Global Carleman Estimates for Solutions of Parabolic Systems Defined by Transposition and Some Applications to Controllability

Enrique Fernández-Cara and Sergio Guerrero

1 Introduction

Let \( \Omega \subset \mathbb{R}^N \) (\( N \geq 1 \)) be a bounded connected open set whose boundary \( \partial \Omega \) is regular enough (e.g., \( \partial \Omega \in C^2 \)). Let \( \omega \subset \Omega \) be a (small) nonempty open subset and let \( T > 0 \). We will use the notation \( Q = \Omega \times (0, T) \) and \( \Sigma = \partial \Omega \times (0, T) \) and we will denote by \( n(x) \) the outward unit normal to \( \Omega \) at the point \( x \in \partial \Omega \).

In this paper we deal with the controllability properties of some nonlinear parabolic equations for which the nonlinear terms are related to time derivatives and/or second-order spatial derivatives. As usual, we will first restrict ourselves to similar linear systems and then we will be concerned with the original nonlinear problems.

For the controllability analysis of the linear systems, the main tool will be a new Carleman estimate that holds for very weak solutions, that is, for solutions that only belong to \( L^2(Q) = L^2(0, T; L^2(\Omega)) \), of appropriate linear parabolic systems. The sense we will give to these solutions comes from the formulation by transposition of the corresponding systems.

More precisely, let us assume that \( \varphi^0 \in H^{-1}(\Omega) \) and \( f, F, G, \) and \( H \) satisfy

\[
f \in L^2(Q), \quad F \in L^2(Q)^N,
\]

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where $D(-\Delta) = H^2(\Omega) \cap H^1_0(\Omega)$ is the domain of the usual Laplace-Dirichlet operator in $\Omega$. Let us mention that by the notation $\sum_{i,j=1}^{N} \partial_{ij} H_{ij}$ we mean the distribution given by

$$\left\langle \sum_{i,j=1}^{N} \partial_{ij} H_{ij}, u \right\rangle = \sum_{i,j=1}^{N} \int_{Q} H_{ij} \partial_{ij} u \, dx \, dt \quad \forall u \in D(Q).$$

Then, by the last condition in (1.3) we mean that one can also consider duality products between $\sum_{i,j=1}^{N} \partial_{ij} H_{ij}$ and functions $u$ which are in the space $L^2(0, T; D(-\Delta))$ in the sense that there exists $\tilde{H} \in L^2(0, T; H^{-1/2}(\partial \Omega))$ such that

$$\left\langle \sum_{i,j=1}^{N} \partial_{ij} H_{ij}, u \right\rangle = \sum_{i,j=1}^{N} \int_{Q} H_{ij} \partial_{ij} u \, dx \, dt$$

$$- \int_{0}^{T} \left\langle \tilde{H}, \frac{\partial u}{\partial n} \right\rangle_2 \, dt \quad \forall u \in L^2(0, T; D(-\Delta)),$$

where $\langle \cdot, \cdot \rangle_2$ stands for the duality product between the spaces $H^{-1/2}(\partial \Omega)$ and $H^{1/2}(\partial \Omega)$. Notice that, if the functions $H_{ij}$ ($1 \leq i, j \leq N$) were smooth, we would have

$$\sum_{i,j=1}^{N} H_{ij} n_i n_j = \tilde{H}.$$  

For the sake of simplicity, in the sequel we will denote $\sum_{i,j=1}^{N} H_{ij} n_i n_j$ instead of $\tilde{H}$.

Then, let us consider the following backwards in time system:

$$- \varphi_t - \Delta \varphi = f - \nabla \cdot F + \sum_{i,j=1}^{N} \partial_{ij} H_{ij} - G_t \quad \text{in } Q,$$

$$\varphi = 0 \quad \text{on } \partial \Omega,$$

$$\varphi(T) = \varphi^0 \quad \text{in } \Omega.$$ 

Under the previous assumptions on the data $\varphi^0$, $f$, $F$, $G$, and $H$, it will be said that $\varphi \in L^2(Q)$ is the (unique) solution by transposition of the parabolic system (1.7) if, for
every $h \in L^2(Q)$, we have
\[
\int_{Q} \varphi h \, dx \, dt = \langle \varphi^0 - G(T), z(T) \rangle_{H^{-1},H^1_0} + \int_{Q} fz \, dx \, dt \\
+ \int_{Q} F \cdot \nabla z \, dx \, dt + \sum_{i,j=1}^{N} \int_{Q} H_{ij} \partial_{ij} z \, dx \, dt \\
- \int_{0}^{T} \left( \sum_{i,j} H_{ij} n_i n_j \frac{\partial z}{\partial n} \right) \, dt + \int_{Q} Gz \, dx \, dt,
\]
where $z$ is the solution of
\[
\begin{align*}
z_t - \Delta z &= h \quad \text{in } Q, \\
z &= 0 \quad \text{on } \partial \Omega, \\
z(0) &= 0 \quad \text{in } \Omega.
\end{align*}
\]
In (1.8), $\langle \cdot , \cdot \rangle_{\partial \Omega}$ stands for the standard duality product coupling the spaces $H^{-1/2}(\partial \Omega)$ and $H^{1/2}(\partial \Omega)$.

From (1.1)–(1.3) and classical regularity properties of the heat equation, it is not difficult to conclude that the previous definition makes sense and there exists a positive constant $C$ such that
\[
\| \varphi \|_{L^2(Q)} \leq C \left( \| f \|_{L^2(Q)} + \| F \|_{L^2(Q)}^N + \| G \|_{L^2(Q)} + \| G(T) \|_{H^{-1}(\Omega)} \right. \\
+ \| H \|_{L^2(Q)}^N + \sum_{i,j=1}^{N} \| H_{ij} n_i n_j \|_{L^2(0,T;H^{-1/2}(\partial \Omega))} + \left. \| G \|_{L^2(Q)}^N + \sum_{i,j=1}^{N} \| H_{ij} n_i n_j \|_{L^2(Q)}^N \right). \tag{1.10}
\]
Observe that the right-hand side of the equation in (1.7) belongs to $L^2(0,T;H^{-2}(\Omega)) + H^{-1}(0,T;L^2(\Omega))$. Consequently, it is reasonable to expect that the solution of 1.7 belongs (only) to $L^2(Q)$.

We will need the weight functions $\alpha$, $\xi$, $\alpha^*$, and $\xi^*$. We set
\[
\begin{align*}
\alpha(x,t) &= \frac{e^{2m\lambda \eta^0} \| \theta \|_{L^\infty} - e^{\lambda(m \eta^0 + \eta^0(x))} t(T-t)}{t(T-t)}, \\
\xi(x,t) &= \frac{e^{\lambda(m \eta^0 + \eta^0(x))} t(T-t)}{t(T-t)},
\end{align*}
\]
where $m > 1$ is a fixed number. Here, $\lambda \geq 1$ is a (sufficiently large) parameter to be chosen below and $\eta^0 = \eta^0(x)$ is a function satisfying
\[
\begin{align*}
\eta^0 &\in C^2(\bar{\Omega}), \quad \eta^0(x) > 0 \quad \text{in } \Omega, \\
\eta^0(x) &= 0 \quad \text{on } \partial \Omega, \quad |\nabla \eta^0(x)| > 0 \quad \text{in } \overline{\Omega \setminus \omega'}, \tag{1.13}
\end{align*}
\]
where $\omega' \subset \subset \omega$ is a nonempty open set. The existence of a function $\eta^0$ satisfying (1.13) is proved in [5]. This particular structure of the weights $\alpha$ and $\xi$ has already been used in several works; see [5, 6].

We will also need the functions $\alpha^*$ and $\xi^*$, with

$$
\alpha^*(t) = \max_{x \in \Omega} \alpha(x, t)|_{\partial \Omega}, \quad \xi^*(t) = \min_{x \in \Omega} \xi(x, t)|_{\partial \Omega}.
$$

(1.14)

The first main result in this paper is a global Carleman inequality for the solution by transposition of (1.7). It is given in the following theorem.

Theorem 1.1. Assume that conditions (1.1)–(1.3) are fulfilled. Then there exist three constants $\overline{\lambda}, \overline{\sigma},$ and $\lambda$ only depending on $\Omega$ and $\omega$ such that, for each $\varphi^0 \in H^{-1}(\Omega)$, the associated solution by transposition of (1.7) satisfies

$$
s^3 \lambda^4 \iint_Q e^{-2s\alpha} \xi^3 |\varphi|^2 dx \, dt \\
\leq C \left( s^3 \lambda^4 \iint_{\omega \times (0, T)} e^{-2s\alpha} \xi^3 |\varphi|^2 dx \, dt \\
+ \iint_Q e^{-2s\alpha} |f|^2 dx \, dt + s^2 \lambda^2 \iint_Q e^{-2s\alpha} \xi^2 |f|^2 dx \, dt \\
+ s^4 \lambda^4 \iint_Q e^{-2s\alpha} (|H|^2 + |G|^2) dx \, dt \\
+ s^4 \lambda^4 \int_0^T e^{-2s\alpha^*} (\xi^*)^4 \left\| \sum_{i,j=1}^N H_{ij} n_i n_j \right\|_{H^{-1/2}(\partial \Omega)}^2 dt \right)
$$

(1.15)

for any $\lambda \geq \overline{\lambda}$ and any $s \geq \overline{s} = \overline{\sigma}(T + T^2)$. \hfill \Box

The proof of Theorem 1.1 is presented in Section 2. It is inspired in the arguments in [6]. In this reference, the authors prove a suitable global Carleman inequality for the weak solutions of heat equations, that is to say, the solutions associated to right-hand sides that belong to the space $L^2(0, T; H^{-1}(\Omega))$ and initial or final data that belong to $L^2(\Omega)$. Theorem 1.1 is an extension of that result as long as right-hand sides in $L^2(0, T; H^{-2}(\Omega))$ and $H^{-1}(0, T; L^2(\Omega))$ and final data in $H^{-1}(\Omega)$ are permitted.

We will deduce from the estimate (1.15) some new controllability results for nonlinear parabolic systems.
First, we will consider the following problem:

\[ \begin{align*}
  y_t - \Delta y - \varepsilon \sum_{i,j=1}^{N} g_{ij}(x,t;y,\nabla y) \partial_{ij} y &= \nu 1_{\omega} \quad \text{in } Q, \\
  y &= 0 \quad \text{on } \Sigma, \\
  y(0) &= y^0 \quad \text{in } \Omega,
\end{align*} \tag{1.16} \]

where \( y^0 \in H^1_0(\Omega) \). Here, we will assume that

\[ g_{ij} \in C^0(Q \times \mathbb{R} \times \mathbb{R}^N), \]

\[ |g_{ij}(x,t;u,w)| \leq C_1 \min \left( 1, \frac{1}{\log^{1/2} \left( \frac{1}{|u|} \right)} \right) \quad \forall (x,t,u,w) \in Q \times \mathbb{R} \times \mathbb{R}^N \tag{1.17} \]

for some positive constant \( C_1 \) and any \( i,j = 1, \ldots, N \).

The null controllability of (1.16) is established in our second main result.

**Theorem 1.2.** Assume that the functions \( g_{ij} \) satisfy (1.17) and let \( y^0 \in H^1_0(\Omega) \). There exists \( \varepsilon_1 \) only depending on \( \Omega, \omega, T, C_1 \), and \( \|y^0\|_{L^2} \) such that, if \( 0 < \varepsilon < \varepsilon_1 \), there exist controls \( \nu \in L^2(\omega \times (0,T)) \) and associated solutions of (1.16) satisfying

\[ y(T) = 0 \quad \text{in } \Omega. \tag{1.18} \]

The proof of Theorem 1.2 is given in Section 3. It relies on an appropriate application of Kakutani's fixed point theorem (see Theorem 3.3). To this end, we will first have to establish a null controllability result for a linear problem similar to (1.16) and this will be a consequence of the Carleman inequality stated in Theorem 1.1.

We will also analyze the null controllability of the nonlinear parabolic system

\[ \begin{align*}
  (1 + \varepsilon b(x,t;y,\nabla y)) y_t - \Delta y &= \nu 1_{\omega} \quad \text{in } Q, \\
  y &= 0 \quad \text{on } \Sigma, \\
  y(0) &= y^0 \quad \text{in } \Omega.
\end{align*} \tag{1.19} \]

We suppose now that

\[ b \in C^0(Q \times \mathbb{R} \times \mathbb{R}^N), \]

\[ |b(x,t,u,w)| \leq C_2 \min \left( 1, \frac{1}{\log^{1/2} \left( \frac{1}{|u|} \right)} \right) \quad \forall (x,t,u,w) \in Q \times \mathbb{R} \times \mathbb{R}^N. \tag{1.20} \]
Our third main result is the following.

**Theorem 1.3.** Assume that the function $b$ satisfies (1.20) and let $y^0 \in H^1_0(\Omega)$. There exists $\varepsilon_2$ only depending on $\Omega$, $\omega$, $T$, $C_2$, and $\|y^0\|_{L^2}$ such that, if $0 < \varepsilon < \varepsilon_2$, there exist controls $\nu \in L^2(\omega \times (0, T))$ and associated solutions of (1.19) satisfying (1.18). □

In fact, Theorem 1.3 is an easy consequence of Theorem 1.2. This will be explained below, in Section 4.

**Remark 1.4.** A similar analysis will permit us to prove the same results in Theorems 1.2 and 1.3 under the hypotheses

$$g_{ij} \in C^0(Q \times \mathbb{R} \times \mathbb{R}^N),$$

$$|g_{ij}(x, t; u, w)| \leq C_3 \min \left(1, \frac{1}{\log^{1/2} \left(\frac{1}{|u|}\right)}\right) \quad \forall (x, t, u, w) \in Q \times \mathbb{R} \times \mathbb{R}^N.$$  \hspace{1cm} (1.21)

**Remark 1.5.** There are several interesting and relevant open questions concerning the controllability of systems (1.16) and (1.19). For instance, can we impose hypotheses on the functions $g_{ij}$ and $b$ so that systems (1.16) and (1.19) are globally null controllable? Observe that the results stated in Theorems 1.2 and 1.3 are local in the sense that the smallness of $\varepsilon$ depends on the largeness of $\|y^0\|_{L^2}$ (see the proof of Theorem 1.2 in Section 3 for more details). On the other hand, can we obtain results similar to Theorems 1.2 and 1.3 with controls acting on (a part of) the boundary? To our knowledge, all this is unknown. The method presented in this paper does not seem in principle sufficiently powerful to deal with these questions.

## 2 Carleman estimates for $L^2$-solutions defined by transposition

In this section, we prove the Carleman inequality stated in Theorem 1.1.

**Remark 2.1.** Observe that the hypotheses imposed in Theorem 1.1 on $H$ and $G$ are exactly the same we need to give a sense to the definition of $\varphi$ as a function of $L^2(Q)$. In fact, we could have defined $\varphi$ in a different way and also get that $\varphi \in C^0([0, T]; H^{-1}(\Omega)).$

In the sequel, $C$ (resp., $K$) denotes generic positive constants only depending on $\Omega$ and $\omega$ (resp., $\Omega$, $\omega$, and $T$). Before starting the proof of Theorem 1.1, let us recall a lemma concerning Carleman estimates for the strong solutions of heat equations.
Lemma 2.2. There exist three constants \( \hat{\mathcal{C}}, \hat{\lambda}, \) and \( \hat{\sigma} \) only depending on \( \Omega \) and \( \omega \) such that, for any \( \lambda \geq \hat{\lambda} \) and any \( s \geq \hat{s} = \hat{\sigma}(T + T^2) \), the next inequality holds:

\[
- \frac{1}{s} \int_Q e^{-2s\alpha} \left( |q_t|^2 + |\Delta q|^2 \right) dx \, dt
+ s\lambda^2 \int_Q e^{-2s\alpha} \xi^2 q^2 dx \, dt + s^3 \int_Q e^{-2s\alpha} \xi^3 |q|^2 dx \, dt
\leq \hat{\mathcal{C}} \left( \int_Q e^{-2s\alpha} |q_t + \Delta q|^2 dx \, dt + s^3 \lambda^4 \int_{(0,T)} e^{-2s\alpha} \xi^3 |q|^2 dx \, dt \right)
\]

for all \( q \in C^2(Q) \) with \( q = 0 \) on \( \Sigma \).

The proof of this lemma can be found in [5]. For details on the dependence of the constants with respect to \( \lambda \) and \( T \), see [3].

Proof of Theorem 1.1. As mentioned above, the proof is inspired in the ideas in [6].

Let us start by introducing the following optimization problem:

\[
\begin{align*}
\text{Minimize} & \quad \frac{1}{2} \left( \int_Q e^{2s\alpha} |z|^2 + s^{-3}\lambda^{-4} \int_{(0,T)} e^{2s\alpha} \xi^{-3} |v|^2 dx \, dt \right) \\
\text{subject to} & \quad v \in L^2(\omega \times (0,T)), \\
& \quad z = s^3 \lambda^4 e^{-2s\alpha} \xi^3 \phi + v_1 \omega \quad \text{in } Q, \\
& \quad z = 0 \quad \text{on } \Sigma, \\
& \quad z(0) = 0, \quad z(T) = 0 \quad \text{in } \Omega.
\end{align*}
\]

As far as this problem is concerned, it is proved in [6] that the unique solution \((\hat{z}, \hat{v})\) of (2.2) satisfies the following estimate of the Carleman kind:

\[
- \frac{1}{s} \int_Q e^{-2s\alpha} \xi^{-2} |\nabla \hat{z}|^2 dx \, dt
+ \int_Q e^{2s\alpha} |z|^2 dx \, dt + s^{-3}\lambda^{-4} \int_{(0,T)} e^{2s\alpha} |v|^2 dx \, dt
\leq C s^3 \lambda^4 \int_Q e^{-2s\alpha} \xi^3 |\phi|^2 dx \, dt.
\]

The main tool to prove (2.3) is Lemma 2.2. For the sake of completeness, we present the proof of (2.3) in the appendix, at the end of the paper.

We divide the rest of the proof of Theorem 1.1 in two steps. In the first one, we will prove that two additional terms can be added to the left-hand side of (2.3): a term involving the time derivative of \( \hat{z} \) and a second term concerning the second-order derivatives in space. This will lead to an improvement of (2.3); see (2.17).
In the second step, we will use Young’s inequality in order to deduce the estimate (1.15) from (2.17).

Step 1 (improvement of (2.3)). Let us consider the system fulfilled by \( \hat{z} \) and \( \hat{v} \):

\[
\hat{z}_t - \Delta \hat{z} = s^3 \lambda^4 e^{-2s\alpha} \xi^3 \varphi + \hat{v}_1 \omega \quad \text{in Q},
\]
\[
\hat{z} = 0 \quad \text{on} \Sigma,
\]
\[
\hat{z}(0) = 0, \quad \hat{z}(T) = 0 \quad \text{in} \Omega.
\]

(2.4)

The partial differential equation satisfied by \( \hat{z} \) can be multiplied by the function \( s^{-4} \lambda^{-4} e^{2s\alpha} \xi^{-4} \hat{z}_t \) and integrated over Q. We obtain the following:

\[
s^{-4} \lambda^{-4} \int_Q e^{2s\alpha} \xi^{-4} |\hat{z}_t|^2 \, dx \, dt = s^{-4} \lambda^{-4} \int_Q e^{2s\alpha} \xi^{-4} \hat{z}_t \Delta \hat{z} \, dx \, dt + s^{-1} \int_Q e^{2s\alpha} \xi^{-4} \hat{z}_t \hat{z} \varphi \, dx \, dt \]
\[
+ s^{-4} \lambda^{-4} \int_{\omega \times (0,T)} e^{2s\alpha} \xi^{-4} \hat{z}_t \hat{v} \, dx \, dt.
\]

(2.5)

The last two terms in the right-hand side can be estimated as follows:

\[
s^{-1} \int_Q \xi^{-1} \hat{z}_t \varphi \, dx \, dt \leq \frac{1}{4} s^{-4} \lambda^{-4} \int_Q e^{2s\alpha} \xi^{-4} |\hat{z}_t|^2 \, dx \, dt + C s^{2} \lambda^{4} \int_Q e^{-2s\alpha} \xi^{-2} |\varphi|^2 \, dx \, dt,
\]

\[
s^{-4} \lambda^{-4} \int_{\omega \times (0,T)} e^{2s\alpha} \xi^{-4} \hat{z}_t \hat{v} \, dx \, dt \leq \frac{1}{4} s^{-4} \lambda^{-4} \int_Q e^{2s\alpha} \xi^{-4} |\hat{z}_t|^2 \, dx \, dt
\]
\[
+ C s^{-4} \lambda^{-4} \int_{\omega \times (0,T)} e^{2s\alpha} \xi^{-4} |\hat{v}|^2 \, dx \, dt.
\]

(2.6)

The last integrals in the inequalities (2.6) can be easily bounded using (2.3), provided we take \( s \geq C T^2 \). Indeed, for such a choice, we have

\[
s^{-1} \xi^{-1} \leq C
\]

(2.7)
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for a positive constant $C$ independent of $T$, $s$, and $\lambda$. In this way we find that

$$s^{-4}\lambda^{-4} \int_{\mathcal{Q}} e^{2s\alpha \xi^{-4}} |\hat{z}_t|^2 \, dx \, dt \leq s^{-4}\lambda^{-4} \int_{\mathcal{Q}} e^{2s\alpha \xi^{-4}} \hat{z}_t \Delta \hat{z} \, dx \, dt$$

$$+ C \left( s^{-3}\lambda^{-4} \int_{\Omega \times (0,T)} e^{2s\alpha \xi^{-3}} |\hat{v}|^2 \, dx \, dt + s^{3}\lambda^{4} \int_{\mathcal{Q}} e^{2s\alpha \xi^{-3}} |\hat{\varphi}|^2 \, dx \, dt \right).$$

Let us now deal with the first term in the right-hand side of (2.8). We integrate by parts with respect to $x$ and we get

$$s^{-4}\lambda^{-4} \int_{\mathcal{Q}} e^{2s\alpha \xi^{-4}} \hat{z}_t \Delta \hat{z} \, dx \, dt = -\frac{1}{2} s^{-4}\lambda^{-4} \int_{\mathcal{Q}} e^{2s\alpha \xi^{-4}} \frac{\partial}{\partial t} |\nabla \hat{z}|^2 \, dx \, dt$$

$$- s^{-4}\lambda^{-4} \int_{\mathcal{Q}} (\nabla (e^{2s\alpha \xi^{-4}}) \cdot \nabla \hat{z}) \hat{z}_t \, dx \, dt.$$

It will be seen in the appendix that

$$t \rightarrow \left( \int_{\Omega} e^{-2s\alpha \xi^{-1}} |\nabla \hat{z}|^2 \, dx \right) (t)$$

belongs to $L^1(0,T)$. Consequently, we can integrate by parts with respect to $t$ in the first term of the right-hand side of (2.9). This yields

$$-\frac{1}{2} s^{-4}\lambda^{-4} \int_{\mathcal{Q}} e^{2s\alpha \xi^{-4}} \frac{\partial}{\partial t} |\nabla \hat{z}|^2 \, dx \, dt = \frac{1}{2} s^{-4}\lambda^{-4} \int_{\mathcal{Q}} (e^{2s\alpha \xi^{-4}})_t |\nabla \hat{z}|^2 \, dx \, dt. \quad (2.11)$$

From the definitions of the weight functions, we see that

$$\left| (e^{2s\alpha \xi^{-4}})_t \right| \leq CsTe^{2s\alpha \xi^{-2}} \quad (2.12)$$

if we take $s$ satisfying $s \geq C T^2$. Therefore,

$$-\frac{1}{2} s^{-4}\lambda^{-4} \int_{\mathcal{Q}} e^{2s\alpha \xi^{-4}} \frac{\partial}{\partial t} |\nabla \hat{z}|^2 \, dx \, dt \leq Cs^{-2}\lambda^{-4} \int_{\mathcal{Q}} e^{2s\alpha \xi^{-2}} |\nabla \hat{z}|^2 \, dx \, dt, \quad (2.13)$$

for $s \geq C(T + T^2)$.

In order to estimate the second term in (2.9), we take into account that

$$|\nabla (e^{2s\alpha \xi^{-4}})| \leq Cs\lambda e^{2s\alpha \xi^{-3}}, \quad (2.14)$$
for \( s \geq C T^2 \), so that using Young’s inequality we obtain

\[
s^{-4} \lambda^{-4} \int_Q (\nabla (e^{2s\alpha \xi^{-4}}) \cdot \nabla \hat{z}) \hat{z}_t \, dx \, dt \\
\leq \frac{1}{4} s^{-4} \lambda^{-4} \int_Q e^{2s\alpha \xi^{-4}} |\hat{z}|^2 \, dx \, dt + Cs^{-2} \lambda^{-2} \int_Q e^{2s\alpha \xi^{-2}} |\nabla \hat{z}|^2 \, dx \, dt. \tag{2.15}
\]

Now, we see from (2.8)–(2.15) that

\[
s^{-4} \lambda^{-4} \int_Q e^{2s\alpha \xi^{-4}} |\hat{z}|^2 \, dx \, dt \\
\leq C \left( s^{-2} \lambda^{-2} \int_Q e^{2s\alpha \xi^{-2}} |\nabla \hat{z}|^2 \, dx \, dt + s^{-3} \lambda^{-4} \int_{\omega \times (0,T)} e^{2s\alpha \xi^{-3}} |\hat{v}|^2 \, dx \, dt \right)
\]

\[
+ s^{-3} \lambda^{-4} \int_{\omega \times (0,T)} e^{2s\alpha \xi^{-3}} |\hat{v}|^2 \, dx \, dt \tag{2.16}
\]

for \( s \geq C(T + T^2) \) and \( \lambda \geq C \), which, combined with (2.3) and the differential equation satisfied by \( \hat{z} \), yields

\[
s^{-4} \lambda^{-4} \int_Q e^{2s\alpha \xi^{-4}} (|\hat{z}|^2 + |\Delta \hat{z}|^2) \, dx \, dt \\
+ s^{-2} \lambda^{-2} \int_Q e^{2s\alpha \xi^{-2}} |\nabla \hat{z}|^2 \, dx \, dt + \int_Q e^{2s\alpha \xi^{-2}} |\hat{z}|^2 \, dx \, dt \\
+ s^{-3} \lambda^{-4} \int_{\omega \times (0,T)} e^{2s\alpha \xi^{-3}} |\hat{v}|^2 \, dx \, dt \leq C s^3 \lambda^4 \int_Q e^{-2s\alpha \xi^3} |\hat{v}|^2 \, dx \, dt,
\]

for \( s \geq C(T + T^2) \) and \( \lambda \geq C \).

Step 2 (last arrangements and conclusion). First, we use that \( \phi \) is the solution by transposition of (1.7) and we take \( h = s^3 \lambda^4 e^{-2s\alpha \xi^3} \phi + \hat{v} 1_{\omega} \) in (1.8). This gives

\[
s^3 \lambda^4 \int_Q e^{-2s\alpha \xi^3} |\phi|^2 \, dx \, dt \\
= - \int_{\omega \times (0,T)} \phi \hat{v} \, dx \, dt + \int_Q f \hat{z} \, dx \, dt + \int_Q F \cdot \nabla \hat{z} \, dx \, dt \\
+ \sum_{i,j=1}^N \left( \int_Q H_{ij} \partial_i \hat{z} \, dx \, dt - \int_0^T \left< H_{ij} n_i n_j, \frac{\partial \hat{z}}{\partial n} \right>_{\partial \Omega} \, dt \right) \\
+ \int_Q G \hat{z}_t \, dx \, dt. \tag{2.18}
\]

As already said, it will suffice to apply Young’s inequality appropriately to the integrals in the right-hand side of (2.18). But, before doing this, we must prove that an estimate of the kind (2.17) also holds for \( \partial_{ij} \hat{z} \), for all \( i \) and \( j \).
Thus, let us set $\hat{w} = s^{-2}\lambda^{-2}e^{s\alpha}\xi^{-2}\hat{z}$. We have
\[
\int\int_{Q} |\Delta \hat{w}|^2 \, dx \, dt \\
\leq C \left( s^{-4}\lambda^{-4} \int\int_{Q} e^{2s\alpha}\xi^{-4}|\Delta \hat{z}|^2 \, dx \, dt \\
+ s^{-2}\lambda^{-2} \int\int_{Q} e^{2s\alpha}\xi^{-2}\nabla \hat{z}^2 \, dx \, dt + \int\int_{Q} e^{2s\alpha}|\hat{z}|^2 \, dx \, dt \right),
\]
for any $s \geq \sqrt{CT^2}$. Since $\hat{w} = 0$ on $\Sigma$, we deduce that
\[
\sum_{i,j=1}^{N} \int\int_{Q} |\partial_{ij}\hat{w}|^2 \, dx \, dt \leq C \int\int_{Q} |\Delta \hat{w}|^2 \, dx \, dt.
\]
Therefore, we also have
\[
\int_{0}^{T} \left\| \frac{\partial \hat{w}}{\partial n}(t) \right\|_{H^{1/2}(\partial\Omega)}^2 \, dt \leq C \int\int_{Q} |\Delta \hat{w}|^2 \, dx \, dt.
\]
From the expressions of $\partial_{ij}\hat{w}$ and $\partial \hat{w}/\partial n$ and the estimates (2.17), (2.19), (2.20), and (2.21), we also have
\[
s^{-4}\lambda^{-4} \int\int_{Q} e^{2s\alpha}\xi^{-4} \left( |\hat{z}|^2 + \sum_{i,j=1}^{N} |\partial_{ij}\hat{z}|^2 \right) \, dx \, dt \\
+ s^{-2}\lambda^{-2} \int\int_{Q} e^{2s\alpha}\xi^{-2}\nabla \hat{z}^2 \, dx \, dt + \int\int_{Q} e^{2s\alpha}|\hat{z}|^2 \, dx \, dt \\
+ s^{-4}\lambda^{-4} \int_{0}^{T} \left\| e^{2s\alpha}(\xi^*)^{-4} \left\| \frac{\partial \hat{z}}{\partial n} \right\|_{H^{1/2}(\partial\Omega)}^2 \right\| \, dt \\
+ s^{-3}\lambda^{-4} \int_{\omega \times (0,T)} e^{2s\alpha}|\hat{z}|^2 \, dx \, dt \leq Cs^{3}\lambda^{4} \int\int_{Q} e^{-2s\alpha}\xi^{3}|\phi|^2 \, dx \, dt,
\]
for $s \geq C(T + T^2)$ and $\lambda \geq C$ (recall that $\alpha^*$ and $\xi^*$ are given by (1.14)).

Now, combining this inequality and (2.18), we obtain the desired estimate (1.15) using Young's inequality several times.

This ends the proof of Theorem 1.1.

For our purposes in the next sections, we concentrate now on obtaining a Carleman inequality similar to (1.15) with weight functions not vanishing at $t = 0$. To this end, we will apply a classical argument that can be found, for instance, in [5].

In the sequel, we take $s = \bar{s}$ and $\lambda = \bar{\lambda}$, where $\bar{s}$ and $\bar{\lambda}$ are furnished by Theorem 1.1.
Let us consider the function
\[
\ell(t) = \begin{cases} 
T^2/4 & \text{for } 0 \leq t \leq T/2, \\
t(T - t) & \text{for } T/2 \leq t \leq T 
\end{cases}
\] (2.23)
and the following associated weights:
\[
\beta(x, t) = e^{2\lambda m\kappa^3} \frac{\ell(t)}{\ell(t)} - e^{\lambda (m\kappa^3 + \kappa^3(x))}, \quad \gamma(x, t) = e^{\lambda (m\kappa^3 + \kappa^3(x))} \frac{\ell(t)}{\ell(t)},
\] (2.24)
\[
\beta^*(t) = \max_{x \in \Omega} \beta(x, t), \quad \hat{\beta}(t) = \min_{x \in \Omega} \beta(x, t), \quad \gamma^*(t) = \min_{x \in \Omega} \gamma(x, t).
\]
Here, \(m\) is a positive number that can eventually be taken large enough.

We provide this new inequality in the following lemma.

**Lemma 2.3.** There exists a positive constant \(K\) only depending on \(\Omega, \omega, \) and \(T\) such that every solution of (1.7) satisfies
\[
\int_Q e^{-2s\beta^3} |\varphi|^2 \, dx \, dt 
\leq K \left( \int_{\omega \times [0, T]} e^{-2s\beta^3} |\varphi|^2 \, dx \, dt + \int_Q e^{-2s\beta^3} |f|^2 \, dx \, dt + \int_Q e^{-2s\beta^3} |G|^2 \, dx \, dt + \int^T_0 e^{-2s\beta^3} \sum_{i,j} H_{ij} n_i n_j |H_{ij}|_{H^{-1/2}(\partial \Omega)}^2 \, dt \right).
\] (2.25)

**Proof.** Let \(\eta \in C^1([0, T])\) be a “cut-off” function satisfying
\[
\eta(t) = 1 \quad \text{in } \left[0, \frac{T}{2}\right], \\
\eta(t) = 0 \quad \text{in } \left[\frac{3T}{4}, T\right], \\
|\eta'(t)| \leq \frac{C}{T}.
\] (2.26)
Then $\eta \varphi$ can be seen as the solution by transposition of (1.7) with right-hand side

$$-\eta' \varphi + \eta' G - \nabla \cdot (\eta F) + \sum_{i,j=1}^{N} \delta_{ij} (\eta H_{ij}) - (\eta G)_t$$

(2.27)

and initial condition zero. Analogously to (1.10), we find that

$$\|\varphi\|_{L^2(0,T/2;L^2(\Omega))} \leq C \left( \|f\|_{L^2(0,3T/4;L^2(\Omega))} + \|G\|_{L^2(0,3T/4;L^2(\Omega)^{N \times N})} + \|H\|_{L^2(0,3T/4;L^2(\Omega))} + \sum_{i,j=1}^{N} H_{ij} n_i n_j L^2(0,3T/4;H^{-1/2}(\partial \Omega)) \right)$$

(2.28)

As a consequence, we obtain a first estimate in $\Omega \times (0,T/2)$:

$$\int_0^{T/2} \int_{\Omega} e^{-2s\beta} \gamma^3 |\varphi|^2 dx \, dt$$

$$\leq K \left( \int_0^{3T/4} \int_{\Omega} e^{-2s\beta} |f|^2 dx \, dt + \int_0^{3T/4} \int_{\Omega} e^{-2s\beta} \gamma^2 |F|^2 dx \, dt 
+ \int_0^{3T/4} \int_{\Omega} e^{-2s\beta} (\gamma^*)^4 \left( \sum_{i,j=1}^{N} H_{ij} n_i n_j \right)^2 H^{-1/2}(\partial \Omega) dt 
+ \int_0^{3T/4} \int_{\Omega} e^{-2s\beta} \gamma^4 (|G|^2 + |H|^2) dx \, dt 
+ \int_{T/2}^{3T/4} \int_{\Omega} e^{-2s\beta} \gamma^3 |\varphi|^2 dx \, dt \right),$$

(2.29)

where $K = K(\Omega, \omega, T)$. Indeed, we have

$$\int_0^{T/2} \int_{\Omega} e^{-2s\beta} \gamma^3 |\varphi|^2 dx \, dt \leq e^{-2s\hat{\beta}(T/2)} \hat{\gamma} \left( \frac{T}{2} \right)^3 \int_0^{T/2} \int_{\Omega} |\varphi|^2 dx \, dt$$

(2.30)

and also

$$\int_0^{3T/4} \int_{\Omega} |g|^2 dx \, dt \leq e^{2s\hat{\gamma}(3T/4)} \gamma^* \left( \frac{3T}{4} \right)^{-\ell} \int_0^{3T/4} \int_{\Omega} e^{-2s\beta} \gamma^\ell |g|^2 dx \, dt,$$

(2.31)

for every $\ell > 0$ and $g \in L^2(Q)$. 
Moreover, since \( \alpha = \beta \) in \( \Omega \times (T/2, T) \), one has

\[
\int_{T/2}^{T} \int_{\Omega} e^{-2s\beta} \gamma^3 |\varphi|^2 \, dx \, dt = \int_{T/2}^{T} \int_{\Omega} e^{-2s\alpha} \xi^3 |\varphi|^2 \, dx \, dt
\]

(2.32)

so that, in view of the Carleman inequality (1.15), we also find that

\[
\int_{T/2}^{T} \int_{\Omega} e^{-2s\beta} \gamma^3 |\varphi|^2 \, dx \, dt \\
\leq K \left( \iint_{\omega \times (0, T)} e^{-2s\alpha} \xi^3 |\varphi|^2 \, dx \, dt \\
+ \iint_{Q} e^{-2s\beta} |f|^2 \, dx \, dt + \iint_{Q} e^{-2s\alpha} \xi^2 |F|^2 \, dx \, dt \\
+ \iint_{Q} e^{-2s\alpha} \xi^4 (|H|^2 + |G|^2) \, dx \, dt \\
+ \int_{0}^{T} e^{-2s\alpha} \gamma^4 \left( \sum_{i,j=1}^{N} H_{ij} \xi_{i} \xi_{j} \right) \, dt \right)
\]

(2.33)

Finally, from the definitions of \( \beta, \beta^*, \gamma, \) and \( \gamma^* \), it is clear that

\[
\int_{T/2}^{T} \int_{\Omega} e^{-2s\beta} \gamma^3 |\varphi|^2 \, dx \, dt \\
\leq K \left( \iint_{\omega \times (0, T)} e^{-2s\beta} \gamma^3 |\varphi|^2 \, dx \, dt \\
+ \iint_{Q} e^{-2s\beta} |f|^2 \, dx \, dt + \iint_{Q} e^{-2s\beta} \gamma^2 |F|^2 \, dx \, dt \\
+ \iint_{Q} e^{-2s\beta} \gamma^4 (|H|^2 + |G|^2) \, dx \, dt \\
+ \int_{0}^{T} e^{-2s \gamma^*} \left( \gamma^* \right)^4 \left( \sum_{i,j=1}^{N} H_{ij} \xi_{i} \xi_{j} \right) \, dt \right)
\]

(2.34)

This, together with (2.29), provides the desired inequality (2.25) for some \( K = K(\Omega, \omega, T) \).
3 Null controllability of (1.16)

In this section we prove Theorem 1.2, which concerns the null controllability of the nonlinear parabolic system

\[
y_t - \Delta y - \epsilon \sum_{i,j=1}^{N} g_{ij}(x, t; y, \nabla y) \partial_{ij} y = v_1 \omega \quad \text{in } Q,
\]

\[
y = 0 \quad \text{on } \Sigma,
\]

\[
y(0) = y^0 \quad \text{in } \Omega.
\]

(3.1)

Here, we assume that \( y^0 \in H^1_0(\Omega) \) and the hypotheses (1.17) on the functions \( g_{ij} \) hold.

On the other hand, \( \epsilon > 0 \) is supposed to be a small constant that may depend on \( \Omega, \omega, T, \) and the constant \( C_1 \) in (1.17). This will be made precise later.

Let us introduce the following associated linear problem:

\[
y_t - \Delta y - \epsilon \sum_{i,j=1}^{N} g_{ij}(x, t; z, \nabla z) \partial_{ij} y = v_1 \omega \quad \text{in } Q,
\]

\[
y = 0 \quad \text{on } \Sigma,
\]

\[
y(0) = y^0 \quad \text{in } \Omega,
\]

(3.2)

where \( z \in L^2(0, T; H^1_0(\Omega)), v \in L^2(\omega \times (0, T)), \) and \( y^0 \in H^1_0(\Omega). \) Let us recall that there exists \( \tau = \tau(\Omega, \omega, T, C_1) > 0 \) such that, for any \( \epsilon \) satisfying \( 0 < \epsilon < \tau, \) (3.2) possesses exactly one solution \( y \) that belongs to the space

\[
Y_2(Q) = L^2(0, T; D(-\Delta)) \cap H^1(0, T; L^2(\Omega)),
\]

(3.3)

where \( D(-\Delta) = H^2(\Omega) \cap H^1_0(\Omega) \) is the domain of the Laplace-Dirichlet operator. We also have

\[
\|y\|_{Y_2} \leq C(\|v_1 \omega\|_{L^2(Q)} + \|y^0\|_{L^2}).
\]

(3.4)

For the proof of these assertions, see, for instance, [7, page 349].
Next, we consider the following adjoint system with right-hand side $f \in L^2(Q)$ and final data $\varphi^0 \in H^{-1}(\Omega)$:

$$
- \varphi_t - \Delta \varphi - \varepsilon \sum_{i,j=1}^N \partial_{ij} \left( g_{ij}(x, t; z, \nabla z) \varphi \right) = f \quad \text{in } Q,
$$

$$
\varphi = 0 \quad \text{on } \partial \Omega,
$$

$$
\varphi(T) = \varphi^0 \quad \text{in } \Omega.
$$

(3.5)

We observe that (3.5) fits the structure of the general backwards system (1.7). By definition, the solution by transposition of (3.5) is the unique function $\varphi \in L^2(Q)$ satisfying

$$
\int_0^T \int_Q \varphi h \, dx \, dt = \langle \varphi^0, w(T) \rangle_{H^{-1}(\Omega), H_0^1(\Omega)} + \int_0^T \int_Q fh \, dx \, dt \quad \forall h \in L^2(Q),
$$

(3.6)

where $w$ is the solution of

$$
w_t - \Delta w - \varepsilon \sum_{i,j=1}^N g_{ij}(x, t; z, \nabla z) \partial_{ij} w = h \quad \text{in } Q,
$$

$$
w = 0 \quad \text{on } \partial \Omega,
$$

$$
w(0) = 0 \quad \text{in } \Omega.
$$

(3.7)

We have the following result.

**Lemma 3.1.** Assume that $f \in L^2(Q)$, the functions $g_{ij}$ verify (1.17), and $z$ is given in $L^2(0, T; H_0^1(\Omega))$. Also suppose that

$$
\|z(t)\|_{L^\infty} \leq M_1 e^{-M_2/(T-t)} \quad \forall t \in \left( \frac{3T}{4}, T \right),
$$

(3.8)

for some $M_1$ and $M_2$. Then, there exist positive constants $K_0 = K_0(\Omega, \omega, T)$ and $\varepsilon_0 = \varepsilon_0(\Omega, \omega, T, C_1, M_1, M_2)$ such that, whenever $0 < \varepsilon < \varepsilon_0$,

$$
\int_0^T \int_Q e^{-2s\theta} \gamma^3 |\varphi|^2 \, dx \, dt 
\leq K_0 \left( \int_{\omega \times (0, T)} e^{-2s\theta} \gamma^3 |\varphi|^2 \, dx \, dt + \int_Q e^{-2s\theta} |f|^2 \, dx \, dt \right),
$$

(3.9)

for all $\varphi^0 \in H^{-1}(\Omega)$. \hfill \Box

**Proof.** Let $\varphi^0 \in H^{-1}(\Omega)$ be given and let $\varphi$ be the associated solution by transposition of (3.5). Notice that $\varphi$ is also the solution by transposition of (1.7) with $F(x, t) \equiv 0$, 

Carleman estimates, transposition, and controllability

\[ G(x, t) \equiv 0, \text{ and} \]

\[ H_{ij}(x, t) = \varepsilon g_{ij}(x, t; z(x, t), \nabla z(x, t)) \varphi(x, t) \quad \forall i, j = 1, \ldots, N. \quad (3.10) \]

Thus, for every \( \tilde{h} \in L^2(Q) \), we have

\[
\iint_Q \varphi \tilde{h} \, dx \, dt = \langle \varphi^0, u(T) \rangle_{H^{-1}(\Omega), H^1(\Omega)}
+ \iint_Q f u \, dx \, dt + \varepsilon \sum_{i,j=1}^N \iint_Q g_{ij}(x, t; z, \nabla z) \partial_{ij} u \varphi \, dx \, dt,
\]

where \( u \) is the solution of

\[
\begin{align*}
    u_t - \Delta u &= \tilde{h} \quad \text{in } Q, \\
    u &= 0 \quad \text{on } \partial \Omega, \\
    u(0) &= 0 \quad \text{in } \Omega.
\end{align*}
\]

It is not difficult to check that we can apply Lemma 2.3 to \( \varphi \) and find that

\[
\iint_Q e^{-2sB \gamma^3 |\varphi|^2} \, dx \, dt
\leq K \left( \iint_{\omega \times (0, T)} e^{-2sB \gamma^3 |\varphi|^2} \, dx \, dt + \iint_Q e^{-2sB \gamma^2 |f|^2} \, dx \, dt \right)
+ \varepsilon^2 \sum_{i,j=1}^N \iint_Q e^{-2sB \gamma^4 |g_{ij}(x, t; z, \nabla z)|^2 |\varphi|^2} \, dx \, dt.
\]

Using the hypotheses (1.17) and the fact that \( ||z(t)||_{L^\infty} \leq M_1 e^{-M_2/(T-t)} \) in \( (3T/4, T) \), it is immediate that

\[
K \varepsilon^2 \sum_{i,j=1}^N \iint_Q e^{-2sB \gamma^4 |g_{ij}(x, t; z, \nabla z)|^2 |\varphi|^2} \, dx \, dt
\leq KC_2 \varepsilon^2 N^2 C_1^2 \iint_Q e^{-2sB (T-t)^{-3}} |\varphi|^2 \, dx \, dt
\leq \frac{1}{2} \iint_Q e^{-2sB \gamma^3 |\varphi|^2} \, dx \, dt,
\]

\[
\]
for some $C_2$ depending on $\Omega$, $\omega$, $T$, $M_1$, and $M_2$. Hence,

$$\sum_{i,j=1}^{N} \int_{Q} e^{-2s_\beta} \gamma^4 |g_{ij}(x,t;z,\nabla z)|^2 |\phi|^2 \ dx \ dt \leq \frac{1}{2} \int_{Q} e^{-2s_\beta} \gamma |\phi|^2 \ dx \ dt,$$  \hspace{1cm} (3.15)

provided $0 < \varepsilon < \varepsilon_0$ and $\varepsilon_0$ is sufficiently small (depending on $\Omega$, $\omega$, $T$, $C_1$, $M_1$, and $M_2$). As a conclusion, we obtain (3.9) with $K_0 = 2K$.  

From the estimate proved in Lemma 3.1, we can now deduce a null controllability result for the linear system (3.2). It is the following.

**Proposition 3.2.** Let the functions $g_{ij}(1 \leq i,j \leq N)$ verify (1.17) and let $z \in L^2(0,T; H_0^1(\Omega))$ satisfy (3.8). Set

$$\varepsilon_1 := \min \{\varepsilon, \varepsilon_0(\Omega, \omega, T, C_1, M_1, M_2)\}.$$  \hspace{1cm} (3.16)

Then there exist positive constants $R_0$, $R$, $R_1$, and $R_2$ only depending on $\Omega$, $\omega$, and $T$ such that, for every $\varepsilon \in (0, \varepsilon_1)$ and every $y^0 \in H_0^1(\Omega)$, there exist controls $v \in L^p(\omega \times (0,T))$ with $p > 1 + N/2$ and associated solutions $y$ of (3.2) satisfying

$$y(T) = 0 \quad \text{in} \quad \Omega,$$  \hspace{1cm} (3.17)

$$\|v\|_{L^p(\omega \times (0,T))} \leq R_0 \|y^0\|_{L^2},$$  \hspace{1cm} (3.18)

$$\|y\|_{Y_2} \leq R \|y^0\|_{H_0^1},$$  \hspace{1cm} (3.19)

$$\|y(t)\|_{L^\infty} \leq R_1 e^{-R_2/(T-t)} \|y^0\|_{L^2} \quad \forall t \in \left(\frac{3T}{4}, T\right).$$  \hspace{1cm} (3.20)

**Proof.** Let us first give an intuitive idea of the way we can find the couple $(y,v)$. We will follow the ideas in [5]. Thus, let us first introduce the solution $\chi$ of

$$\chi_t - \Delta \chi - \varepsilon \sum_{i,j=1}^{N} g_{ij}(x,t,z,\nabla z) \partial_{ij} \chi = 0 \quad \text{in} \quad Q,$$  \hspace{1cm} (3.21)

$$\chi = 0 \quad \text{on} \quad \Sigma,$$

$$\chi(0) = y^0 \quad \text{in} \quad \Omega.$$
Let \( \mu = \mu(t) \) be a function in \( C^1([0, T]) \) such that
\[
\begin{align*}
\mu(t) &\equiv 1 \quad \text{in } \left[0, \frac{T}{4}\right], \\
\mu(t) &\equiv 0 \quad \text{in } \left[\frac{3T}{4}, T\right].
\end{align*}
\]
(3.22)

We set \( y = \mu \chi + w \) and we try to find \( v \) and \( w \) with the regularity properties indicated above and satisfying
\[
\begin{align*}
\frac{\partial w}{\partial t} - \Delta w - \epsilon \sum_{i,j=1}^{N} g_{ij}(x, t, z, \nabla z) \partial_{ij} w &= -\mu' \chi + v \omega \quad \text{in } Q, \\
w &= 0 \quad \text{on } \Sigma, \\
w(0) &= 0 \quad \text{in } \Omega, \\
w(T) &= 0 \quad \text{in } \Omega.
\end{align*}
\]
(3.23)

Let us introduce the extremal problem
\[
\inf \left\{ \frac{1}{2} \left( \int_{Q} e^{2s\beta} |w|^2 \, dx \, dt + \int_{\omega \times (0, T)} e^{2s\beta} \gamma^{-3} |v|^2 \, dx \, dt \right) \right\}
\]
subject to \( v \in L^2(\omega \times (0, T)) \),
\[
\begin{align*}
\frac{\partial w}{\partial t} - \Delta w - \epsilon \sum_{i,j=1}^{N} g_{ij}(x, t, z, \nabla z) \partial_{ij} w &= -\mu' \chi + v \omega \quad \text{in } Q, \\
w &= 0 \quad \text{on } \Sigma, \\
w(0) &= 0, \quad w(T) = 0 \quad \text{in } \Omega,
\end{align*}
\]
(3.25)

where \( s = \bar{s}, \lambda = \bar{\lambda}, \) and \( \epsilon \) are respectively chosen as in Theorem 1.1 and Lemma 3.1.

Assume that (3.25) possesses a unique solution \((\hat{w}, \hat{v})\). Then, in view of Lagrange’s principle, there exists a dual variable \( \hat{p} \) such that
\[
\begin{align*}
\hat{w} &= e^{-2s\beta} \mathcal{L}^* \hat{p} \quad \text{in } Q, \\
\hat{v} &= -e^{-2s\beta} \gamma^3 \hat{p} \quad \text{in } \omega \times (0, T), \\
\hat{p} &= 0 \quad \text{on } \Sigma,
\end{align*}
\]
(3.26)

where \( \mathcal{L}^* \) is the (formally) adjoint operator of \( \mathcal{L} \), that is,
\[
\mathcal{L}^* p = -p_t - \Delta p - \epsilon \sum_{i,j=1}^{N} \partial_{ij}(g_{ij}(x, t, z, \nabla z)p).
\]
(3.27)
Let us now set
\[ P_0 = \{ p \in C^\infty(Q) : p = 0 \text{ on } \Sigma \}, \]  
\[ a(p_1, p_2) = \int_Q e^{-2s\beta} L^* p_1 L^* p_2 \, dx \, dt \]  
\[ + \int_{\omega \times (0,T)} e^{-2s\beta} \gamma^3 p_1 p_2 \, dx \, dt, \quad \forall p_1, p_2 \in P_0. \]  

Then, if the functions \( \hat{w} \) and \( \hat{v} \) given by (3.26) satisfy the parabolic problem in (3.25), we must have
\[ a(\hat{p}, q) = -\int_Q \mu' \chi q \, dx \, dt \quad \forall q \in P_0. \]  

The key idea in this proof is to demonstrate that there exists exactly one \( \hat{p} \) satisfying (3.30) in an appropriate class. We will then define \( \hat{w} \) and \( \hat{v} \) using (3.26) and we will check that \( (\hat{w}, \hat{v}) \) fulfills the desired properties.

Thus, consider the linear space \( P_0 \) and the bilinear form \( a(\cdot, \cdot) \) on \( P_0 \) defined by (3.29). Observe that the Carleman inequality (3.9) holds for all \( p \in P_0 \). Consequently,
\[ \int_Q e^{-2s\beta} \gamma^3 |p|^2 \, dx \, dt \leq K_0 a(p, p) \quad \forall p \in P_0 \]  
and, in particular, \( a(\cdot, \cdot) \) is a scalar product in \( P_0 \).

Let us now consider the space \( P \) given by the completion of \( P_0 \) for the norm associated to \( a(\cdot, \cdot) \) (which we denote by \( \| \cdot \|_P \)). This is a Hilbert space and the right-hand side of (3.30) defines a linear continuous form on \( P \). More precisely, in view of (3.9) and the definition of \( \chi \), it is clear that
\[ \left| \int_Q \mu' \chi p \, dx \, dt \right| \leq K_0^{1/2} e^{s\beta(3T/4)} \| y^0 \|_{L^2} \| p \|_P \quad \forall p \in P \]  
(recall that \( s = \bar{s} \) is fixed from Lemma 2.3). Hence, in view of Lax-Milgram's lemma, there exists one and only one \( \hat{p} \) satisfying
\[ a(\hat{p}, q) = -\int_Q \mu' \chi q \, dx \, dt \quad \forall q \in P, \quad \hat{p} \in P. \]  

Let us set
\[ \hat{w} = e^{-2s\beta} L^* \hat{p}, \quad \hat{v} = -e^{-2s\beta} \gamma^3 \hat{p} 1_\omega. \]
With these definitions, it is easy to check that \( \hat{w} \) and \( \hat{v} \) verify
\[
\int_Q e^{2s\beta}\hat{w}^2 \, dx \, dt + \int_{\omega \times (0,T)} e^{2s\beta} \gamma^{-2} |\hat{v}|^2 \, dx \, dt \leq K_0 e^{2s\beta} (3T/4) \|y_0\|_{L^2}^2
\] (3.35)
and solve the heat system in (3.23). Consequently, \( \hat{v} \) is a control in \( L^2(\omega \times (0,T)) \) that drives the state \( \hat{w} \) exactly to zero at time \( T \).

From this, it is readily seen that (3.19) holds for some positive \( R = R(\Omega, \omega, T) \). Let us see that, in fact, this control and this state are more regular. More precisely, let us show that \( \hat{v} \) and \( \hat{y} = \mu \chi + \hat{w} \), respectively, satisfy (3.18) and (3.20) for appropriate \( R_0, R_1, \) and \( R_2 \). Obviously, this will achieve the proof.

In the sequel, \( K_1, K_2, \ldots \) denote different constants only depending on \( \Omega, \omega, \) and \( T \). To fix ideas, we will assume that \( N \geq 3 \).

In view of (3.26), we see that \( \hat{p} \) satisfies
\[
-\hat{p}_t - \Delta \hat{p} - \varepsilon \sum_{i,j=1}^N \partial_{i,j} (g_{ij}(x, t; z, \nabla z)\hat{p}) = e^{2s\beta} \hat{w} \quad \text{in } Q,
\] (3.36)
\[
\hat{p} = 0 \quad \text{on } \Sigma.
\]

Let \( \delta > 0 \) be a small number and let us introduce the functions \( \eta_0, \eta_1, \ldots \in C^\infty([0, T]) \), with \( \eta_k \equiv 0 \) in \([0, (T/4) + k\delta] \), \( \eta_k \geq 0 \), and \( \eta_k \equiv 1 \) in \([3T/4, T] \).

Let us set \( q_0 = \eta_0 e^{s\beta} \gamma^{-2} \hat{w} \). We have
\[
q_{0,1} - \Delta q_0 - \varepsilon \sum_{i,j=1}^N g_{ij}(x, t; z, \nabla z)\partial_{i,j} q_0 = F_0 \quad \text{in } Q,
\] (3.37)
\[
qu_0 = 0 \quad \text{on } \Sigma,
\]
\[
qu_0(0) = 0 \quad \text{in } \Omega,
\]
where
\[
F_0 = -\eta_0 e^{s\beta} \gamma^{-2} \mu \chi + \eta_0 e^{s\beta} \gamma^{-2} \hat{v}_1 \omega + (\eta_0 e^{s\beta} \gamma^{-2})_i \hat{w} - 2\nabla \cdot (\eta_0 \hat{w} \nabla (e^{s\beta} \gamma^{-2})) + \eta_0 \Delta (e^{s\beta} \gamma^{-2}) \hat{w}
\]
\[
- \varepsilon \eta_0 \sum_{i,j=1}^N \left( -\partial_{i,j} (e^{s\beta} \gamma^{-2}) \hat{w} + \partial_i \left( \partial_j (e^{s\beta} \gamma^{-2}) \hat{w} \right) + \partial_j \left( \partial_j (e^{s\beta} \gamma^{-2}) \hat{w} \right) \right).
\] (3.38)

From (3.35), we deduce that \( F_0 \in L^2(0, T; H^{-1}(\Omega)) \) and consequently
\[
q_0 \in \bar{Y}_2(\Omega) := L^2(0, T; H^1(\Omega)) \cap H^1(0, T; H^{-1}(\Omega)) \hookrightarrow L^{2(N+2)/N}(Q),
\] (3.39)
where the imbedding is continuous. Furthermore,

$$\|q_0\|_Y \leq K_1 K_0^{1/2} e^{s_\beta y (3/4)} \|y^0\|_{L^2}.$$  \hspace{1cm} (3.40)

Notice that

$$\tilde{Y}_r(Q) := L^r(0, T; W^{1,r}(\Omega)) \cap W^{1,r}(0, T; W^{-1,r}(\Omega)) \subset L^{(N+2)/(N+2-r)}(Q),$$  \hspace{1cm} (3.41)

for \( r < N + 2 \). Furthermore, when \( r = N + 2 \), \( \tilde{Y}_r(Q) \) is imbedded in \( L^c(Q) \) for every finite \( c \) and, when \( r > N + 2 \), \( \tilde{Y}_r(Q) \) is imbedded in \( L^\infty(Q) \). All these imbeddings are continuous.

Then, let us introduce \( q_1 = \eta_1 e^{s_\beta (y-3)} \tilde{w} \). This function fulfills a system similar to (3.37), with right-hand side

$$F_1 = -\epsilon_1 e^{s_\beta (y-3)} \mu \chi + \eta_1 e^{s_\beta (y-3)} \partial_i w + (\eta_1 e^{s_\beta (y-3)})_t \tilde{w} - 2\eta_1 \Delta (e^{s_\beta (y-3)}) \tilde{w}$$

$$-\epsilon_1 \sum_{i,j=1}^{N} g_{ij} (\partial_i (e^{s_\beta (y-3)}) \tilde{w} + \partial_i (e^{s_\beta (y-3)}) \partial_j \tilde{w} + \partial_i (e^{s_\beta (y-3)}) \partial_j \tilde{w}).$$  \hspace{1cm} (3.42)

Thanks to the regularity of \( q_0 \), the fifth, seventh, and eighth term in the expression of \( F_1 \) belong to \( L^2(Q) \). The other terms, thanks to (3.35), also belong to \( L^2(Q) \). Consequently,

$$q_1 \in Y_2(Q) := L^2(0, T; H^2(\Omega)) \cap H^1(0, T; L^2(\Omega)) \subset L^{(N+2)/(N-2)}(Q),$$  \hspace{1cm} (3.43)

$$\|q_1\|_{L^{(N+2)/(N-2)}(Q)} \leq K_2 K_0^{1/2} e^{s_\beta y (3/4)} \|y^0\|_{L^2}.$$  \hspace{1cm} (3.44)

Notice also that

$$Y_r(Q) := L^r(0, T; W^{2,r}(\Omega)) \cap W^{1,r}(0, T; L^r(\Omega)) \subset L^{(N+2)/(N+2-r)}(Q),$$  \hspace{1cm} (3.45)

for \( r < 1 + N/2 \). Furthermore, when \( r = 1 + N/2 \), \( Y_r(Q) \) is continuously imbedded in \( L^c(Q) \) for every finite \( c \) and, when \( r > 1 + N/2 \), \( Y_r(Q) \) is continuously imbedded in \( L^\infty(Q) \).

Let us set \( \tilde{p}_1 = e^{-s_\beta (y-4)} \partial_i \). This function satisfies

$$-\partial_1 + \Delta \tilde{p}_1 - \epsilon \sum_{i,j=1}^{N} \partial_i (g_{ij}(x, t; z, \nabla z) \tilde{p}_1) = G_1 \text{ in } Q,$$

$$\tilde{p}_1 = 0 \text{ on } \Sigma,$$

$$\tilde{p}_1(T) = 0 \text{ in } \Omega,$$  \hspace{1cm} (3.46)
where
\[
G_1 = e^{s\beta} \gamma^{-4} \hat{\nu} + (e^{s\beta} \gamma^{-4}) \hat{p} - 2\nabla : (\hat{p} \nabla (e^{-s\beta} \gamma^{-4})) + \Delta (e^{-s\beta} \gamma^{-4}) \hat{p} - \varepsilon \sum_{i,j=1}^{N} ( - \partial_{ij} (e^{-s\beta} \gamma^{-4}) g_{ij} \hat{p} + \partial_{i} (\partial_{j} (e^{-s\beta} \gamma^{-4}) g_{ij} \hat{p}) + \partial_{j} (\partial_{i} (e^{-s\beta} \gamma^{-4}) g_{ij} \hat{p})).
\] (3.47)

Thanks to the Carleman inequality (3.9) and (3.35), \(G_1 \in L^2(0, T; H^{-1}(\Omega))\). Accordingly, \(\hat{p}_1\) can be viewed as the solution by transposition of (3.46) in the space \(L^{2(N+2)/N}(Q)\). In other words, \(\hat{p}_1\) is the unique function in \(L^{2(N+2)/N}(Q)\) satisfying
\[
\iint_{Q} \hat{p}_1 h \, dx \, dt = \iint_{Q} G_1 u \, dx \, dt \quad \forall h \in L^{2(N+2)/(N+4)}(Q),
\] (3.48)
where \(u\) is the solution of
\[
u t - \Delta u - \varepsilon \sum_{i,j=1}^{N} g_{ij}(x, t; z, \nabla z) \partial_{ij} u = h \quad \text{in } Q,
\]
\[
u = 0 \quad \text{on } \Sigma,
\]
\[
u(0) = 0 \quad \text{in } \Omega.
\] (3.49)
Furthermore,
\[
\|\hat{p}_1\|_{L^{2(N+2)/N}(Q)} \leq K_3 K_0^{1/2} e^{s\beta^{*}(3T/4)} \|y^0\|_{L^2}.
\] (3.50)

We now introduce the function \(q_2 = \eta_2 e^{s\beta} \gamma^{-4} \hat{\nu}\), which verifies a system similar to (3.37) with right-hand side
\[
F_2 = -\eta_2 e^{s\beta} \gamma^{-4} \mu \chi + \eta_2 e^{s\beta} \gamma^{-4} v_1 \omega + (\eta_2 e^{s\beta} \gamma^{-4}) \hat{\nu} \cdot \nabla \hat{\nu} + \eta_2 \Delta (e^{s\beta} \gamma^{-4}) \hat{\nu} - \varepsilon \eta_2 \sum_{i,j=1}^{N} g_{ij} (\partial_{ij} (e^{s\beta} \gamma^{-4}) \hat{\nu} + \partial_{i} (e^{s\beta} \gamma^{-4}) \partial_{j} \hat{\nu} + \partial_{j} (e^{s\beta} \gamma^{-4}) \partial_{i} \hat{\nu}).
\] (3.51)

From (3.50), the second term in the expression of \(F_2\) belongs to \(L^{2(N+2)/N}(Q)\). From (3.40), we deduce that the third, fifth, and sixth terms also belong to this space. Moreover, we see from (3.40) and (3.44) that the fourth term belongs to \(L^{2(N+2)/(N-2)}(0, T; W^{-1, 2(N+2)/(N-2)}(\Omega))\). Therefore,
\[
q_2 \in Y_{2(N+2)/N}(Q) + \bar{Y}_{2(N+2)/N-2}(Q) \hookrightarrow L^{2(N+2)/(N-4)}(Q),
\] (3.52)
\[
\|q_2\|_{L^{2(N+2)/(N-4)}(Q)} \leq K_4 K_0^{1/2} e^{s\beta^{*}(3T/4)} \|y^0\|_{L^2}.
\] (3.53)
Let us now introduce \( \hat{\gamma}_2 = e^{-s\beta}y^5\). This function verifies a system similar to (3.46) with

\[
G_2 = e^{s\beta}y^5\hat{\gamma} + (e^{s\beta}y^5)\gamma\hat{p} - 2\nabla : (\hat{\gamma} \nabla (e^{-s\beta}y^5)) + \Delta (e^{-s\beta}y^5)\hat{\gamma}
- \varepsilon \sum_{i,j=1}^N \left( - \partial_{ij}(e^{-s\beta}y^5)g_{ij}\hat{\gamma} + \partial_i(\partial_j(e^{-s\beta}y^5)g_{ij}\hat{\gamma}) + \partial_j(\partial_i(e^{-s\beta}y^5)g_{ij}\hat{\gamma}) \right).
\]

(3.54)

From (3.35), the first, second, fourth, and fifth terms belong to \( L^2(\Omega) \). Thanks to (3.50), the remaining two terms belong to \( L^2(N+2)/N(0,T;W^{1,2(N+2)/N}(\Omega)) \). Thus, as in the case of \( \hat{\gamma}_1, \hat{\gamma}_2 \) can be viewed as the solution by transposition in \( L^2(N+2)/(N-2)(\Omega) \) and

\[
\| \hat{\gamma}_2 \|_{L^2(N+2)/(N-2)(\Omega)} \leq K_5 K_0^{1/2} e^{s\beta^* (3T/4)} \| y^0 \|_{L^2}.
\]

(3.55)

Observe that any function in \( Y_2(\Omega) \) or \( \bar{Y}_{2(N+2)/N}(\Omega) \) belongs to the space \( L^2(N+2)/(N-2)(\Omega) \).

Proceeding in this way, after \( \ell \) steps we see that the function

\[
\hat{q}_\ell := \eta \gamma_\ell e^{s\beta}y^{-(\ell+2)}\hat{\gamma}
\]

(3.56)

belongs to the space \( L^2(N+2)/(N-2\ell)(\Omega) \) and

\[
\| q_\ell \|_{L^2(N+2)/(N-2\ell)(\Omega)} \leq K_2 \ell \gamma \ell e^{s\beta^* (3T/4)} \| y^0 \|_{L^2}.
\]

(3.57)

Then, introducing \( \hat{p}_\ell = \eta e^{-s\beta}y^{-(\ell+3)}\hat{\gamma} \), we have \( \hat{p}_\ell \in L^2(N+2)/(N-2(\ell-1))(\Omega) \) and

\[
\| \hat{p}_\ell \|_{L^2(N+2)/(N-2(\ell-1))(\Omega)} \leq K_{2\ell+1} \gamma \ell e^{s\beta^* (3T/4)} \| y^0 \|_{L^2}.
\]

(3.58)

Obviously, if \( \ell \) is large enough (depending on \( N \)), we have \( q_\ell \in L^\infty(\Omega) \) and

\[
\| q_\ell \|_{L^\infty(\Omega)} \leq K_{2\ell} \gamma \ell e^{s\beta^* (3T/4)} \| y^0 \|_{L^2}.
\]

(3.59)

In particular, we see that \( \hat{\gamma} \in L^p(\omega \times (0,T)) \) with \( p > 1 + N/2, e^{s\beta}y^{-(\ell_0+3)}\hat{\gamma} \in L^\infty(3T/4, T; L^\infty(\Omega)) \) and, furthermore, the following estimates hold for some positive constants only depending on \( \Omega, \omega, \) and \( T \):

\[
\| \hat{\gamma} \|_{L^p(\omega \times (0,T))} \leq R_0 \| y^0 \|_{L^2}
\]

(3.60)
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\[ \left\| e^{s \beta} \gamma^{-(I_0 + 3)} e \right\|_{L^\infty(3T/4, T; L^\infty(\Omega))} \leq \bar{K} \left\| y^0 \right\|_{L^2}, \]  

(3.61)

for some \( \bar{K}(\Omega, \omega, T) > 0 \).

From this last estimate, we deduce that

\[ \left\| \hat{y}(t) \right\|_{L^\infty} \leq \bar{K}e^{-s \beta(t)} \left\| y^0 \right\|_{L^2} \quad \forall t \in \left( \frac{3T}{4}, T \right), \]  

(3.62)

for some \( \bar{K}(\Omega, \omega, T) > 0 \).

Consequently, there exist \( R_1 \) and \( R_2 \) such that \( \hat{y} \) satisfies (3.20).

This ends the proof of Proposition 3.2.

We can now give the proof of Theorem 1.2. As in many other previous works dealing with the controllability of nonlinear systems, we will use a fixed-point argument. This strategy was introduced in [8] in the context of the exact controllability of the semilinear wave equation. See also [2, 4, 5] for other similar results concerning the approximate and null controllability of semilinear heat equations with Dirichlet or Neumann boundary conditions.

Thus, let \( y^0 \) be given in \( H^1_0(\Omega) \) and let \( \varepsilon_1 \) be the (possibly small) constant associated given by (3.16) with \( M_1 = R_1 \left\| y^0 \right\|_{L^2} \) and \( M_2 = R_2 \) (where \( R_1 \) and \( R_2 \) are furnished by Proposition 3.2). Let us set

\[ X = L^2(0, T; H^1_0(\Omega)) \]  

and, for each \( z \in X \), let us denote by \( \tilde{z} \) the function defined as follows:

\[ \tilde{z}(x, t) = \min \left( \max \left( z(x, t), -R_1 \left\| y^0 \right\|_{L^2} e^{-R_2/(T-t)} \right), R_1 \left\| y^0 \right\|_{L^2} e^{-R_2/(T-t)} \right). \]  

(3.63)

Let \( A(z) \) be the family of all controls \( v \in L^p(\omega \times (0, T)) \) which satisfy the estimate (3.18) and drive the solution of (3.2) with \( z \) replaced by \( \tilde{z} \) to zero at time \( T \). In view of Proposition 3.2, this set is not empty.

Let us now introduce the set-valued mapping \( \Lambda : X \rightarrow X \), with

\[ \Lambda(z) = \{ y : (y, v) \text{ solves (3.2) with } z \text{ replaced by } \tilde{z}, \ v \in A(z), \ y \text{ satisfies (3.19)} \} \]  

(3.64)

for all \( z \in X \).

To prove Theorem 1.2, it will suffice to show that \( \Lambda \) possesses at least one fixed point. To this purpose, we will use Kakutani’s theorem, that we recall now in the following theorem.
Theorem 3.3. Let $Z$ be a Banach space and let $\Lambda : Z \rightrightarrows Z$ be a set-valued mapping satisfying the following assumptions.

1. $\Lambda(z)$ is a nonempty closed convex set of $Z$ for every $z \in Z$.
2. There exists a nonempty convex compact set $B \subset Z$ such that $\Lambda(B) \subset B$.
3. $\Lambda$ is upper-hemicontinuous in $Z$, that is, for each $\sigma \in Z'$ the single-valued mapping

$$z \mapsto \sup_{y \in \Lambda(z)} \langle \sigma, y \rangle_{Z', Z}$$

is upper-semicontinuous.

Then $\Lambda$ possesses a fixed point in the set $B$, that is, there exists $z \in B$ such that $z \in \Lambda(z)$.

For a proof of this result, see, for instance, [1].

We apply Kakutani's theorem with $Z = X$ and

$$B = \left\{ z \in Y_2(Q) : \| z(t) \|_{L^\infty} \leq R_1 e^{-R_2/(T-t)} \text{ in } (\frac{3T}{4}, T) \right\}.$$  \hspace{1cm} (3.66)

In view of Proposition 3.2, $\Lambda(z)$ is a nonempty closed convex subset of $X$ for every $z \in X$ and $\Lambda(B) \subset B$. Furthermore, since $Y_2(Q) \hookrightarrow X$ with a compact embedding, $B$ is a convex compact subset of $X$. Consequently, in order to apply Theorem 3.3, all we have to do is to check that $\Lambda$ is upper-hemicontinuous.

Thus, assume that $\sigma \in Z'$ is given and let $\{z_n\}$ be a sequence satisfying $z_n \rightarrow z_0$ strongly in $X$. We must prove that

$$\limsup_{n \rightarrow +\infty} \left( \sup_{w \in \Lambda(z_n)} \langle \sigma, w \rangle_{Z', Z} \right) \leq \sup_{w \in \Lambda(z_0)} \langle \sigma, w \rangle_{Z', Z}.$$  \hspace{1cm} (3.67)

Let $\{z_{n'}\}$ be a subsequence of $\{z_n\}$ such that

$$\limsup_{n \rightarrow +\infty} \sup_{w \in \Lambda(z_n)} \langle \sigma, w \rangle_{Z', Z} = \lim_{n' \rightarrow +\infty} \sup_{w \in \Lambda(z_{n'})} \langle \sigma, w \rangle_{Z', Z}.$$  \hspace{1cm} (3.68)

Since each $\Lambda(z_{n'})$ is a compact set of $X$, for every $n'$, we have

$$\sup_{w \in \Lambda(z_{n'})} \langle \sigma, w \rangle_{Z', Z} = \langle \sigma, w_{n'} \rangle_{Z', Z}$$  \hspace{1cm} (3.69)

for some $w_{n'} \in \Lambda(z_{n'})$. On the other hand, since all the states $w_{n'}$ belong to $Y_2(Q)$ (which is compactly imbedded in $X$), at least for a new subsequence (again indexed by $n'$) we must have $w_{n'} \rightarrow w_0$ strongly in $X$. 


In order to conclude, we have to prove that \( w_0 \in \Lambda(z_0) \).

First, the weak \( L^2 \)-limit of \( g_{ij}(x, t; z_n', \nabla z_n') \) is \( g_{ij}(x, t, z_0, \nabla z_0) \), since \( z_n' \to z_0 \) strongly in the space \( L^2(0, T; H^1_0(\Omega)) \) (observe that we are using here the continuity of the functions \( g_{ij} \)).

Moreover, in view of the estimate (3.18), we can assume that \( v_n' \) converges to a function \( v_0 \) weakly in \( L^2(\omega \times (0, T)) \). Then, \( w_0 \) solves (3.2) with \( z \) replaced by \( z_0, v = v_0 \) and we have \( w_0(T) = 0 \). Consequently, it is immediate that \( w_0 \) is the solution to (3.2) associated to the control \( v_0 \).

Furthermore, \( v_0 \) also satisfies (3.18) and so \( v_0 \in \Lambda(z_0) \). This shows that \( w_0 \in \Lambda(z_0) \) and, therefore, \( \Lambda \) is upper hemicontinuous.

Hence, the set-valued mapping \( \Lambda \) possesses at least one fixed point. This ends the proof of Theorem 1.2.

### 4 Null controllability of (1.19)

This section is devoted to the proof of Theorem 1.3, which deals with the null controllability of the nonlinear problem

\[
(1 + \varepsilon b(x, t; y, \nabla y)) y_t - \Delta y = v1_\omega \quad \text{in } Q,
\]
\[
y = 0 \quad \text{on } \Sigma, 
\]
\[
y(0) = y^0 \quad \text{in } \Omega. 
\]

We recall that, here, we are assuming that \( b \) satisfies (1.20), \( \varepsilon > 0 \) is a small constant, and \( y^0 \in H^1_0(\Omega) \).

First, notice that (4.1) can be (at least formally) rewritten in the form

\[
y_t - \Delta y + \varepsilon \frac{b(x, t; y, \nabla y)}{1 + \varepsilon b(x, t; y, \nabla y)} \Delta y = \frac{1}{1 + \varepsilon b(x, t; y, \nabla y)} v1_\omega \quad \text{in } Q, 
\]
\[
y = 0 \quad \text{on } \Sigma, 
\]
\[
y(0) = y^0 \quad \text{in } \Omega. 
\]

Let us consider the auxiliary system

\[
y_t - \Delta y + \varepsilon \frac{b(x, t; y, \nabla y)}{1 + \varepsilon b(x, t; y, \nabla y)} \Delta y = u1_\omega \quad \text{in } Q, 
\]
\[
y = 0 \quad \text{on } \Sigma, 
\]
\[
y(0) = y^0 \quad \text{in } \Omega. 
\]
From a very easy adaptation of Theorem 1.2, we deduce the existence of $\varepsilon$ such that, for any $y^0 \in H^1_0(\Omega)$, there exist controls $u \in L^2(\omega \times (0, T))$ and associated solutions of (4.3) satisfying $y(T) = 0$. Hence, if we take

$$v = (1 + \varepsilon b(x, t; y, \nabla y)) u,$$

we see that the null controllability of (4.2) holds.

This proves Theorem 1.3.

Appendix

Proof of (2.3)

In this section, we reproduce the proof of the inequality (2.3) that was given in [6]; see also [3] for some additional comments.

Let $s$ and $\lambda$ be as in Lemma 2.2 and let us introduce the following fourth-order problem, which will be justified below:

$$\begin{align*}
\mathcal{L}(e^{-2s\alpha}L^*p) + s^3\lambda^4 e^{-2s\alpha} \xi^3 p &= -s^3\lambda^4 e^{-2s\alpha} \xi^3 p_1 \omega & \text{in } Q, \\
p &= 0, & e^{-2s\alpha}L^*p = 0 & \text{on } \Sigma, \\
(e^{-2s\alpha}L^*p)(0) &= (e^{-2s\alpha}L^*p)(T) = 0 & \text{in } \Omega.
\end{align*}$$

(A.1)

Here, we have used the notation $\mathcal{L}q \equiv q_t - \Delta q$, $\mathcal{L}^*q \equiv -q_t - \Delta q$. The partial differential problem (A.1) possesses exactly one (weak) solution $p$ with

$$s^{-1}\iint_Q e^{-2s\alpha} \xi^{-1} \left(|p_t|^2 + |\Delta p|^2\right) \, dx \, dt + s\lambda^2 \iint_Q e^{-2s\alpha} \xi |\nabla p|^2 \, dx \, dt + s^3\lambda^4 \iint_Q e^{-2s\alpha} \xi^3 |p|^2 \, dx \, dt < +\infty.$$

(A.2)

Indeed, let $P_0$ be the linear space

$$P_0 = \{z \in C^2(\overline{Q}) : z = 0 \text{ on } \Sigma\}$$

(A.3)

and let us set

$$\begin{align*}
\kappa(p, p') &= \iint_Q e^{-2s\alpha}L^*pL^*p' \, dx \, dt + s^3\lambda^4 \iint_{\omega \times (0, T)} e^{-2s\alpha} \xi^3 pp' \, dx \, dt & \forall p, p' \in P_0, \\
l(p) &= -s^3\lambda^4 \iint_Q e^{-2s\alpha} \xi^3 qp \, dx \, dt & \forall p \in P_0.
\end{align*}$$

(A.4)

Then, $\kappa(\cdot, \cdot)$ is a strictly positive and symmetric bilinear form in $P_0$. 

Let \( P \) be the completion of \( P_0 \) for the norm \( \| p \|_P = (\kappa(p, p))^{1/2} \). Then \( P \) is a Hilbert space for the scalar product \( \kappa(\cdot, \cdot) \) and, in view of the Carleman inequality (2.1), we have that the functions in \( P \) satisfy (A.2). It is also clear from (2.1) that \( l \) is a continuous linear form on \( P \):

\[
|l(p)| \leq C \left( s^3 \lambda^4 \int_Q e^{-2s\alpha \xi^3 |q|^2} dx \, dt \right)^{1/2} \| p \|_P \quad \forall p \in P. \tag{A.5}
\]

Consequently, in view of Lax-Milgram’s lemma, the following variational equation possesses exactly one solution \( p \):

\[
\kappa(p, p') = l(p') \quad \forall p' \in P, \, p \in P. \tag{A.6}
\]

It is not difficult to see that the unique solution to (A.6) also solves the fourth-order problem (A.1) in the distributional sense.

Now, let us set

\[
\tilde{z} = -e^{-2s\alpha \xi^3} p, \quad \tilde{v} = s^3 \lambda^4 e^{-2s\alpha \xi^3} p_1 \omega. \tag{A.7}
\]

It is readily seen from (A.1) that \( \tilde{z} \) is, together with \( \tilde{u} \), a solution to the null controllability problem

\[
\tilde{z}_t - \Delta \tilde{z} = s^3 \lambda^4 e^{-2s\alpha \xi^3} \varphi + \tilde{v} \omega \quad \text{in } Q,
\]

\[
\tilde{z} = 0 \quad \text{on } \Sigma, \tag{A.8}
\]

\[
\tilde{z}(0) = \tilde{z}(T) = 0 \quad \text{in } \Omega
\]

(see system (2.2) above).

Let us multiply the equation in (A.1) by \( p \). After some integrations by parts, we have

\[
\kappa(p, p) = -s^3 \lambda^4 \int_Q e^{-2s\alpha \xi^3} \varphi p dx \, dt, \tag{A.9}
\]

which combined with the Carleman inequality (2.1) provides the desired inequality for the two last terms in the left-hand side of (2.3):

\[
\int_Q e^{2s\alpha |z|^2} dx \, dt + s^{-3} \lambda^{-4} \int_{\omega \times (0,T)} \xi^{-3} e^{2s\alpha |\tilde{v}|^2} dx \, dt 
\leq Cs^3 \lambda^4 \int_Q e^{-2s\alpha \xi^3 |\varphi|^2} dx \, dt. \tag{A.10}
\]
In order to get an estimate of $|\nabla \hat{z}|^2$, we multiply the equation satisfied by $\hat{z}$ by $s^{-2} \lambda^{-2} e^{2s \alpha} \xi^{-2} \hat{z}$. Then, we integrate by parts with respect to $x$, we also integrate with respect to $t$ and we get

$$s^{-2} \lambda^{-2} \int_Q e^{2s \alpha} \xi^{-2} \hat{z} \hat{t} \, dx \, dt$$

$$+ s^{-2} \lambda^{-2} \int_Q e^{2s \alpha} \xi^{-2} |\nabla \hat{z}|^2 \, dx \, dt$$

$$- 2s^{-1} \lambda^{-1} \int_Q e^{2s \alpha} \xi^{-1} \nabla \eta^0 \cdot \nabla \hat{z} \, dx \, dt$$

$$- 2s^{-2} \lambda^{-1} \int_Q e^{2s \alpha} \xi^{-2} \nabla \eta^0 \cdot \nabla \hat{z} \, dx \, dt$$

$$= s^{\lambda^2} \int_Q \xi \varphi \hat{z} + s^{-2} \lambda^{-2} \int_{\omega \times (0, T)} e^{2s \alpha} \xi^{-2} \hat{v} \hat{z} \, dx \, dt.$$ (A.11)

This time, let us integrate by parts with respect to the time variable in the first term. We obtain the following:

$$s^{-2} \lambda^{-2} \int_Q e^{2s \alpha} \xi^{-2} \hat{z} \hat{t} \, dx \, dt = - \frac{1}{2} s^{-2} \lambda^{-2} \int_Q (e^{2s \alpha} \xi^{-2}) |\nabla \hat{z}|^2 \, dx \, dt$$

$$\leq C s^{-1} \lambda^{-2} \int_Q e^{2s \alpha} |\hat{z}|^2 \, dx \, dt \leq C \int_Q e^{2s \alpha} |\hat{z}|^2 \, dx \, dt,$$ (A.12)

for $s \geq CT$ and $\lambda \geq 1$.

Finally, we use Young's inequality for the other terms of (A.11) and we obtain

$$- 2s^{-1} \lambda^{-1} \int_Q e^{2s \alpha} \xi^{-1} \nabla \eta^0 \cdot \nabla \hat{z} \, dx \, dt - 2s^{-2} \lambda^{-1} \int_Q e^{2s \alpha} \xi^{-2} \nabla \eta^0 \cdot \nabla \hat{z} \, dx \, dt$$

$$\leq C \int_Q e^{2s \alpha} |\hat{z}|^2 \, dx \, dt + \frac{1}{2} s^{-2} \lambda^{-2} \int_Q e^{2s \alpha} \xi^{-2} |\nabla \hat{z}|^2 \, dx \, dt,$$ (A.13)

$$s\lambda^2 \int_Q \xi \varphi \hat{z} \, dx \, dt \leq C \left( \int_Q e^{2s \alpha} |\hat{z}|^2 \, dx \, dt + s^3 \lambda^4 \int_Q e^{-2s \alpha} \xi^3 |\varphi|^2 \, dx \, dt \right),$$ (A.14)

$$s^{-2} \lambda^{-2} \int_{\omega \times (0, T)} e^{2s \alpha} \xi^{-2} \hat{v} \hat{z} \, dx \, dt$$

$$\leq C \left( \int_Q e^{2s \alpha} |\hat{z}|^2 \, dx \, dt + s^{-3} \lambda^{-4} \int_{\omega \times (0, T)} e^{2s \alpha} \xi^{-3} |\hat{v}|^2 \, dx \, dt \right),$$ (A.15)

for $s \geq CT^2$. 
As a conclusion, we deduce (2.3) directly from (A.11) and (A.12)–(A.15).

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Enrique Fernández-Cara: Department of Differential Equations and Numerical Analysis, University of Sevilla, P.O. Box 1160, 41080 Sevilla, Spain
E-mail address: cara@us.es

Sergio Guerrero: Jacques-Louis Lions Laboratory, University Pierre and Marie Curie (Paris 6), P.O. Box 187, 75252 Paris Cedex 05, France
E-mail address: guerrero@ann.jussieu.fr