Dilatation, Specialconformal and Superconformal Symmetries at Finite Temperature

Hirohumi SAWAYANAGI

Department of Physics, Hokkaido University, Sapporo 060

(Received May 15, 1986)

Dilatation, specialconformal and superconformal symmetries are investigated at finite temperature. It is shown that these symmetries are spontaneously broken without Goldstone particles at the tree level. We consider the one loop case briefly and find that the tree level result persists at the one loop level.

§ 1. Introduction

There are two ways of studying field theories at finite temperature. One way is the imaginary time formalism. This formalism is adequate to the use of high temperature expansions. However, since the time argument (i.e., the temperature variable) is restricted to the imaginary finite domain, it is not convenient to deal with time-dependent quantities by this formalism. Another method is the real time formalism. This formalism treats the temperature and the real time separately. The time integration extends over the entire real domain as in \( T = 0 \) field theories. Thus this formalism is convenient to calculate time-dependent quantities and to see the zero temperature limit. Furthermore any symmetry associated with the time variable is manifested even at finite temperature.

A few years ago, supersymmetry (SUSY) at finite temperature was investigated by using the real time formalism. The imaginary time formalism was used to study this subject before and led some confusion. But the real time formalism could clarify the behavior of SUSY at finite temperature. SUSY is spontaneously broken even at the tree level. Though there exists a Goldstone mode, a Goldstone particle does not appear in general. In some models, this Goldstone mode is transferred to a fermion (Goldstone fermion) at the one loop level.

If we use the real time formalism, the above results, i.e., the spontaneous symmetry breaking and the existence of a Goldstone mode at finite temperature, seem to be expected for other symmetries. In fact, Lorentz symmetry at finite temperature was studied recently and the same results were obtained.

In this paper, we study dilatation, specialconformal and superconformal symmetries at finite temperature by using the real time formalism. In § 2, we briefly review the finite temperature field theory and SUSY breaking at the tree level. The dilatation is studied in § 3 and the specialconformal and the superconformal symmetries are examined in § 4 at the tree level. We extend our tree level consideration to the one loop level in § 5. Section 6 is devoted to discussion.

§ 2. Real time formalism and SUSY breaking at finite temperature

As mentioned in § 1, the real time formalism is more convenient to study the
symmetries associated with space-time. In this formalism, all fields must be doubled by adding their tilde fields. At $T=0$, let us consider the action $\int d^4x (\mathcal{L}_0 + \mathcal{L}_I)$ with the free Lagrangian $\mathcal{L}_0$ and the interaction $\mathcal{L}_I$. Then, at $T\neq 0$, the total action is given by $\int d^4x (\mathcal{L}_0 + \mathcal{L}_I - \mathcal{L}_0 - \mathcal{L}_I)$, where $\mathcal{L}_0$ and $\mathcal{L}_I$ are the free and the interaction Lagrangians of the tilde fields, respectively. In performing the perturbative calculation, $\mathcal{L}_I$ and $\mathcal{L}_I$ are treated as interaction. The temperature only affects propagators through the asymptotic boundary condition. The propagators for scalar bosons and spin-1/2 fermions are given as follows:\textsuperscript{43,53,93}

$$D(k) = \begin{pmatrix}
\frac{-2\pi isinh^2 \theta \delta(\Lambda^2 - m^2)}{\Lambda^2 - m^2 + i\epsilon} & \frac{-2\pi isinh \theta \delta(\Lambda^2 - m^2)}{\Lambda^2 - m^2 - i\epsilon} \\
-2\pi isth \theta \delta(\Lambda^2 - m^2) & \frac{-2\pi isth \delta(\Lambda^2 - m^2)}{\Lambda^2 - m^2 - i\epsilon}
\end{pmatrix},$$ (2·1)

$$S(k) = (\hat{k} + m) \times \begin{pmatrix}
\frac{1}{\Lambda^2 - m^2 + i\epsilon} + 2\pi isinh^2 \theta \delta(\Lambda^2 - m^2) & -2\pi isth \theta \delta(\Lambda^2 - m^2) \\
-2\pi isth \theta \delta(\Lambda^2 - m^2) & \frac{1}{\Lambda^2 - m^2 - i\epsilon} - 2\pi isth \delta(\Lambda^2 - m^2)
\end{pmatrix},$$ (2·2)

where

$$\sinh \theta = \frac{e^{-\beta \Lambda^2/2}}{\sqrt{1 - e^{-\beta \Lambda^2}}}, \quad \cosh \theta = \frac{1}{\sqrt{1 - e^{-\beta \Lambda^2}}},$$ (2·3)

$$\sin \theta = \frac{e^{-\beta \Lambda^2/2}}{\sqrt{1 + e^{-\beta \Lambda^2}}}, \quad \cos \theta = \frac{1}{\sqrt{1 + e^{-\beta \Lambda^2}}}.\quad (2·4)$$

The (11) component of each propagator is the propagator of a usual field and the (22) component is that of a tilde one. Off-diagonal components appear, because the usual field and the tilde field mix at $T\neq 0$. In this paper, we consider the one loop level, at most. So the (11) component of each propagator and the interaction $\mathcal{L}_I$ are relevant below.

For simplicity, we use $n_B$ and $n_F$ defined by

$$n_B(k) = \sinh^2 \theta = \frac{1}{e^{\beta u^2} - 1},$$ (2·5)

$$n_F(k) = \sin^2 \theta = \frac{1}{e^{\beta u^2} + 1},$$ (2·6)

where $u^\mu$ is the four-vector satisfying $u^2 = 1$. (In (2·3) and (2·4), $u^\mu = (1, 0)$ is chosen.) This vector makes the thermal distribution functions covariant form.\textsuperscript{10}\textsuperscript{10}

Next we review the SUSY breaking of the free Wess-Zumino model at finite temperature.\textsuperscript{53,7}\textsuperscript{53,7}\textsuperscript{7} The Lagrangian

$$\mathcal{L} = \partial_\mu a^* \partial^\mu a + \bar{\Phi}(i \hat{\sigma} - m) \psi/2 + f^I f^I + mf a - mf^I a^I$$ (2·7)

is supersymmetric and its supercurrent is given by
Dilatation, Specialconformal and Superconformal Symmetries

\[ Q_\mu = -\sqrt{2} \left( \partial_\mu \gamma_\rho \phi_\rho + \partial_\mu \gamma_\nu \phi_\nu + im a \gamma_\mu \phi_+ + im a \gamma_\mu \phi_- \right), \quad (2.8) \]

where \( \phi_\pm = \sigma_\pm \phi \). Since this model is free, the first non-trivial global Ward-Takahashi (WT) identity is

\[ i \int d^4 z \partial_\mu \left< Q^\mu(z) A(x) \phi(y) \right>_\beta = \left< \phi(x) \phi(y) \right>_\beta + \left< A(x) [F(y) + i \partial_\gamma A(y)] \right>_\beta. \quad (2.9) \]

Here \( A = (a + a^\dagger)/\sqrt{2} \) and \( F = (f + f^\dagger)/\sqrt{2} \), and \( \left< \cdot \right>_\beta \) means the thermal average. It is simple to calculate the RHS of (2.9) by using the propagators (2.1) and (2.2):

\[ \text{(RHS)} = -2\pi \int \frac{d^4 k}{(2\pi)^4} \frac{\partial}{\partial k^0} \left[ n_f(k) + n_B(k) \right] e^{-ik \cdot (x-y)}. \quad (2.10) \]

The factor \( n_f + n_B \) comes from the difference of the fermion propagator \( S^{11}(k) \) from the boson propagator \( D^{11}(k) \). (The index 11 means the (11) component.) Since the (LHS) of (2.9) is the total divergence, (2.10) implies that SUSY is spontaneously broken even at the tree level because of the difference of the thermal distributions. To see zero-mass modes, let us calculate the (LHS) of (2.9). It becomes as follows:

\[ \text{(LHS)} = i \int d^4 z \partial_\mu \left[ \left( \partial_\mu + im \right) A(x) A(x) \gamma_\mu \phi(z) \phi(y) \right]_\beta \]

\[ = \lim_{p \to 0} \int \frac{d^4 k}{(2\pi)^4} i(k + p/2 + m) \not{p} \left[ S^{11}(k-p/2) D^{11}(k+p/2) \right] e^{-ik \cdot (x-y)}. \quad (2.11) \]

Using (2.1) and (2.2), we find that \( S^{11}(k-p/2) D^{11}(k+p/2) \) includes the terms

\[ \delta \left( \left( k - \frac{p}{2} \right)^2 - m^2 \right) n_f(k-p/2) - \delta \left( \left( k + \frac{p}{2} \right)^2 - m^2 \right) n_B(k+p/2) \]

\[ = \frac{1}{2p \cdot k} \left[ \delta \left( \left( k - \frac{p}{2} \right)^2 - m^2 \right) n_f(k-p/2) + \delta \left( \left( k + \frac{p}{2} \right)^2 - m^2 \right) n_B(k+p/2) \right]. \quad (2.12) \]

These terms, which are the \((T=0) \times (T=0)\) parts, bring the desired singularities.

Before closing this section, we remark on the model studied below. We employ the massless Wess-Zumino model\(^{12}\)

\[ \mathcal{L} = \partial_\mu a^\dagger \partial^\mu a + \bar{\phi} i \not{D} \phi \frac{1}{2} + f^\dagger f + f f f + P(\alpha) + f^\dagger P(\alpha^\dagger) \]

\[ - (1/2) \bar{\phi} \left[ P(\alpha) \gamma_+ + P^\dagger(\alpha) \gamma_- \right] \phi \quad (2.13) \]

with \( P(\alpha) = (g/6) \alpha^3 \). This model is the simplest one that has the dilatation, the specialconformal and the superconformal symmetries. However, in §§ 3 and 4.1, we consider the actions of a real scalar and a Majorana spinor for simplicity, because these sections are devoted to the bosonic symmetries of free fields.

§ 3. Dilatation at finite temperature

The previous example of the SUSY breaking indicates that other symmetries may
break spontaneously at the tree level. In this section, we study the dilatation at the tree level.

3.1. Scalar

A dilatation is the change of the space-time coordinates. They transform as $\delta x_{\mu} = -x_{\mu}$. The scalar field transforms under the dilatation as $\delta \varphi(x) = (d + x \cdot \partial) \varphi(x)$, where $d = 1$ is the scale dimension of $\varphi$. The action of a free massless scalar

$$\int d^4x (\partial \varphi)^2/2$$

is invariant under the dilatation. The Noether current is

$$D^\mu(x) = x^\nu \partial^\mu \varphi + \varphi \partial^\mu \varphi = x^\nu \partial_e^\mu + \partial_e X^{\mu \nu}.$$  

Here $\partial_e^\mu$ and $\partial_e^\mu$ are the canonical and the improved energy momentum tensor, respectively:

$$\partial_e^\mu = \partial^\mu \varphi \partial_e \varphi - \delta_e^\mu \mathcal{L}(x),$$

$$\partial_e^\mu = \partial_e^\mu + (1/6)\left( \delta_e^\mu \Box - \partial_e \partial_e \right) \varphi^2(x).$$

Since $X^{\mu \nu}$ is an antisymmetric tensor, it can be discarded in considering the divergence of $D^\mu$. The WT identities are obtained from

$$\int d^4x \delta D^\mu(x) - J(x)(1 + x \cdot \partial) \varphi(x) = 0$$

by taking derivatives with respect to the source $J$. The first non-trivial identity comes from the second derivative:

$$\int d^4x \delta D^\mu(z) \varphi(x) \varphi(y) \delta^\mu = -i(2 + Y \cdot \partial) \varphi(x) \varphi(y),$$

with $Y_\mu = x_\mu - y_\mu$.

The calculation of the RHS of (3·4) is simple. Using (2·1) with $m = 0$, we obtain

$$\text{(RHS)} = \int d^4k (2\pi)^4 e^{-ik \cdot y} 2\pi i \delta(k^2) k \cdot u \delta_n(k) \frac{du(k)}{dk} \cdot u,$$

where $n_B$ and $u_\mu$ are defined in (2·5). In deriving (3·5), we used $k^2 \delta(k^2) = 0$. To check the LHS, the divergence of $D^\mu$ is divided into two parts:

$$\partial_\mu D^\mu = \partial_\mu (x^\nu \partial_e^\nu) + (1/2) \Box \varphi^2.$$  

The contribution of the first term is

$$\int d^4x \partial_\mu \delta Z \partial_e^{\mu \nu}(z) \varphi(x) \varphi(y) \delta$$

$$= -\int d^4\frac{d^4k}{(2\pi)^4} \frac{d^4k}{2\pi} e^{ip \cdot x} e^{-ik \cdot y} P^\mu \frac{\partial}{\partial^\nu} \left[ D^{11}(\frac{p}{2} - k)D^{11}(\frac{p}{2} + k) \right]$$

$$\times \left\{ (\frac{p}{2} - k)^\mu (\frac{p}{2} + k)^\nu + (\frac{p}{2} - k)^\nu (\frac{p}{2} + k)^\mu - g^{\mu \nu} (\frac{p}{2} - k) \cdot (\frac{p}{2} + k) \right\}.$$
where \( X_\mu = (x_\mu + y_\mu) / 2 \). Performing the \( \partial_{\mu} \) differentiation and using the identity 
\[
(p/2 \pm k)^2 \delta((p/2 \pm k)^2) = 0,
\]
we obtain
\[
\int d^4 z \partial_\mu \langle z \theta_\mu (z) \varphi(x) \varphi(y) \rangle = \int \frac{d^4 k}{(2\pi)^4} e^{-ik \cdot y} 2\pi i \delta(k^2) k \cdot u \frac{dn_\mu(k)}{dk \cdot u} 
- \lim_{p \to 0} \int \frac{d^4 k}{(2\pi)^4} p^2 e^{-ip \cdot y} D^{11}(p/2 - k) D^{11}(p/2 + k) .
\]
(3.8)

The contribution of the second term of (3.6) is
\[
\int d^4 z \langle (1/2) \Box \varphi^2(z) \varphi(x) \varphi(y) \rangle = \lim_{p \to 0} \int \frac{d^4 k}{(2\pi)^4} p^2 e^{-ip \cdot y} D^{11}(p/2 - k) D^{11}(p/2 + k) ,
\]
(3.9)
which cancels the second term of (3.8). Thus LHS equals RHS. In (3.7), the singularities \( \delta((p/2 \pm k)^2) \) in \( D^{11}(p/2 + k) D^{11}(p/2 - k) \) are canceled by the factor \( p \cdot k \) in the bracket. We note this factor comes from the first term of \( \theta_\mu \).

3.2. Spinor

The action of a Majorana spinor \( \int d^4 x (1/2) \bar{\psi} i \gamma \psi \) is also invariant under the dilatation. The spinor field transforms as \( \delta \psi = (d + x \cdot \partial) \psi \) with \( d = 3/2 \). The dilatation current is given by
\[
D^\mu(x) = x^\mu \theta^\nu + (d/2) \bar{\psi} i \gamma^\mu \psi = x^\mu \theta^\nu - \partial_\alpha(X^{\alpha \nu} x_\nu) ,
\]
(3.10)
where the symmetric energy momentum tensor \( \theta^{\mu \nu} \) is related to the canonical one \( \theta_\mu \) as
\[
\theta^{\mu \nu} = \theta_\mu + \partial_\alpha X^{\alpha \nu} , \quad \theta_\mu = (1/2) \bar{\psi} i \gamma^\mu \partial_\nu \bar{\psi} - g^{\mu \nu} \mathcal{L}(x) ,
\]
\[
X^{\alpha \nu} = (i/4) \bar{\psi} (\gamma^\alpha S^{\mu \nu} - \gamma^\mu \gamma^\sigma S^{\alpha \sigma} - \gamma^\nu \gamma^\sigma S^{\alpha \sigma}) \psi , \quad S^{\alpha \sigma} = [\gamma^\alpha , \gamma^\sigma] / 4 .
\]
As in the scalar case, the first non-trivial WT identity is
\[
\int d^4 z \partial_\mu \langle D^\mu(x) \varphi(x) \bar{\psi}(y) \rangle = - i (2d + Y \cdot \partial_Y) \langle \varphi(x) \bar{\psi}(y) \rangle ,
\]
(3.11)
with \( Y_\mu = x_\mu - y_\mu \). Now let us study the LHS. The non-vanishing term of the LHS comes from
\[
(1/2) \int d^4 z \partial_\mu \langle z_\mu \varphi(z) i \gamma^\nu \partial_\nu \varphi(z) \varphi(x) \bar{\psi}(y) \rangle .
\]
(3.12)
Equation (3.12) includes the following:
\[
\lim_{p \to 0} \int \frac{d^4 k}{(2\pi)^4} e^{-ik \cdot y} \left( \frac{p}{2} - k \right) \partial_\nu \left( \frac{p}{2} + k \right) E^{11}(\frac{p}{2} - k) E^{11}(\frac{p}{2} + k) .
\]
where $E^{11}(q) = S^{11}(q)$. From $\mathcal{A}E^{11}(q) = S^{11}(q)$, we can obtain the factors to cancel the singularities in $E^{11}(p/2-k)E^{11}(p/2+k)$ as follows:

\[ (p/2-k)E^{11}(p/2+k) = (p/2-k)^2(p/2+k)^2(p/2-k). \]

The final result of (3-11) is

\[ \text{(LHS)} = \text{(RHS)} = -\int \frac{d^4k}{(2\pi)^4}e^{-ik\cdot\gamma}2\pi\delta(k^2)k\cdot u \frac{dn_B(k)}{dk\cdot u}. \quad (3\cdot13) \]

### 3.3. The meaning of these results

From (3-5) and (3-13), we find that the factor

\[ k\cdot u \frac{dn_B,F(k)}{dk\cdot u} \quad (3\cdot14) \]

is specific to the spontaneous breaking of the dilatation at finite temperature. We consider the meaning of (3-14) by using the scalar propagator $D^{11}(Y)$. As is shown in (2-1), $D^{11}(Y)$ is divided into the $T=0$ and the $T\neq0$ part:

\[ D^{11}(Y) = D^{11}_0(Y) + D^{11}_{s}(Y) \]

\[ \begin{align*}
&= \int \frac{d^4k}{(2\pi)^4}e^{-ik\cdot\gamma}2\pi\delta(k^2)k\cdot u \frac{dn_B(k)}{dk\cdot u}. \quad (3\cdot15)
\end{align*} \]

Changing the integration variable $k_\mu$ to $e^\rho k_\mu$ with infinitesimal $\rho$, (3-15) gives

\[ \left(2 + Y\cdot\partial_Y + \beta \frac{d}{d\beta}\right)D^{11}_s(Y) = 0. \quad (3\cdot16) \]

(The $T=0$ part satisfies $(2 + Y\cdot\partial_Y)D^{11}_0=0$.) From (3-16), we obtain

\[ (2 + Y\cdot\partial_Y)D^{11}_s(Y) = -\beta \frac{d}{d\beta}D^{11}_s(Y) \]

\[ \begin{align*}
&= \int \frac{d^4k}{(2\pi)^4}e^{-ik\cdot\gamma}2\pi\delta(k^2)k\cdot u \frac{dn_B(k)}{dk\cdot u}. \quad (3\cdot17)
\end{align*} \]

Equation (3-17) implies that the dilatation must be compensated by the transformation of the temperature scale. The expression (3-14) is peculiar to this transformation.

### § 4. Specialconformal and superconformal symmetries at $T\neq0$

In this section, we study the specialconformal symmetry for a scalar field and the superconformal symmetry for the Wess-Zumino model at the tree level.

#### 4.1. Specialconformal symmetry

A specialconformal transformation is the space-time transformation whose infinitesimal form is given by $\delta^a x^a = 2x^a\theta^a - g^{ab}x^b \cdot \cdot \cdot$ (11). A scalar field transforms as $\delta^a \varphi(x) = (2x^a\theta^a - g^{ab}x^b)\varphi(x) + 2x^a d\varphi(x)$ with $d = 1$ and the action (3-1) is invariant under this transformation. The current associated with this symmetry is found to be $K^{\mu}\varphi = (2x^\mu x^\nu - \delta^\mu_{\mu} x^2)\theta^\nu$, where $\theta^\nu$ is given by (3-3). The WT identity to be studied is.
Dilatation, Specialconformal and Superconformal Symmetries

\[ \int d^4z \partial_\mu K^{\mu\nu}(z) \varphi(x) \varphi(y) = -i [2x^\mu + 2y^\mu + (2x^\alpha x^\nu - g^{\alpha\nu} x^2) \partial_\mu + (2y^\alpha y^\nu - g^{\alpha\nu} y^2) \partial_\nu] \varphi(x) \varphi(y) . \]  \hspace{1cm} (4.1)

As in § 3, the term \( \partial^\mu \varphi \partial^\nu \varphi \) in \( \theta^\mu \) produces the factor \( p \cdot k \) to cancel the singularity. The identity (4.1) finally becomes as follows:

\[ \text{(LHS)} = \text{(RHS)} \]
\[ = \int \frac{d^4k}{(2\pi)^4} e^{-ik\cdot Y} 4\pi i \delta(k^2) \left[ X^a k^a u^a \frac{dn_{\mu}(k)}{dk \cdot u} + X^a (k^\nu u^\mu - k^\mu u^\nu) \frac{dn_{\nu}(k)}{dk \cdot u} \right] . \]  \hspace{1cm} (4.2)

Here we set \( X_\nu = (x_\nu + y_\nu)/2 \) and \( Y_\mu = x_\mu - y_\mu \). The first term of (4.2) contains the factor (3.14) which is specific to the dilatation. The factor

\[ (k^\nu u^\alpha - k^\alpha u^\nu) \frac{dn_{\nu}(k)}{dk \cdot u} \]  \hspace{1cm} (4.3)

in the second term corresponds to the breaking of Lorentz-symmetry.8) To see this point explicitly, we consider the propagator (3.15) again. If we change \( k_\mu \) to \( k_\mu + \omega_\mu k_\nu \) with infinitesimal \( \omega_\mu \), we obtain

\[ Y^\mu \omega_\mu \partial_\nu D_{\nu}^{11}(Y) + \beta_\mu \omega_\nu \partial_{\mu} - \beta_\nu \partial_{\mu} \mu \nu \partial_{\nu} D_{\nu}^{11}(Y) = 0 , \]  \hspace{1cm} (4.4)

where \( \beta_\mu = \beta u_\mu \) and \( Y^\mu \omega_\mu \partial_\nu D_{\nu}^{11}(Y) = 0 \) was used. From (4.4), the following expression which contains the factor (4.3) is obtained:

\[ (Y_\mu \partial_\nu - Y_\nu \partial_\mu ) D_{\nu}^{11}(Y) = - (\beta_\mu \frac{\partial}{\partial \beta^\nu} - \beta_\nu \frac{\partial}{\partial \beta^\mu} ) D_{\nu}^{11}(Y) \]
\[ = \int \frac{d^4k}{(2\pi)^4} e^{-ik\cdot Y} 2\pi i \delta(k^2) (u_\mu k_\nu - u_\nu k_\mu) \frac{dn_{\nu}(k)}{dk \cdot u} . \]  \hspace{1cm} (4.5)

The result (4.2) is related to the fact that the conformal current \( K^{\mu\nu} \) can be expressed in terms of \( D^\mu \) and the Lorentz current \( M^{\mu\nu} \) as \( K^{\mu\nu} = x_\nu (g^{\nu\rho} D_{\rho} - M^{\nu\rho} ) \).11)

4.2. Superconformal symmetry

The Wess-Zumino model (2.13) is invariant under a superconformal transformation.12) Since we consider the tree level WT identity here, we set \( g = 0 \) in (2.13) for simplicity. Then the superconformal transformation is given by

\[ \delta a = \sqrt{2} \varepsilon(x) \phi_+ , \quad \delta f = - \sqrt{2} \varepsilon(x) i \partial \phi_+ , \]
\[ \delta \phi_+ = \sqrt{2} f_-(x) - i \sqrt{2} \partial a \varepsilon_-(x) - (1/2) a i \partial \varepsilon_-(x) . \]  \hspace{1cm} (4.6)

Here \( \phi_\pm \) is \( (1 \pm \gamma_5)/2 \cdot \phi \) and the parameter \( \varepsilon(x) = i \varepsilon_1 \) is the Majorana spinor. The transformations for \( a^\nu, f^\nu \) and \( \phi_- \) are obtained from (4.6) easily. The superconformal current becomes

\[ S_\mu = \sqrt{2} f_+ (\partial a \gamma_\mu \phi_+ + \partial a^\nu \gamma_\mu \phi_- ) - 2\sqrt{2} \gamma_\mu (a \phi_- + a^\nu \phi_+ ) \]
\[ = \partial Q_\mu , \]

where the supercurrent with the modified term13) is
Now let us study the WT identity
\[ \int d^4z \partial^\mu S^\mu(z) A(x) \langle \bar{\psi}(y) \rangle = -i \langle -\bar{\psi}(x) \psi(y) \rangle \]
with \( A = (a + a^\dagger) / \sqrt{2} \). By using (2·1) and (2·2), the LHS becomes
\[ \langle \text{LHS} \rangle = \lim_{\rho \to 0} i \int \frac{d^4k}{(2\pi)^4} e^{-ik\cdot r} \left[ \frac{\partial}{\partial \rho^\mu} \left( e^{i\rho \cdot x} (p/2 - k) \gamma^\mu (p/2 + k) \right) \right] \]
\[ \times D^{11}(p/2 - k)E^{11}(p/2 + k) - 2\rho (p/2 + k) D^{11}(p/2 - k)E^{11}(p/2 + k) \]
\[ = - \int \frac{d^4k}{(2\pi)^4} e^{-ik\cdot r} 2\pi i \delta(k^2) \left\{ (X^\mu \gamma_\mu - i) (n_B(k) + n_F(k)) \right\} \]
\[ + (i/2) k \cdot u \frac{d}{dk} (n_B(k) - n_F(k)) \]
\[ + (i/8) [\gamma^\mu, \gamma^\nu] (u_{\mu} k_{\nu} - u_{\nu} k_{\mu}) \frac{d}{dk} (n_B(k) - n_F(k)) \].

It is easy to check that the RHS of (4·7) equals (4·8). Equation (4·8) shows the three origins of the spontaneous breaking of the superconformal symmetry at finite temperature. They are (1) the difference of the thermal distributions of bosons and fermions (the first term), (2) the breaking of the dilatation by temperature (the second term) and (3) the breaking of the Lorentz symmetry by temperature (the third term). This result is expected because the superconformal symmetry is the “square root” of the special-conformal symmetry.

§ 5. One loop level

In the preceding sections, we investigated the tree WT identities. Following Ref. 7), we extend our consideration to the one loop level here. As an example, we consider the dilatation of the model (2·13). The amputated WT identity for fermions is
\[ \int d^4z d^4u d^4w \partial^\mu S^\mu(z) \phi(u) \bar{\phi}(w) \langle \psi(w) \bar{\psi}(y) \rangle = -i (2d - 8 - Y \cdot \partial_Y) \langle \phi(x) \bar{\phi}(y) \rangle \]
with \( d = 3/2 \). In (5·1), a trace anomaly is neglected for the time being. The diagram which contributes to the RHS of (5·1) is shown in Fig. 1(a). The temperature-dependent part of the last term of the RHS is
\[ - Y \cdot \partial_Y \langle \phi(x) \bar{\phi}(y) \rangle = g^2 \int \frac{d^4k}{(2\pi)^4} \frac{d^4l}{(2\pi)^4} e^{-ik\cdot r} \partial_k \cdot k [S^{11}(k - l) D^{11}(l) - S_0^{11}(k - l) D_0^{11}(l)] \]
\[ = g^2 \int \frac{d^4k}{(2\pi)^4} \frac{d^4l}{(2\pi)^4} e^{-ik\cdot r} \left[ 5S^{11}(k - l) D^{11}(l) - S_0^{11}(k - l) D_0^{11}(l) \right] \]
Dilatation, Special conformal and Superconformal Symmetries

The one loop diagrams which contribute to the WT identity (5.1). The diagram (a) contributes to the RHS and the diagrams (b) and (c) contribute to the LHS. The solid line represents the fermion and the dotted line represents the boson. The diagram (d) is given to see that the singularity of the one loop diagram (b) is related to the singularity of the tree diagram.

\begin{align*}
&+2\pi i(k-I)\delta((l-k)^2)(k-l)\cdot u \frac{dn_F(k-l)}{d(k-l)\cdot u} D^{11}(l) \\
&-2\pi i\delta(l^2) l\cdot u \frac{dn_B(l)}{d(l)\cdot u} S^{11}(k-l) \quad (5.2)
\end{align*}

By substituting (5.2) into the RHS of (5.1), the temperature-dependent part becomes

\begin{align*}
&-ig^2\int \frac{d^4k}{(2\pi)^4}e^{-i(k-y)} \int \frac{d^4l}{(2\pi)^4} \left[2\pi i(k-I)\delta((l-k)^2) u\cdot(k-l) \frac{dn_F(k-l)}{d(k-l)\cdot u} D^{11}(l) \\
&-2\pi i\delta((k-l)^2) u\cdot(k-l) \frac{dn_B(k-l)}{d(k-l)\cdot u} S^{11}(l) \right] \\
&= (5.3)
\end{align*}

In deriving (5.3), we have changed the variable \( l \) of the second term to \( k-l \), since the \( T=0 \) part is finite.

Next we study the LHS. In the interacting case, there are additional current vertices which are proportional to the coupling constant. However it is easy to understand that they do not contribute, because they do not produce any singularity as the limit \( p \to 0 \) is taken. Thus the diagrams which contribute to the LHS are shown in Figs. 1(b) and (c). As to the \( T=0 \) part, we can use the tree level results (3.5) and (3.13) by changing the momentum variable \( k \) to \( k-l \). (Compare Fig. 1(b) with Fig. 1(d)) Multiplying (3.13) by a scalar propagator and (3.5) by a fermion propagator appropriately, we can prove (LHS) = (RHS) = (5.3). Equation (5.3) is connected with the fact that the fermion acquires the mass proportional to \( gT \) at the one loop level. 16)

For other symmetries, we can also prove the non-trivial WT identities at the one loop level in the same way.

We must remark on the \( T=0 \) part here. It diverges at the one loop level, so an appropriate regularization must be done. As is well-known, a careful treatment produces anomalies. 11,13)

The model (2.13) contains massless particles and has an \( R \)-symmetry \( (a \to e^{i\alpha}a, \psi \to e^{-i\alpha} \psi, f \to e^{-2i\alpha}f) \). As was discussed in Ref. 7, this symmetry prevents \( a \) and \( f \) from
having the thermal averages $\langle a \rangle_\beta$ and $\langle f \rangle_\beta$. Thus there is no Goldstone particle. It is easy to check this point at the one loop level. Following Ref. 14, under the condition $|\sqrt{2F/gA^4}|<1$, we obtain the one loop effective potential

$$V_{\text{eff}} \approx \frac{1}{1 - a \ln(A^2/M^2) + 2af(b)}$$

$$+ \left( \frac{2}{\beta} \right) \int \frac{d^3k}{(2\pi)^3} \left[ \ln(1 - e^{-\beta E}) - \ln(1 + e^{-\beta E}) \right].$$

(5·4)

Here we wrote $E = \sqrt{k^2 + g^2 A^2/2}$, $b^2 = g^2 A^2/2$, $a = g^2/(32\pi^2)$, $A = \langle (a + a^\dagger)/\sqrt{2} \rangle_\beta$, $F = \langle (f + f^\dagger)/\sqrt{2} \rangle_\beta$, and $M$ is the renormalization parameter.\(^{14}\) $I(b)$ is given by\(^{14}\)

$$I(b) \approx (\pi/2b) + (1/2)\ln(b/4\pi) + \gamma/2 + \mathcal{O}(b^2)$$

with $\gamma \approx 0.577$. The factor $\ln A$ in the denominator, which comes from the infrared singularity, is canceled by the $\ln b$ term in $I(b)$. Then by differentiating (5·4) with respect to $A$, we find that $A = F = 0$ is the minimum of (5·4).

§ 6. Discussion

(1). In §§ 3 and 4, we showed that the dilatation, the special conformal and the superconformal symmetry are spontaneously broken at finite temperature. The massless Goldstone mode appears though it is not a single particle. These symmetries are peculiar to massless fields. But the origin of the Goldstone mode is not the vanishing mass of a field. It is the cross terms ($T = 0 \times T = 0$) of two propagators with the same mass. In terms of thermo field dynamics,\(^{3,4}\) the creations of a particle and a tilde particle generate this mode.\(^{5,13}\) As an example, let us consider a free real scalar field, as in § 3.1. Since this is the free theory, its dilatation current is conserved. Thus, in thermo field dynamics, the non-trivial WT identity is now

$$\int d^4z \partial_\tau \langle 0(\beta) | TD^\tau(z) \varphi(x) \varphi(y) | 0(\beta) \rangle = \langle 0(\beta) | T [D, \varphi(x) \varphi(y)] | 0(\beta) \rangle = 0,$$

where $|0(\beta)\rangle$ is the temperature-dependent vacuum (see (6·2)) and the dilatation charge $D$ is $\int d^3z D^0(z)$. This expression shows that the dilatation charge produces some Goldstone mode. To see this mode explicitly, we expand a free real scalar field as

$$\varphi(x) = \frac{1}{(2\pi)^{3/2}} \int \frac{d^3k}{\sqrt{2\omega_k}} \left[ a(k)e^{-i(\omega_k t - kx)} + a^\dagger(k)e^{i(\omega_k t - kx)} \right].$$

(6·1)

The operators and the vacuum which are temperature-dependent are obtained by a Bogoliubov transformation:

$$a(k) = \cosh \theta_k a(k, \beta) + \sinh \theta_k a^\dagger(k, \beta),$$

$$a(k, \beta) |0(\beta)\rangle = \tilde{a}(k, \beta) |0(\beta)\rangle = 0.$$ (6·2a)

Here $\cosh \theta_k$ and $\sinh \theta_k$ are defined by (2·3). Using (6·1) and (6·2), we find that the state

$$\lim_{\beta \to 0} \int d^4z e^{ipz} \varphi(z, \beta) |0(\beta)\rangle$$
includes the states $\bar{a}^\dagger(k,\beta)a^\dagger(k,\beta)|0(\beta)\rangle$ and $a^\dagger(k,\beta)\bar{a}^\dagger(k,\beta)|0(\beta)\rangle$ with $\delta(p)$. These are the Goldstone modes mentioned above.

(2). There are three factors that break tree WT identities spontaneously at finite temperature: i.e., (2°10), (3°14) and (4°3). The factors (3°14) and (4°3) appear because $e^{i\mathbf{u}\cdot\mathbf{k}}$ in $n_{B,F}(k)$ is not invariant under the corresponding transformations. In other words, symmetries which do not commute with $\mathbf{u}\cdot\mathbf{P}$, i.e., the translation along $\mathbf{u}_\mu$, break at finite temperature.15

(3). In § 5, we briefly considered the one loop effective potential of the Wess-Zumino model. The $T=0$ part produced the factor $\ln A$ and the $T\neq0$ part brought the same factor with the opposite sign. So they were canceled.

At high temperature, the same cancellation of the infrared singular term happens also in the case of the $\varphi^4$ theory,19 which has no chiral invariance. Then the one loop effective potential takes the minimum at $\langle\varphi\rangle_0=0$.

From these examples, we expect, in the case of massless theories, there is no Goldstone particle at the one loop level. On the contrary, the particles that are massless at the tree level acquire the masses proportional to $gT$.

(4). It is not hard to convince ourselves that the contribution of the Goldstone mode mentioned above exists at higher-loop orders. For example, we can obtain higher-loop contributions to the LHS of (5·1) by adding fermion lines and boson lines in Figs. 1(b) and (c). At the two loop level, one of such diagrams is shown in Fig. 2(a). The current vertex part of Fig. 2(a), i.e., Fig. 2(b), produces the Goldstone mode as before. In fact, after taking the limit $p_\mu\to0$, Fig. 2(b) gives

$$i/2\left[\delta^{\gamma\gamma}\mathbf{u}\cdot\mathbf{q}\frac{\partial}{\partial\mathbf{u}\cdot\mathbf{q}} S^{\alpha}(q) + \delta^{\alpha\alpha}\mathbf{u}\cdot\mathbf{q}\frac{\partial}{\partial\mathbf{u}\cdot\mathbf{q}} S^{\gamma}(q)\right]$$

in the momentum representation. Here $\alpha$ and $\gamma$ are thermal indices. This contribution exists at higher-loop levels.

Acknowledgements

The author would like to thank Professor S. Araki for his information on Ref. 15.

References

H. Umezawa, H. Matsumoto and M. Tachiki, Thermo Field Dynamics and Condensed States (North-Holland, Amsterdam, 1982).