this problem in some detail, and an estimate of the "relaxation" of the residual stresses resulting from this "secondary" yielding was reported in a paper published some time ago. Fig. 1 here-with is a reproduction of Fig. 10 of that paper and illustrates how the residual stresses were reduced by this secondary yielding below those found if secondary yielding was neglected. The authors might be interested to note the similarity in their method of attack. In particular, Prof. Ford and the writer simulated the release of the expanding pressure by adding a tensile stress at the periphery of the hole to give the superimposed elastic stresses during recovery, much in the same way as the authors have superimposed a bending moment of opposite sign in the problem under discussion.

**Authors' Closure**

The authors wish to thank Dr. Alexander for his kind comments, which are particularly welcome in view of his own significant contribution to the solution of an allied problem dealing with bending of wide plates and with residual stresses. It should be noted that Dr. Alexander's and the present plate analyses provide solutions for different aspects of the same problem and thus tend to complement each other.

**Approximate Synthesis of Spatial Linkages**

D. P. ADAMS. The authors refer to their method as being a generalization of that of Freudenstein, but it would appear that it could stand completely on its own merits. Freudenstein's contribution lay primarily in obtaining equation (1) as the simplest possible equation for the four-bar linkage. It was an important advance, but a comparable development does not appear to have been necessary for the authors. The authors' system of linear equations is so basic an attack that the parallelism with Freudenstein appears as an inevitable result rather than a motivating factor.

The importance of such an approach cannot be overemphasized in view of advances in computing techniques, machining processes, and materials in recent years. With respect to the first of these, the authors have acknowledged their indebtedness for facilities and programming suggestions. It is doubtful that such a paper would have been considered practical 15 years ago. Today the authors are able to include specific results, and the paper is able to illuminate clearly the very great possibilities of the general technique which they have already set forth in earlier publications of which the present paper presents such good examples. Clearly for the kinematician, as for so many engineers and scientists, the computer has become a way of life.

One of the great advantages of the plane four-bar mechanism is its simplicity of manufacture and operation. The corresponding lower-pair space mechanisms retain much of this simplicity. The spherical is the most difficult of the six motions, yet with modern materials, tooling, and lubrication, the possibility of good three-point contact is available by a linear, compensable adjustment, and its potentialities loom greater than ever before.

Perhaps the major difficulty with these mechanisms is due to binding. There may be situations in which vibration of suitable magnitude and frequency can be introduced whenever binding occurs to create forces especially adapted to relieve it. The authors are to be congratulated on such a stimulating paper. In addition to its direct utility, it offers the immediate possibility of corresponding optimizing programs plus the extensions where feasible to higher orders of agreement between the desired and derived functions. Each moving member can be regarded as an extended body and the path of one of its points synthesized into approximate agreement with some desired path. An effective pattern in two dimensions for expression in three dimensions has recently been set up for several of these projects by papers from Columbia University.

FERDINAND FREUDENSTEIN. The authors are to be commended for this further substantial contribution in the field of three-dimensional kinematics, in which field they have already established themselves by previous work on matrix notation and analysis. This is believed to be the first American paper—and one of the first anywhere—on three-dimensional synthesis, and the methods described appear to be well suited to the use of programmed computers.

R. A. BEYER. Both authors, well known for their research work done in the analysis and synthesis of plane and spatial mechanisms—in Europe and particularly in Germany—have made an excellent contribution in this paper. It is of high caliber, not only as to its contents concerning progress in kinematical topics, but also in connection with the general point of view applied in their investigations and by the skillfulness of their methods. In these they derive properties of 3-D mechanisms and spherical mechanisms from those of plane mechanisms. They choose the "four-bar linkage of plane kinematics" as the starting-point for...
DISCUSSION

generalization of the displacement equations to the solutions of 3-D-problems.

The results which the authors obtain in their practical applications for function generation, for angular velocities and for angular accelerations, and so on (programmed moreover for computer use) indicate that the authors are on the right track for continuing further research work in this direction.

With respect to these general remarks the American reader perhaps will be interested in some research made by the discussers concerning similar problems in three-dimensional kinematics:

(b) "Constructions for the Spatial Slider-Crank Mechanism," Maschinenbau/Betrieb, vol. 23, 1944, p. 133.

Additional references on the subject are to be found in R. Beyer and E. Schörner, "Raumkinematische Grundlagen," Munich, Barth, 1953.

Summarizing, it will be observed that this paper represents a remarkable step forward toward knowledge about the synthesis of spatial mechanisms and toward interesting the designer in applying these methods.

T. P. GOODMAN. The straightforwardness of the authors' approach to this formidable problem and the accuracy of their results, as shown in Examples (b) and (c), are indeed impressive. In view of the great generality of their method, it would be of interest to know whether any general statement can be made at this time as to which additional mechanisms are susceptible to this approach; i.e., which additional mechanisms have displacement equations which can be put in the form of equation (2) of the paper.

To obtain additional accuracy points, it appears that either the analysis could be extended in the hope of finding a spatial analog for Freudenstein's five-point synthesis of four-bar linkages, or a scanning procedure could be used to repeat the computer solution while systematically varying the arbitrary parameters such as starting angles. It would be of interest to have the authors' opinion as to which of these alternatives seems more promising.

A third matter on which the authors' comments would be of interest is the range of transmission angles encountered in the spatial mechanisms which the authors have designed. If the transmission angles are favorable in the mechanisms of Examples (b) and (c), these might well be the first two entries in an atlas of spatial linkages.

Authors' Closure

We wish to thank the several discussers for their interest, kind remarks, and informative contributions.

It is true, as Professor Adams notes, that binding may be a problem with spatial linkages, especially those with cylinder pairs. Recent developments in ball-bushings and ball-splines may eliminate most of the anticipated difficulties; the ball-bearing screw is already a very useful kinematic pair.

The questions raised by Dr. Goodman involve matters to which we can give only partial answers at this time.

1 Can a general statement be made as to which mechanisms are susceptible to this approach? The success of the method depends on being able to write the displacement equation of the mechanism in the form of the linear equation (2) with n coefficients k, themselves functions of n linkage parameters. It would seem therefore that the method can always be applied—in principle at least—provided that n is taken sufficiently small. However, since n is also the number of accuracy points, the procedure consists in writing the displacement equation with all possible linkage parameters and then giving arbitrary values to some parameters until an equation of the type (2) may be written. We have applied this trial procedure to a number of mechanisms, but know of no general rule to predict its outcome.

2 How can additional accuracy points be obtained? A spatial analog of Freudenstein's four and five-point syntheses of the planar four-bar linkage may in some cases be obtained without too much difficulty. For example, a four-revolute spherical mechanism may be designed with four accuracy points if μ is (then initial crank angle) is taken as a design parameter. In this case, a second-degree compatibility equation is required, but the problem may be handled on a desk calculator. In other cases, however, a higher-degree compatibility equation, or more than one compatibility equation, may be required. Although less elegant, the scanning procedure in which a solution is repeated while systematically varying an arbitrary parameter has two advantages. It uses a simpler, and therefore cheaper, computer program, and provides a family of solutions from which the designer may choose, not only from the viewpoint of accuracy but also that of practicability. It must not be forgotten that while the present method yields mathematically exact results, these results may be completely impractical. We have, for example, had experience with a problem involving a logarithmic function generator of the two-revolute, two-spheric pair type in which the coupler was to be 150 times the length of the crank! With a different initial follower angle, more acceptable dimensions were obtained.

3 What are the transmission angles in Examples (b) and (c)? The exact meaning of the transmission angle in the case of spatial linkages seems difficult to define at this time, and we would prefer to keep our statement somewhat vague and to speak of force transmission qualities. Using breadboard components, a model of the mechanism of Example (c) was built and operates within its useful range without binding or excessive errors. No model was built of the mechanism of Example (b), but drawings in different positions indicate that good force transmission may also be expected of this mechanism.

When the crank of a planar four-bar linkage has a specified position, the mechanism may be closed for two different positions of the follower, forming either an "open" or a "crossed" linkage. Should the linkage proportions be such that only one follower position is possible, the transmission angle is then either zero or 180 deg, and no force may be transmitted from the crank to the follower. For other linkage proportions, there is a relation between the transmission angle and the two possible follower angles. It seems that a similar property holds for spatial linkages, the problem being more involved since there may be more than two ways of closing a spatial linkage. We feel that a rigorous study of the force transmission properties of spatial linkages is one of the most challenging problems in kinematics today.