Magnetic Domain Wall in uudd $^3$He

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As seen in the c-w NMR experiments of uudd $^3$He, at least three magnetic domains each having the anisotropic axis perpendicular to each other always seem to coexist in single crystals. We first determine the structure of the magnetic domain walls taking into account the exchange interaction, the dipole interaction and an applied magnetic field. Next we study scattering of spin waves by the domain wall and show that spin waves with small wave number $k \ll \xi^{-1}$, where $\xi$ is the dipole length, are almost totally reflected by the dipole potential near the wall. This result can be related to the NMR experiments: The spin precession in a resonant domain is in general not coupled to spins in other non-resonant domains, which may provide a plausible explanation for the observed coexistence of three sharp NMR peaks.

§ 1. Introduction

As shown by Osheroff, Cross and Fisher (OCF), the spin alignment of bcc $^3$He in the ordered phase is of the sequence up-up-down-down(uudd) along one of the cubic axis, which is denoted by vector $\vec{l}$ (Fig. 1). In stabilizing the uudd phase, four-spin exchange interactions, in particular, the planar four-spin interaction play an important role.$^{2,3}$

In the c-w NMR experiments$^{1,6}$ using the single crystal of bcc $^3$He three resonance lines are almost always observed in the uudd phase. This implies three magnetic domains, each of which has $\vec{l}$ perpendicular to each other, always coexists in the samples. This fact leads us to consider problems concerning the magnetic domains in the uudd phase, especially possible structures of the domain walls.

The problems are also related with the recent pulsed NMR experiments in the uudd phase by Kusumoto et al.$^4$ The decay rate of free induction signals (FID) was observed to be non-exponential and dependent on the initial tipping angle; the signals decayed more rapidly, the larger the initial tipping angle. One can think of possible mechanisms for the anomalous decay, such as the onset of chaotic motions of the order parameters$^5$ or the instability of the uniform mode against the excitation of the spin waves,$^6$ as discussed previously. Due to spin coupling between magnetic domains, it is also possible for energy currents to carry away the magnetic energy from a resonant domain to adjacent non-resonant domains.

In §§ 2 and 3 we discuss the equilibrium structure of domain walls. In § 4 we study the scattering of spin waves at the domain wall. Section 5 is devoted to conclusion and discussion.
§ 2. Equilibrium magnetic domain wall

To understand the spin alignment of bcc $^3$He it is necessary to use the spin Hamiltonian, consisting of two-spin interactions up to third neighbors and planar (P) cyclic four-spin interaction terms (Fig. 1):

$$H = -\frac{1}{2} \sum_{n=1}^{3} J_n \sum_{\langle i,j \rangle} \mathbf{\sigma}_i \cdot \mathbf{\sigma}_j - \frac{1}{4} K_P \sum_{\langle i,j,k,l \rangle} \delta_{ijkl},$$

(2.1)

where

$$\delta_{ijkl} = (\mathbf{\sigma}_i \cdot \mathbf{\sigma}_j)(\mathbf{\sigma}_k \cdot \mathbf{\sigma}_l) + (\mathbf{\sigma}_i \cdot \mathbf{\sigma}_l)(\mathbf{\sigma}_j \cdot \mathbf{\sigma}_k) - (\mathbf{\sigma}_i \cdot \mathbf{\sigma}_k)(\mathbf{\sigma}_j \cdot \mathbf{\sigma}_l).$$

(2.2)

Here $i$ and $j$ refer to nearest neighboring spins if $n=1$, to next neighbors if $n=2$ and to third neighbors if $n=3$.

Recently Stipdonk and Hetherington suggested a new model using only three kinds of exchange interactions, namely, the direct nearest-neighbor exchange $J_{nn}$, the three-spin exchange $J_t$, and the planar four-spin exchange $K_P$. The magnitude of the respective exchange constants are

$$J_{nn} = -0.377\text{mK}, \quad J_t = -0.155\text{mK}, \quad K_P = -0.327\text{mK}.$$  

(2.3)

If we adopt this model, the effective two-spin exchange parameters in (2.1) are given in terms of those parameters as

$$J_t = J_{nn} - 6J_t + 3K_P = -0.428\text{mK},$$

$$J_2 = -4J_t + K_P = 0.293\text{mK},$$

$$J_3 = K_P/2 = -0.1635\text{mK}.$$  

(2.4)

[2.1] In this subsection we investigate the spin configuration near the planar boundary between two domains and show that the (110) domain wall of the type illustrated in Fig. 4(a) has the lowest boundary energy per unit area.

For convenience we choose the boundary plane separating the two domains in such a way that no lattice point lies on it. We assume that the spin configuration far from the boundary plane is the uniform uudd alignment characterized by the vectors $\hat{d}_1$ and $\hat{d}_2$ on one side and by $\hat{d}_2$ and $\hat{d}_1$ on the other side, $\hat{d}_i$ being the unit vector parallel to the sublattice magnetization. The type of the domain wall is then specified by these four vectors and, when necessary, by the relative position of the uudd sequences on both sides. Since the characteristic length of our system is of the order of the lattice constant, we expect the spin alignment to vary only in a few layers near the boundary. As the first step to find the stable domain walls, therefore, we assume that the two uniform domains meet each other at the boundary plane and that both have the same $\hat{d}$ vector, $\hat{d}_1 = \hat{d}_2$, and calculate the energy associated with the boundary. In the next step we try to lower the energy by changing the spin configuration near the boundary. When doing this, we relax the second condition if necessary; $\hat{d}$ of the domain-1 may be different from that of the domain-2.

In calculating the energy it is worthwhile to note that the planar four-spin interaction is most important in stabilizing the uudd spin alignment and that $\delta_{ijkl}$ given by (2.2) is
Fig. 2. Spin configuration on (110) plane near the [001] boundary line, represented by the solid line. The symbols □ and × denote up and down spin respectively. The wrong planar rings which have \( \delta = 1 \) are depicted by closed solid lines. Arrows denote the projected \( \hat{\mathbf{I}} \) vectors of adjacent domains on the (110) plane: (a) \( \hat{\mathbf{I}}_1 \parallel [001] \), (b) \( \hat{\mathbf{I}}_2 \parallel [001] \), (c) \( \hat{\mathbf{I}}_2 \parallel [001] \), and (d) \( \hat{\mathbf{I}}_1 \parallel \hat{\mathbf{I}}_2 \parallel [001] \).

Fig. 3. Spin configuration on (110) plane near [111] boundary line. The configuration (a) has no wrong ring but (b) has wrong rings depicted by closed lines.

equal to \(-1\) for every planar ring in the bulk uudd phase. Therefore, the guiding principle in finding the most stable spin configuration of the domain wall is to minimize the number of the planar ring with \( \delta = 1 \) that occur at the boundary. Since all planar rings lie on (110) planes or on its equivalent, we have to know spin configurations on these planes when two domains, assumed to be uniform in the first step, meet at the boundary. The intersection of a (110) plane with the boundary plane will be called the boundary line. When the boundary line is parallel to [001] direction, there are four types of spin configuration shown in Fig. 2. In these figures the planar rings depicted by solid lines have \( \delta = 1 \), and will be called wrong rings. When the boundary plane is parallel to a (110) plane, the boundary line does not appear on the (110) plane and hence there is no wrong ring on it. The configurations of Fig. 3(a) without wrong rings occur when the boundary plane is parallel to (101), (011), etc. These considerations suggest the (110) (or its equivalent) domain walls to be most stable. There are three types of the (110) walls shown in Figs. 4(a)~(c); the first two differ from each other by the translation of the
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uudd sequence on one side. Among these the one shown in Fig. 4(a) has the wrong rings only on the (110) plane as is given in Fig. 2(a), and hence the most stable. In Table I we tabulate the boundary energy per unit area associated with the various domain walls shown in Figs. 4(a)~(c) ((110)), Fig. 5 ((100)) and Fig. 6 ((111)) under the assumption mentioned above. As seen in Table I, the boundary energy of other domain walls is at least about two times as large as that of the (110) domain wall illustrated in Fig. 4(a) (called the wall (110)a hereafter). Even if we adopt other set of values for the exchange

Fig. 4. (110) domain walls. (a) The wall (110)a. (b) The wall (110)b. The boundary is represented by the solid line plane in (c).

Fig. 5. (100) domain walls.
parameters, we expect that the wall (110)\(a\) is the most stable as long as \(|K_p|\) is comparable to \(|J_i|\).

One must note the following points in constructing possible domain patterns. First, when the two domains with \(\hat{l}_1\) and \(\hat{l}_2\) are separated by the wall (110)\(a\), all three vectors \(\hat{l}_1\), \(\hat{l}_2\) and the normal to the wall must be co-planar. Secondly, if the domain boundary takes the form such as shown in Fig. 7(a), one of the (110) wall turns out to be the wall (110)\(b\) illustrated in Fig. 4(b), hence the wrong wall. The configuration in Fig. 7(b) avoids this. From these consideration we conclude that in the bulk each domain can take only a concave polygonal form bounded by (110) or equivalent planes. The simplest example of the domain patterns fulfilling this requirement is the bcc lattice made up of octahedral domains with \(\hat{l}_1 = \hat{x}, \hat{y}, \hat{z}\), no adjacent octahedrons possessing the same \(\hat{l}\). If the (110)\(a\) domain walls are actually dominant, it should be unlikely to find only two types of the magnetic domains. In other

<table>
<thead>
<tr>
<th>Domain Wall</th>
<th>Spin Configuration and Number of Wrong Rings on (110) or Equivalent Planes</th>
<th>Boundary Energy* (mK)</th>
</tr>
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<tbody>
<tr>
<td>(100)</td>
<td>(110) (110) (101) (011) (011)</td>
<td></td>
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<tr>
<td>Fig. 5(a)</td>
<td>3(a^0) 3(a) 3(b) 3(b) 3(b) 3(b) 2(b) 2 2 2</td>
<td>(-J_1 - 2J_3 - 3K_p)</td>
</tr>
<tr>
<td>Fig. 5(b)</td>
<td>3(b) 3(b) 3(b) 3(b) uudd, 3(c), 3(d) uudd, 3(c), 3(d)</td>
<td>(-J_1 - 2J_3 - 3K_p)</td>
</tr>
<tr>
<td>Fig. 4(a)</td>
<td>uudd(a) 3(a) 4(a) 4(a) 4(a) 4(a) 4(a) 0 2 0 0 0 0 0</td>
<td>1/(2\sqrt{3})((-2J_3 - K_p))</td>
</tr>
<tr>
<td>Fig. 4(b)</td>
<td>uudd udde 3(a) 4(b) 4(b) 4(b) 4(b) 0 1 (\sqrt{3}/3) (\sqrt{3}/3) (\sqrt{3}/3) (\sqrt{3}/3)</td>
<td>(-J_1 - 2J_3 - 5K_p)</td>
</tr>
<tr>
<td>Fig. 4(c)</td>
<td>uudd udde 3(b) 4(a) 4(b) 4(b) 4(b) 4(a) 0 2 (\sqrt{3}/3) (\sqrt{3}/3) (\sqrt{3}/3) (\sqrt{3}/3)</td>
<td>(-J_1 - 2J_3 - 5K_p)</td>
</tr>
<tr>
<td>Fig. 6</td>
<td>3(b) step(a) 3(d) uudd, step 3(b) step</td>
<td>1/(2\sqrt{3})((-J_1 + J_3))</td>
</tr>
<tr>
<td></td>
<td>2 (\sqrt{3}/3) 2 (\sqrt{3}/3) (\sqrt{3}/3) (\sqrt{3}/3)</td>
<td>(-2J_3 - 5K_p)</td>
</tr>
</tbody>
</table>

a) Boundary energy per unit area calculated by making use of the exchange parameters (2-4) with the lattice constant \(a = 1\).

b) Spin configuration given by the figure of this number.

c) Number of wrong rings per the lattice constant along the boundary line.

d) "uudd" means that the boundary line does not appear on the plane.

e) "step" means that the boundary line is step-like.
words, when there exist many domains in the samples, it is likely to observe all three resonance lines.

[2.2] We now try to lower the boundary energy by relaxing the condition of the two domains on both sides of the boundary being uniform, in other words, by letting directions of spins vary near the boundary. We will do this for the wall (110)a. As shown in Fig. 8, there are four kinds of planes, denoted by a, b, c and d, perpendicular to the boundary on which the configurations of spins is different from each other. We specify the directions of spins $\sigma_a(i)$ as

$$\sigma_a(i) = (\sin \phi_a(i) \cos \theta_a(i), \sin \phi_a(i) \sin \theta_a(i), \cos \phi_a(i)),$$

where $i$ is the distance to the spin from the boundary shown by the dotted line in Fig. 8 and $a = a$, $b$, $c$, or $d$ also shown in Fig. 8. We now assume that the spin configurations possess the following symmetry with respect to the reflection of the spins,

$$\sigma_a(i) = -\sigma_c(i), \quad \sigma_b(i) = -\sigma_d(i).$$

It will be shown later numerically that the reflection symmetry is a necessary condition for extremum of the boundary energy; we cannot find extremum configurations which do not have the symmetry.

When calculating the boundary energy we must note in Eq. (2.2) that provided, for example,

$$\sigma_i = \sigma_j, \quad \sigma_k = -\sigma_i,$$

the planar exchange energy of a planar ring does not depend on the angle between $\sigma_i$ and $\sigma_k$ and

$$\delta_{ijkl} = -1.$$
We can easily show that for the extremum the directions of all spins must be confined to a plane. We choose it to be the $x$-$y$ plane, hence putting $\phi_d(i) = \pi/2$ in (2.5). Here we introduce new variable $\theta_d'(i)$ which specify the sublattice magnetization $\vec{d}$. Note that, if $\vec{d}$ is parallel to the $x$-axis, $\theta_d(i)$ and $\theta_b(i)$ for the uudd alignment is given by $i(i-1)\pi/2$ and $i(i+1)\pi/2$, respectively. Therefore, we introduce new variables defined as

$$
\theta_d'(i) = \theta_d(i) - \frac{1}{2} i(i-1)\pi ,
$$

$$
\theta_b'(i) = \theta_b(i) - \frac{1}{2} i(i+1)\pi .
$$

The boundary conditions for these variables are simply

$$
\theta_d'(i) \rightarrow 0 \quad \text{for} \quad i \rightarrow -\infty .
$$

In terms of $\theta_d'(i)$ the boundary energy is given by

$$
E = \sum_{i=-\infty}^{\infty} \left[ -J_1(-1)^{i+1}(\cos(\theta_d'(i+1) - \theta_d'(i)) - \cos(\theta_b'(i+1) - \theta_b'(i))) 
+ \sum_{a=a, b} \left\{ \frac{J_3}{2}\cos(\theta_d'(i+2) - \theta_d'(i)) 
+ \frac{K_p}{4}\cos(2\theta_d'(i+1) - \theta_d'(i) - \theta_d'(i+2)) \right\} 
+ K_r(-\cos^3(\theta_d'(i) - \theta_b'(i))) 
+ 2(\cos(\theta_d'(i) - \theta_d'(i+1))\cos(\theta_b'(i) - \theta_b'(i+1)) 
+ \cos(\theta_d'(i) - \theta_b'(i))\cos(\theta_d'(i+1) - \theta_b'(i+1)) 
- \cos(\theta_d'(i) - \theta_b'(i+1))\cos(\theta_d'(i+1) - \theta_b'(i))) \right\} .
$$

Here the terms with $J_2$ vanish because of the symmetry with respect to the reflection of the spins (2.6). The variation with respect to $\theta_d'(i)$ leads to

$$
J_1(-1)^{i+1}(\sin(\theta_d'(i) - \theta_d'(i-1)) - \sin(\theta_d'(i) - \theta_d'(i+1))) 
+ \sum_{m=\pm 1} \left[ \frac{J_3}{2}\sin(\theta_d'(i) - \theta_d'(i+2m)) 
+ \frac{K_p}{4}\{-\sin(2\theta_d'(i+m) - \theta_d'(i) - \theta_d'(i+2m)) 
+ 2\sin(\theta_d'(i) - \theta_d'(i+m))\cos(\theta_d'(i) - \theta_d'(i-m)) 
- 4\sin(\theta_d'(i) - \theta_b'(i))\cos(\theta_d'(i) - \theta_b'(i)) 
+ 8(\sin(\theta_d'(i) - \theta_d'(i+m))\cos(\theta_b'(i) - \theta_b'(i+m)) 
+ \sin(\theta_d'(i) - \theta_b'(i))\cos(\theta_d'(i+m) - \theta_b'(i+m)) 
- \sin(\theta_d'(i) - \theta_b'(i+m))\cos(\theta_d'(i+m) - \theta_b'(i))) \right\} = 0
$$
and a similar equation with the subscripts $a$ and $b$ interchanged.

For the particular case $J_1=0$, we have

$$\theta_a'(i) = \theta_b'(i) \quad (2.11)$$

and can solve Eq. (2.10) analytically. As shown in Appendix A, we get the solutions with an arbitrary angle between the $\vec{d}$ vectors of two domains, i.e., $\theta_0 = \theta_a'(\infty) - \theta_a'(-\infty)$. For the solutions the boundary energy does not depend on the angle $\theta_0$. The width of the domain wall $l_{a,w}$ tends to zero in the limit $2J_3/K_F \rightarrow 1$, that is, the spin alignment of the two domains is uniform on both sides of the boundary. In this limit, it is easy to see from (2.7) and (2.8) that the boundary energy does not change when the uniform spin alignment of $i \geq 1$ is rotated rigidly from that of $i \leq 0$. Moreover, we can show that, when $\theta_0=0$, the solution with $l_{a,w}=0$ still satisfies Eq. (2.10) even if $J_1 \neq 0$. It means $J_1$ does not contribute to the boundary energy for $\theta_0=0$ and $2J_3/K_F = 1$. For a finite $J_1$, however, the condition (2.11) is lost and the first nearest-neighbor exchange energy becomes lower, so that Eq. (2.10) can no longer be solved analytically.

We consider $5 \times 2$ spins near the boundary, i.e., $\theta_a'(i)$ for $-2 \leq i \leq 2$ with conditions $\theta_a'(i) = 0$ for $i \leq -3$ and $\theta_a'(i) = \theta_0$ for $i \geq 3$, and numerically determine the configuration of the spins which satisfies Eq. (2.10) by using the exchange parameters given in (2.4). The boundary energy thus obtained depends on the angle $\theta_0$ and becomes minimum when $\theta_0 = \frac{\pi}{2}$. In order to confirm this conclusion we have repeated the same calculation for $9 \times 2$ spins and obtained the same result. The reason why we need not let the direction of a large number of spins vary is that the obtained wall thickness $l_{a,w}$ is of only a few layers. We have also investigated the problem of $9 \times 4$ spins in which $\theta_a(i)$ and $\theta_b(i)$ can vary independently of $\theta_a(i)$ and $\theta_b(i)$ to justify the assumption of the reflection symmetry (2.6). The solution without the reflection symmetry cannot be found.

We discuss in the next section the variation of $\vec{d}$ over the distance of the order of the dipole length $\xi$ near the boundary region, taking into account the effect of an applied magnetic field and the dipole interaction. Then, since the exchange interaction between two spins is much stronger than the dipole interaction and $l_{a,w}$ is only a few layers thick, the results obtained in this section reduces to the boundary condition of $\vec{d}$.

§ 3. The domain boundary in the presence of the dipole interaction and an external field

In the uudd phase the dipole interaction and an external field $H\vec{h}$ tend to confine the direction of the sublattice magnetization to the plane perpendicular to $\vec{I}$ and to $\vec{h}$, respectively. In their presence, therefore, the $\vec{d}$-vectors in the two domains can no longer take arbitrary directions far from the boundary. We will here study the slow spatial variation of $\vec{d}$ connecting the equilibrium directions inside the two domains, using the equations of motion in the continuum limit for the magnetization and sublattice magnetization density which were given in our preceding paper.69

We assume $\vec{d}$ varies one-dimensionally perpendicular to the (110) boundary plane, i.e., along the [110] direction. Here we introduce the $\xi$-coordinate along [110]. Let us consider adjacent domains separated by the boundary plane at $\xi=0$; the domain-1 with $\vec{I}_1 = \vec{x} = (1, 0, 0)$ for $\xi < 0$ and the domain-2 with $\vec{I}_2 = \vec{y} = (0, 1, 0)$ for $\xi > 0$.

We will consider the boundary condition on $\vec{d}$. First, as discussed in the last section,
we have the $\pi/2$ jump of $\vec{d}$ at the boundary,
\[
\vec{d}_1(-0) \perp \vec{d}_2(+0).
\]

(3.1)

Next, in order to consider the boundary condition on $\nabla \vec{d}$, let us study the scattering of spin waves at the wall (110)a obtained in the previous paragraph. Although it is difficult to solve the scattering problem for general cases, we can easily show that the transmission coefficient $T$ tends to unity in the limit of long wave length $\lambda \gg l_{a,w}$, because the difference between $\delta S_{e}(i)$ and $\delta S_{e}(i+1)$ is of the order $O(a/\lambda)$ even at the domain wall, where $\delta S_{e}(i)$ is the small oscillations of $S_{e}(i)$. If $T=1$, we get the boundary condition
\[
\frac{\partial}{\partial \nu} \vec{d}_1(-0) = \frac{\partial}{\partial \nu} \vec{d}_2(+0)
\]

(3.2)

from the conservation of energy current at the boundary
\[
J \propto \dot{\vec{d}} \frac{\partial}{\partial \nu} \vec{d},
\]

(3.3)

where the dot represents the time derivative and $\partial/\partial \nu$ refers to derivative parallel to the current. The derivation of Eq. (3.3) is described in Appendix B.

For simplicity we confine ourselves to the limit of the strong magnetic field $\omega_{L} \gg \Omega_{b}$, where $\omega_{L} = \gamma_{0} H$ is the Larmor frequency with the gyromagnetic ratio $\gamma_{0}$ and $\Omega_{b}$ the resonance frequency at zero field. In this limit, $\vec{d}$ is confined to the plane perpendicular to $\vec{h}$. Therefore, it is convenient to introduce the rectangular coordinates $\vec{x'} = \hat{l}_1 \times \vec{h}$, $\vec{y'} = \hat{l}_1 \times \vec{h} / ||\hat{l}_1 \times \vec{h}||$, $\vec{z'} = \vec{h}$ to describe the variation of $\vec{d}$ (Fig. 9). We will specify $\vec{d}$ by an angle $\phi$:
\[
\vec{d} = \cos \phi \vec{x'} + \sin \phi \vec{y'}.
\]

(3.4)

The equilibrium configuration of the $\vec{d}$ vector is described by the sine-Gordon equation,$^6$
\[
c_{s}^{2} \partial_{\tau}^{2} \phi = \frac{\Omega_{b}^{2}}{2} \sin^{2} \theta \sin 2\phi
\]

(3.5)

with $\xi = (1/\sqrt{2})(x+y)$, where $c_{s}^{2} = (1/2)(c_{s}^{2} + c_{\perp}^{2})$, $c_{s}$ and $c_{\perp}$ are the spin wave velocities in the direction parallel and perpendicular to $\vec{l}$ respectively and $\theta$ is the angle between $\vec{h}$ and $\vec{l}$.

If we represent the $\vec{l}$ vectors in the $(x', y', z')$ coordinates by
\[
\vec{l}_1 = \sin \theta_1 \vec{y'} + \cos \theta_1 \vec{z'}
\]

(3.6a)

and
\[
\vec{l}_2 = \sin \theta_2 \cos \delta \vec{x'} + \sin \theta_2 \sin \delta \vec{y'} + \cos \theta_2 \vec{z'},
\]

(3.6b)

the condition
\[
\sin \delta = -\cot \theta_1 \cot \theta_2
\]

(3.7)
must be satisfied because of $\hat{I}_1 \perp \hat{I}_2$. Since $\hat{d} // \pm (\hat{h} \times \hat{I})$ for $\xi \to \pm \infty$, we adopt the boundary conditions given by (Fig. 10)

$$\phi(-\infty)=0 \quad \text{or} \quad \pi,$$

$$\phi(+\infty)=\delta \pm \frac{\pi}{2},$$

which yields

$$\cos\phi = \pm \tanh \frac{\xi-a_1}{\xi_1} \quad \text{for} \quad \xi<0,$$

$$\sin(\delta-\phi) = \pm \tanh \frac{\xi-a_2}{\xi_2} \quad \text{for} \quad \xi>0,$$

where

$$\xi_i = \frac{C_s}{Q_0} \sin \theta_i, \quad (i=1, 2)$$

and $a_1$ and $a_2$ are constants which must be determined by the matching conditions at the domain boundary.

The conditions (3·1) and (3·2) yield

$$\lim_{\xi \to 0} \phi = \lim_{\xi \to +\infty} \phi \pm \frac{\pi}{2} = \phi_0,$$

$$\lim_{\xi \to 0} \partial_\xi \phi = \pm \lim_{\xi \to +\infty} \partial_\xi \phi.$$  

From (3·9) and (3·10) we obtain

$$\tan \phi_0 = \pm \frac{\cos \theta_1 \cos \theta_2}{\sin^2 \theta_1 \pm \sqrt{\sin^2 \theta_1 - \cos^2 \theta_2}},$$

$$\cos \phi_0 = -\tanh(-a_1/\xi_1),$$

$$\cos(\delta-\phi_0) = \tanh(-a_2/\xi_2).$$

Here we comment on the signs in Eq. (3·11a). We have four possible combinations of signs on the right-hand side of (3·11a), i.e., $(+, +)$, $(+, -)$, $(-, +)$ and $(-, -)$, which give four different configurations of $\hat{d}$ vector. The relative orientations of $\hat{I}_1$, $\hat{I}_2$ and $\hat{h}$ determine which of the four is most stable.

§ 4. Scattering of spin wave at the domain wall

In the limit of a strong magnetic field $\omega_t \gg \Omega_0$, the transverse spin wave mode decouples from the longitudinal mode and the equation of motion for the transverse mode simplifies to

$$\omega^2 \delta d_z = [\omega^2 + \Omega_0^2 \cos^2 \theta + \Omega_0^2 \sin^2 \theta \cos 2\phi - \cos^2 \theta \Omega_0^2] \delta d_z$$

(4·1)
and for the longitudinal mode\(^{\star}\)

\[
\omega^2 \delta S_z = \left[ \Omega_0^3 \sin^2 \theta \cos 2\phi - c_s^2 \partial^2 \xi^2 \right] \delta S_z .
\]

(4.2)

From (3.9), the dipole potential \( V = \Omega_0^2 \times \sin^2 \theta \cos 2\phi \) is given by (Fig. 11)

\[
V = \Omega_0^2 \sin^2 \theta \left( 1 - 2 \text{sech}^2 \frac{\xi - \alpha_i}{\xi_i} \right),
\]

(4.3)

where, \( i = 1(2) \) for \( \xi < (>) 0 \). Here we discuss only the scattering of the longitudinal mode; a similar discussion can easily be applied to the transverse mode and we get the same result.

If the incident wave propagates from \( \xi = -\infty \), the solution to (4.2) for \( \xi < 0 \) (see Appendix C for derivation) is

\[
\delta S_z = A \frac{2^{-ik_1\xi_1}}{-ik_1 \xi_1 + 1} e^{ik_1(\xi - a_1)} \left( -ik_1 \xi_1 + \tanh \frac{\xi - a_1}{\xi_1} \right) \]

\[+ B \frac{2^{-ik_1\xi_1}}{ik_1 \xi_1 + 1} e^{-ik_1(\xi - a_1)} \left( ik_1 \xi_1 + \tanh \frac{\xi - a_1}{\xi_1} \right),
\]

(4.4)

while for \( \xi > 0 \),

\[
\delta S_z = \frac{2^{-ik_2\xi_2}}{-ik_2 \xi_2 + 1} e^{ik_2(\xi - a_2)} \left( -ik_2 \xi_2 + \tanh \frac{\xi - a_2}{\xi_2} \right),
\]

(4.5)

where \( k_1 \) and \( k_2 \) are related to \( \omega \) as

\[
\omega^2 = \Omega_0^2 \sin^2 \theta_1 + c_s^2 k_1^2 = \Omega_0^2 \sin^2 \theta_2 + c_s^2 k_2^2 .
\]

(4.6)

From the boundary condition that \( \delta S_z \) and \( \partial \delta S_z \) are continuous at \( \xi = 0 \), we obtain the transmission coefficient as follows:

\[
T = \frac{1}{|A|^2} \frac{k_2}{k_1} \]

\[= 4k_1k_2(1 + k_1^2 \xi_1^2)(1 + k_2^2 \xi_2^2) \]

\[\times \left[ k_1^2 k_2^2 \xi_1^2 \xi_2^2 (k_1 + k_2)^2 + c_s^2 k_1^4 \xi_1^4 + c_s^2 k_2^4 \xi_2^4 \right. \]

\[+ k_2^2(2(2 + 2c_1^2) + \xi_2^2(2 - c_2^2)) + 2k_1k_2(k_2^2 \xi_1^2 + k_2^2 \xi_2^2 + 1) \]

\[+ k_1^2 \left( c_s^2 (2 - c_1^2) + \frac{\xi_1^2}{\xi_2} (1 - c_2^2)^2 \right) + k_2^2 \left( c_s^2 (2 - c_2^2) + \frac{\xi_2^2}{\xi_1} (1 - c_1^2)^2 \right) \]

\[+ \left\{ \frac{c_2}{\xi_2} (1 - c_1^2) + \frac{c_1}{\xi_1} (1 - c_2^2) \right\} \right]^{-1},
\]

(4.7)

where \( c_1 = \cos \phi = -\tanh(-a_1/\xi_1) \) and \( c_2 = \cos(\delta - \xi_0) = \tanh(-a_2/\xi_2) \). In the long wave length limit of \( k_1 \), \( T \) behaves for \( k_1 \approx k_2 \) as

\(^{\star}\) Equation (5.13b) in Ref. 6 has an error of the sign and should read as (4.2).
Magnetic Domain Wall in uudd $^3$He

\[ \lim_{k_1 \to 0} T \propto k_1 \]

and for $k_1 = k_2$ as

\[ \lim_{k_1 \to 0} T \propto k_1^2. \] (4.8b)

In conclusion, the spin wave with the long wave length is reflected almost perfectly by the dipole potential at the domain wall.

§ 5. Conclusion and discussion

We have studied the magnetic domain wall in uudd $^3$He. First, we obtained the equilibrium domain wall shown in Fig. 4(a) by minimizing only the exchange energy. Next, taking into account a magnetic field and the dipole interaction, we determined the $\vec{d}$ field near the boundary. Using the continuum approximation we then studied scattering of spin waves by the domain wall and showed that spin waves with small wave number $k \ll \xi_0^{-1}$, where $\xi_0$ is the dipole length, are almost totally reflected by the dipole potential associated with the $\vec{d}$-field structure near the wall. As a consequence, the spin precession in a resonant domain is in general not coupled to spins in other non-resonant domains. This result may be related to the fact that three sharp resonant peaks are always observed in the c-w NMR experiments.

On the other hand, the FID observed by Kusumoto et al. cannot be explained by the spin current carrying the Zeeman energy from the resonant to non-resonant domains in the linear approximation. If nonlinear effects, such as, three or four-magnon process at the domain wall, are taken into account, it might be possible to explain their experiment.

The localized modes trapped in the dipole potential (4.3) may exist, as discussed in Ref. 6). However, since the characteristic width of the domain wall is very small, i.e., $\xi_0 \sim \xi_0 / \xi_0 \sim 10^{-6}$ cm, it might be difficult to detect it.

Since the magnons have a wave length comparable to the lattice constant, they are scattered by the exchange coupling at the domain wall which is different from the bulk exchange coupling. Therefore, we expect that the thermal conductivity in the spin ordered phase could yield information concerning the domain walls.

In order to examine the effects of magnetic domains on the spin dynamics of uudd $^3$He more closely we need a crystal of single domain. We hope a future progress in experiment along this line.

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Appendix A

We shall show that Eq. (2.10) has the analytic solution for $J_1 = 0$. In this limit, Eq. (2.10) (rewritten in terms of $\theta_d(i)$) is satisfied when $\theta_d(i) = \theta_s(i)$ for odd $i$ and $\theta_d(i) = \theta_s(i) + \pi$ for even $i$, and reduces to
\[ J \sin \frac{\delta_{i+1} + \delta_{i-1} + 2 \delta_i}{2} - K_p \sin \frac{\delta_{i+1} + \delta_{i-1} - 2 \delta_i}{2} = 0, \]  \hspace{1cm} (A.1)

where \( \delta_i = \theta_{i+1} - \theta_i \). Equation (A.1) has a nontrivial solution

\[ \cos \delta_i = \tanh \chi (i + \Delta_i) \]  \hspace{1cm} (A.2)

with

\[ \alpha = \frac{2 J_3}{K_p} = \frac{\cosh \chi - 1}{\cosh \chi + 1}, \]  \hspace{1cm} (A.3)

where \( \Delta_i \) is a constant and \( \chi^{-1} \) is the thickness of the domain wall. For \( \Delta_i = 0 \), because of the symmetry of the solution (A.2), we obtain

\[ \theta_{i+1} = \theta(\infty) - \theta(-\infty) = \sum_{i=-\infty}^{\infty} \cos^{-1} \tanh \chi i = \frac{\pi}{\alpha}. \]  \hspace{1cm} (A.4)

We can calculate \( \theta_0 \) for \( \Delta_i \neq 0 \) in the limit \( \chi \to 0 \), i.e., \( J_3 \to 0 \).

By changing the summation to an integration in this limit, we get

\[ \theta_0 = \frac{\pi}{2} + \lim_{\chi \to 0} \int_{-\infty}^{\infty} dx \left( \cos^{-1} \tanh \chi (i + \Delta_i) - \cos^{-1} \tanh \chi i \right) \]

\[ = \frac{\pi}{2} + \pi \Delta_i. \]  \hspace{1cm} (A.5)

Next, we calculate the \( \theta_0 \)-dependence of the boundary energy. We rewrite the energy (2.9) as

\[ E = -\frac{K_p}{8} \sum_i \left[ \cos(\delta_{i+1} - \delta_i) + \cos(\delta_i - \delta_{i-1}) \right] \]

\[ - \frac{J_3}{4} \sum_i \left[ \cos(\delta_{i+1} + \delta_i) + \cos(\delta_i + \delta_{i-1}) \right] \]

\[ = -\frac{K_p}{8} \sum_i \left[ \cos \delta_{i+1} \cos \delta_{i-1} + (\sin \delta_{i+1} + \sin \delta_{i-1}) \sin \delta_i \right] \]

\[ - \frac{J_3}{4} \sum_i \left[ \cos \delta_{i+1} \cos \delta_{i-1} - (\sin \delta_{i+1} + \sin \delta_{i-1}) \sin \delta_i \right]. \]  \hspace{1cm} (A.6)

By substituting (A.2) into (A.6) and using (A.3), the energy (A.6) is reduced to

\[ E = \sum_i \frac{2 \cosh \alpha (\cosh \alpha - 1)}{\cosh \chi (i + 1 + \Delta_i) \cosh \chi (i - 1 + \Delta_i)}. \]  \hspace{1cm} (A.7)

Finally, using the relation

\[ \tanh \chi (i + 1 + \Delta_i) - \tanh \chi (i - 1 + \Delta_i) = \frac{2 \sinh \alpha \cosh \alpha}{\cosh \chi (i + 1 + \Delta_i) \cosh \chi (i - 1 + \Delta_i)}, \]  \hspace{1cm} (A.8)

we obtain

\[ E = -\frac{K_p}{2} \tanh \frac{\chi}{2}, \]  \hspace{1cm} (A.9)

which is independent of \( \theta_0 \).
Appendix B

In this appendix we refer to the derivation of Eq. (3 · 3).

By including gradient terms we have generalized the nonlinear equations of the spin dynamics in uudd $^3$He, proposed by OCF, for spatially inhomogeneous cases (Eqs. (3 · 1) and (3 · 2) of Ref. 6):

\[
\dot{\mathbf{d}} = \mathbf{d} \times (\gamma_0 \mathbf{H} - \gamma_0^2 \chi_0^{-1} \mathbf{S}),
\]

\[
\mathbf{S} = \gamma_0 \mathbf{S} \times \mathbf{H} - \lambda (\mathbf{d} \cdot \mathbf{l}) (\mathbf{d} \times \mathbf{l}) + \chi \gamma_0^{-2} [c_s^2 (\mathbf{d} \times \nabla_s \mathbf{d}) + c_\perp^2 (\mathbf{d} \times \nabla_\perp \mathbf{d})],
\]

where $\mathbf{S}$ is the total spin and $\lambda$ is the coefficient of the dipole energy $(\lambda (\mathbf{d} \cdot \mathbf{l})^2 / 2)$. For simplicity we transform space coordinates so that $c_s = c_\perp = c$.

Total magnetic energy of a domain is

\[
E = \iiint_V \left[ \frac{1}{2} \gamma_0^2 \chi_0^{-1} S^2 - \gamma_0 \mathbf{S} \cdot \mathbf{H} + \frac{\lambda}{2} (\mathbf{d} \cdot \mathbf{l})^2 
\right.
\]

\[
+ \frac{1}{2} \gamma_0^{-2} \chi_0 c^2 (\nabla \cdot \mathbf{d})^2 + (\nabla_d \mathbf{d})^2 + (\nabla \cdot \mathbf{d})^2 \right] dv.
\]

Using Eqs. (B · 1) and (B · 2), we obtain the time derivative of the energy

\[
\dot{E} = \gamma_0^{-2} \chi_0 c^2 \iint_S \frac{\partial}{\partial \mathbf{n}} \cdot \mathbf{d} d\sigma,
\]

where the integration is taken over the whole of the closed boundary surface surrounding the domain and $\partial / \partial \mathbf{n}$ refers to the derivative normal to the boundary surface. The vector

\[
\mathbf{J} = \gamma_0^{-2} \chi_0 c^2 \mathbf{d} \cdot \frac{\partial}{\partial \mathbf{n}} \mathbf{d}
\]

is the energy current.

Appendix C

In this appendix we refer briefly to the derivation of Eq. (4 · 4). The scattering solution of the Schrödinger equation,

\[
E \phi = \left( - \frac{\hbar^2}{2m} \frac{d^2}{dx^2} + U_0 \text{sech}^2 \beta x \right) \phi
\]

is expressed as follows:

\[
\phi = (1 - \eta^2)^{-ik/2\beta} F \left( - \frac{ik}{\beta} s, - \frac{ik}{\beta} s + 1, \frac{1 - \eta}{2} \right),
\]

where $\eta = \tanh \beta x$, $k = (1 / \hbar) \sqrt{2mE}$ and $s = \frac{1}{2} \left( -1 + \sqrt{1 - 8mU_0 / \beta^2 \hbar^2} \right)$. Substituting (3 · 9c) into (4 · 1), one finds that $s = 1$ and the solution (C · 2) reduces to (4 · 4).
References

9) L. D. Landau and E. M. Lifshitz, Quantum Mechanics, revised third ed. (Pergamon Press).