac{1}{\bar{X}^2}\left(2A^2\left(1 - \sqrt{1 - \frac{\bar{X}\bar{Y}}{A^2}}\right) - \bar{X}\bar{Y}\right)$$

and

$$\beta(\bar{X}, \bar{Y}) = \bar{Y} - a(\bar{X}, \bar{Y})\bar{X} = \frac{2A^2}{\bar{X}}\left(1 - \sqrt{1 - \frac{\bar{X}\bar{Y}}{A^2}}\right).$$

When $\bar{X}\bar{Y} < A^2$, $a$ and $\beta$ are real and (23) is nothing but the Lorentzian solution which passes $(\bar{X}, \bar{Y})$ and is tangential to the hyperbola $XY = A^2$. (Note that the relation in (19) holds.) In this sense the wave function in this region describes the set of solutions (19) we start with. The relations $\partial_X S = \pi_X$ and $\partial_Y S = \pi_Y$ hold as is expected.

When $\bar{X}\bar{Y} > A^2$, however, $a$ and $\beta$ become complex. (23) is neither Lorentzian nor Euclidean. The solution intersects with the physical region $(X, Y; \text{real and positive})$.
only once when \( X = \bar{X} \) and \( Y = \bar{Y} \). Therefore the wave function does not admit ordinary classical interpretation.

In this example Hartle and Hawking's initial point \( X = Y = 0 \) is in the Lorentzian region. For convenience of the next section we construct a toy model in which this point lies in a Euclidean one. Suppose that the Hamiltonian (3) would have a form

\[
H = -6(XY)^{1/4}\left(\frac{1}{9} \pi_x \pi_y - 1\right).
\]

(24)

The last term comes from the spatial curvature and we would have obtained this Hamiltonian if we had started with the hyperbolic space. If one wishes, the closedness of the space could be recovered by considering the topology of the torus type. If we use (24), (4a) changes to

\[
X\dot{Y}/N^2 = +4(XY)^{1/2},
\]

and we get \( a = +4\beta^2/c^2 \) instead of (6). Therefore Lorentzian solutions and Euclidean solutions are interchanged, which is the only modification in the classical level due to (24).

Now it is easy to see how this change affects the wave function discussed in this section. The net effect is obtained by a simple replacement \( S \to iS \). The region \( XY < A^2 \) now becomes Euclidean and classically forbidden. (There is a controversy\(^8\) whether we should take \( e^{-|S|} \) or \( e^{-|S|} \) in the Euclidean region but we do not touch this problem here.) The wave function in the region \( XY > A^2 \) becomes

\[
\Psi(XY > A^2) = \frac{Y_0}{36} \left( \frac{Y}{X} \right)^{1/2} e^{i\theta},
\]

where

\[
\theta = \tan^{-1}\left(\frac{XY}{A^2} - 1\right)^{1/2},
\]

and \( C_\pm \) are constant coefficients. Though this \( \Psi \) is oscillatory in the region \( XY > A^2 \), it is not described by Lorentzian solutions but complex ones. Curiously this wave function takes a form which admits a simple classical interpretation when \( XY \gg A^2 \),

\[
\Psi(XY \gg A^2) = C_+ \sqrt{\frac{Y}{X}} e^{i\frac{1}{2}(XY + \frac{1}{2}\pi)} + C_- \sqrt{\frac{Y}{X}} e^{-i\frac{1}{2}(XY - \frac{1}{2}\pi)}.
\]

From the argument of § 3, we can say that this wave function describes a superposition of the universes \( Y = aX + \beta \) with \( a, \beta \) real and \( \beta \ll X, Y \), with the amplitude \( C_+ \sqrt{a} \) (expanding solutions) and \( C_- \sqrt{a} \) (contracting solutions). I am not sure whether this is an accidental result or not. It is true, however, that the saddle points which give this wave functions are unphysical complex solutions and it does not make sense to trace the above universes (with \( a, \beta \) real) to the region \( XY \approx A^2 \).

§ 6. The tunneling picture of the universe and the massive scalar model

Hartle and Hawking's proposal is closely related to Vilenkin's idea\(^2\) that the universe is created from nothing through the tunneling process. His idea is based on a pure gravity model. In this section we first explain this model in terms of the path integral
Quantum-Classical Correspondence in Wave Functions

The classical solution of the Einstein-Hilbert action with the cosmological constant $\Lambda$ in the closed space Robertson-Walker metric is obtained from

$$\frac{a^2}{N^2} + 1 - \frac{\Lambda}{3} a^2 = 0.$$  

(25)

Assuming $\dot{N} = 0$ again we get

$$a(\tau) = \sqrt{\frac{3}{\Lambda}} \cosh \sqrt{\frac{\Lambda}{3}} (\tau - \tau_0),$$

(26)

where $\tau \equiv N t$ and $\tau_0$ is an integration constant. $N$, $\tau$ and $\tau_0$ are complex in general. The path integral representation of the wave function $\Psi(a)$ of this model is

$$\Psi(a) = \int \mathcal{D} a dN \exp (i \int_0^T dt \mathcal{L}).$$

(27)

The variations of the action with respect to $a(t)$ and $N$ give Eq. (25) and therefore the saddle point for (27) is given by the classical solution (26). Then what will happen if we assume Hartle and Hawking’s condition in (27)? The relevant saddle point is such that $a = 0$ at $t = 0$ and $a = (a$ in $\Psi(a))$ at $t = T$. From the former we get

$$\tau_0 = i \frac{\pi}{2} \sqrt{\frac{3}{\Lambda}}.$$

The latter condition determines the value $N (= \tau / T)$. When $a < \sqrt{3/\Lambda}$ $N$ is imaginary while $N$ is complex when $a > \sqrt{3/\Lambda}$

$$NT = \tau_a + i \frac{\pi}{2} \sqrt{\frac{3}{\Lambda}},$$

where

$$\tau_a = \sqrt{\frac{3}{\Lambda}} \cosh^{-1} \left( \sqrt{\frac{\Lambda}{3}} a \right).$$

The path of $\tau$ is shown in Fig. 4 when $t$ varies from 0 to $T$ along the real axis. The saddle point $a(\tau)$ is a complex function which is real only at $t = 0$ and $t = T$. However $a(\tau)$ is an analytic function of $\tau$ and when we calculate the action, we can distort the path to the one which goes to $\tau_0$ first and later moves horizontally to $NT$. Along this path $a(\tau)$ is always real and positive. For $0 < |\tau| < |\tau_0|$ it corresponds to a Euclidean solution (i.e., a bounce solution) while, for $|\tau| > |\tau_0|$, it corresponds to a Lorentzian solution which is nothing but the de Sitter metric in the closed space. In this sense, we can say that the above wave

![Fig. 4](https://example.com/fig4.png)
function describes the de Sitter universe which is created at $a(\tau=\tau_0)=\sqrt{3/\Lambda}$ by the tunneling process from $a(\tau=0)=0$ (i.e., "nothing").

In Ref. 4) the authors discussed a model in which the massive minimal scalar field couples to the above system. In terms of the variables defined in (2) the Hamiltonian is written as

$$H = -6(XY)^{1/4}\left\{\frac{1}{9}\pi_x \pi_y + V\right\},$$

$$V = 1 - \frac{m^2}{16}(XY)^{1/2}\left(\log \frac{X}{Y}\right)^2.$$  (28)

An essential difference from the massless case is that the potential term changes its sign inside the physical region. The lines $V=0$ are shown in Fig. 5. In the third of Ref. 4) the authors solved the Wheeler-DeWitt equation numerically by using the boundary conditions on the light cone $XY=0$ given by Hartle and Hawking's initial condition (see below). The result shows that $\Psi$ is nonoscillatory in the region including $XY\approx 0$ and that $\Psi$ is oscillatory when $XY$ is large (see Fig. 5).

The emergence of the oscillatory region is understood at least when $Y/X$ (or $X/Y$) $\gg 1$ as follows: Consider the set of the classical solutions which start from Hartle and Hawking's initial point $X=Y=0$ and goes into the physical region. They are Euclidean solutions, because the mass term in (26) can be ignored when $XY\approx 0$ and $H$ is equal to that of the massless case. In the massless case which was discussed in the previous sections, these classical trajectories go straight without intersecting each other and cover the whole physical region. Therefore they define the whole wave function in the saddle point approximation, as was shown in Fig. 2. In the massive case, however, the classical solutions bend and intersect with each other as is depicted in Fig. 5. We can write the contours of the action from these Euclidean solutions (nearly?) up to the maximum of $XY = a^4$. However we cannot determine the action from them beyond their envelope, which becomes the boundary between the nonoscillatory region and the oscillatory one.

The nonoscillatory region which is given by the Euclidean solutions is an ordinary classically forbidden region. Then how can we interpret the oscillatory region? Does it describe a superposition of universes created by the tunneling? We suggest that this is not the case, in an ordinary sense at least.

In order that the above interpretation is possible, the wave function should be written as

$$\Psi \simeq \sum_i \exp(iS_i/h),$$

and, in the leading order in $h$, $\mathcal{P} S_i$ should coincide with Lorentzian solutions. $S_i$ in the leading order is calculated from classical solutions which give the stationary phase
Quantum-Classical Correspondence in Wave Functions

(saddle points) for the path integral formula. By considering the above pure gravity example, it is natural to believe that the saddle points in the oscillatory region of the massive scalar model are given by complex classical solutions, as was also the case in the example of the previous section. (Note also that there is a similarity between the two examples; their Euclidean regions, or at least a part of them, are bounded by an envelope of the Euclidean trajectories.) If there were only one physical degree of freedom, a complex solution were transformed to a sum of a Lorentzian and a Euclidean one by distorting the path of $Nt$ in its complex plane. This was in fact the case in the pure gravity example and $\partial_a S$ is nothing but the classical de Sitter solution. When there are two or more physical degrees of freedom, however, this trick fails. We cannot put two complex functions into the physical region simultaneously by distorting the path. In the example of the previous section, for example, $\mathcal{P} S$ in the oscillatory region was not real and did not give Lorentzian solutions.

Another feature which indicates the difference between the pure gravity model and a model with multiple physical degrees of freedom is that, in the former, the Euclidean solution bounces exactly where the potential vanishes ($a=\Lambda/3$) while in the latter Euclidean solutions cross the line $V=0$ because the multiple kinetic terms do not vanish simultaneously and therefore the boundary of the oscillatory region does not coincide with $V=0$. As a result, in the former, the Euclidean and Lorentzian solutions are smoothly connected at $a=\sqrt{\Lambda}/3$ (i.e., $a=0$ for both solutions). In the latter, on the other hand, they are not smoothly connected in the minisuperspace, since $\dot{X}\dot{Y}$ is nonvanishing (because $V \neq 0$) and its sign is opposite (because $dt^2=-(d\tau)^2$).

From these considerations we speculate that the oscillatory region in the massive minimal scalar model is not Lorentzian and a straightforward classical interpretation is not possible. However there is a possibility that a classical interpretation is recovered in a certain asymptotic region as was the case in the previous section. This is sufficient in practice because the WKB approximation itself is not good in the region $V \approx 0$, even if $\mathcal{P} S$ is real.

§ 7. Conclusion

In this paper we discussed some examples of wave functions of the universe in the semiclassical limit. The classical interpretation of the examples shown here are the following:

1. Superposition of universes (§ 3). When the action is real, it is given by (classical) Lorentzian solutions. When the action is imaginary, it is given by Euclidean solutions.
2. A special case of (1) (§ 4). The action is imaginary and the whole physical region is classically forbidden. The wave function describes 'no universe'.
3. In some region the action is real (imaginary) and is given by Lorentzian solutions (Euclidean solutions). Elsewhere the action is complex and is given by complex classical solutions. In this region the wave function is oscillatory but does not describe classical universes in an ordinary sense (§ 5). There is a possibility, however, that the interpretation as a superposition of universes is recovered in an asymptotic region.
4. The action in some region is given by complex solutions which, however, can be interpreted as a sum of Euclidean solutions and Lorentzian solutions. Therefore the wave
function describes a classical universe (the first example in § 6).

We speculate that the wave function of the massive minimal scalar model in Ref. 4) belongs to (3).

The author acknowledges stimulating discussion with H. Kodama, S. Watamura and T. Yoneya.

References

1) For a review, see
6) S. Wada, Class. Quantum Grav. 2 (1985), L57.
7) S. Wada, University of Tokyo-Komaba Preprint (1985).