bles since we regarded the ideally straight column as a compression creep specimen up to the introduction of the test disturbance. As a result, the creep instability criterion depended on uniform compression creep properties and not on nonuniform stress, strain, and hence metallurgical histories. However, as we pointed out in the paper, considerable time may elapse between the onset of instability and ultimate collapse, so that the complete creep buckling process will undoubtedly be path-dependent and thus will be governed by both mechanical and metallurgical processes.

The Elastic Moduli of Heterogeneous Materials[1]

B. PAUL.[2] The author has succeeded in finding improved upper and lower bounds on the elastic constants of heterogeneous materials with spherical inclusions. The purpose of this discussion is to show how these bounds compare with the writer's previous bounds (reference [19] of the paper) and the experimental work of Kieffer and Schwartzkopf [1][3] and to call attention to some more recent experimental work [2].

Fig. 1 of this discussion shows the various bounds obtained by Hashin and Paul. It also shows the experimental points that were obtained by Nishimatsu and Gurland (reference [17] of the paper) and Kieffer and Schwartzkopf [1]. The broken line represents an approximate "strength-of-materials" type of solution based upon a cube-shaped inclusion. It is to be observed that the majority of experimental points reported by Kieffer and Schwartzkopf lie slightly above the author's theoretical upper bound. Although the difference is slight and may be attributed either to photographic distortion of printed figures or to the unavoidable scatter in works of this type, it is interesting to note that the author's upper bound shown in the figure is strictly correct only for a spherical inclusion. In a later work [3] Hashin and Shtrikman dispense with the assumption of a spherical inclusion and find bounds which are apparently independent of inclusion geometry. The upper bound so found coincides almost exactly with Paul's approximate solution which, as may be seen in the figure, is compatible with the data of Kieffer and Schwartzkopf. In view of the inherent uncertainties in measurements of elastic constants, the writer feels that any of the author's bounds, as well as Paul's approximate solution, are sufficiently close for most practical purposes. However, it is worth noting that the bounds presented in the paper under discussion are strictly valid only for spherical particles; and the experimental data although questionable seem to reflect this fact.

The writer also would like to call attention to Gurland's paper [2] in which he reports the values for Young's modulus E given in Table 1.

[1] By Zvi Hashin, published in the March, 1962, issue of the JOURNAL OF APPLIED MECHANICS, vol. 29, TRANS. ASME, vol. 84, Series E, pp. 143–150.

[2] Member of Technical Staff, Bell Telephone Laboratories, Murray Hill, N. J.

[3] Numbers in brackets designate References at end of discussion.

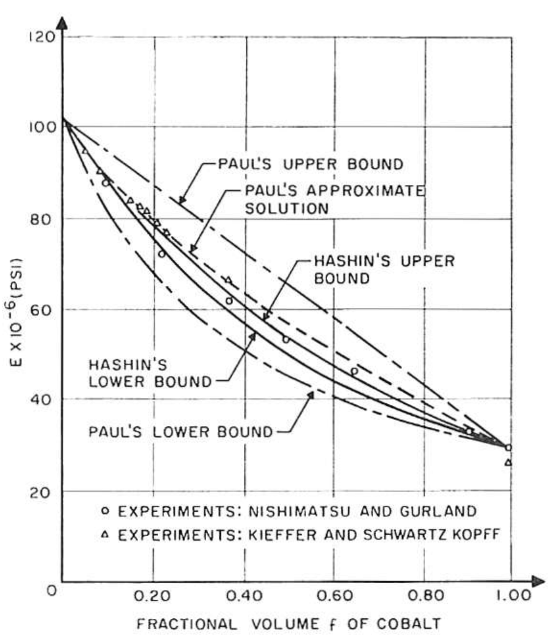

Fig. 1 Comparison of theory with experiment

Table 1 Values for Young's modulus E

Material	$E \times 10^6$ (psi) experiment	$E \times 10^6$ (psi) Paul's approximate theory
100% Ag....................	11.1	11.1
85% Ag-15% W............	14.5	15
85% Ag-15% MO...........	15.2	15
85% Ag-15% WC...........	17.8	17.4
85% Ag-15% Ni............	12.1	13.4

Although Gurland did not specify the elastic constants of the dispersed phases, one could compute the values of E used for the dispersed phase from the given data and equation (21) of reference [19] of the paper. Would the author care to apply his theory to these cases (estimating, if necessary, reasonable values of Poisson's ratio)? In view of the scarcity of experimental data it is perhaps worth noting the paper of Brooks, et al. [4] which, according to H. Brooks [5], tends to confirm the approximate solution of reference [19].

References

1 R. Kieffer and P. Schwartzkopf, "Hartstoffe und Hartmetalle," Springer, Vienna, Austria, 1953.

2 J. Gurland, "Comparison of Dispersion Hardening in Four Silver Base Alloys of Equivalent Composition," *Trans. AIME*, vol. 22, 1961, p. 407.

3 Z. Hashin and S. Shtrikman, "On Some Variational Principles in Elasticity and Their Application to the Theory of Two Phase Materials," Technical Report No. 1, Nonr 551(42), Towne School of Engineering, University of Pennsylvania, July, 1961.

4 H. Brooks, G. I. Lewis, and J. I. M. Forsyth, "Elastic Moduli and Tensile Properties of Titanium-Carbon and Titanium-Aluminium-Carbon Alloys," Technical Note Met. 265, Royal Aircraft Establishment, Farnborough, Hants, England, June, 1957.

5 H. Brooks, private communication, March, 1959.

Author's Closure

I wish to thank Dr. Paul for his detailed comments.

It seems to me that the practical value of results and agreement with experimental measurements are perhaps overemphasized in the discussion. Expressions for effective elastic moduli, or other physical constants of multiphase materials, will be of lasting value only if, besides agreeing with experimental results, they are also based on exact theory. The concept of an effective physical constant is, of course, based on an approximation, but once such a constant has been properly defined one should strive to derive expressions for it by sound theory.

The bounds derived by Dr. Paul are very important theoretical results, especially since they hold for arbitrary phase geometry. His approximate expression, however, is admittedly based on crude "strength of materials" assumptions. No account is taken of boundary conditions at particle-matrix interfaces. The Euler-Bernoulli assumption, which is known to be a good approximation only for slender cylinders, is applied to the deformation of plane sections of cubical particles embedded in a matrix.

If practical value and agreement with experiments are considered as being of sole importance, then one may just as well use the average of Dr. Paul's bounds, which does agree quite well with the experiments cited.

The field of the analysis of the gross behavior of multiphase systems has known a large number of investigations based on rough approximations, especially in the analysis of electric and viscous properties of such systems. As a result, many different and sometimes contradictory expressions have been derived. In view of this, it seems to me that exact results, even when holding only for special spherical geometry, are of value.

Of special importance is the result for the effective bulk modulus given in the paper. On the basis of this it has been later shown (reference [3] in Dr. Paul's discussion) that bounds obtained for the bulk modulus of quasi-isotropic and quasi-homogeneous two-phase materials of arbitrary phase geometry are the most restrictive ones that can be found in terms of phase moduli and volume fractions. [A stronger argument for this can be given on the basis of results obtained by R. Hill (private communication).] This shows that an effective elastic modulus of a two-phase material is not determinate in terms of phase moduli and volume fractions, which is a conclusion of considerable practical importance.

Before applying the present theory to check the experimental data given in reference [2] of the discussion (as suggested by Dr. Paul), it will be necessary to have full information about the elastic moduli of the constituting phases.

Bending Deflection of a Circular Shaft Terminating in a Semi-Infinite Body[1]

J. G. LEKKERKERKER.[2] The authors state that, in view of the inaccuracy of the experimental results, it cannot be decided whether the analytically obtained value of the intercept distance is too high or too low. However, using the theorem of minimum energy of the stresses, it can be proved that the assumption of linearly distributed tractions at the contact area between the beam and the elastic half-space leads to an upper bound.

On the other hand, presuming linear displacements within the contact area, that is, preventing this circular surface from being curved whereby the construction is made more rigid, the writer obtained a lower bound[3] (apart from a probably small error as a result of the neglect of radial and tangential tractions).

In an analogous way the case of a beam of circular cross section, clamped in an elastic half-space and subject to torsion, has been treated. In this case no error occurs in determining the lower bound of the intercept distance in consequence of the symmetry which prevents either radial or axial tractions within the contact area.

The results are given in Table 1 of this discussion.

Table 1

	Bending	Torsion
Upper bound (linear tractions)	$0.34\ (1 - \mu^2)d$	$0.17\,d$
Lower bound (linear displacements)........	$0.295(1 - \mu^2)d$	$0.15\,d$

It may be remarked that some experiments bearing on the subject have been carried out in the Laboratory of Technical Mechanics at Delft; relating to torsion, in 1939,[4] and, relating to bending, recently by Prof. J. J. Koch and P. Hoedemaker. The results of the latter authors have not yet been published. It may be worthwhile to draw attention to the fact that there are three quantities which are of essential significance: (*a*) The fillet radius, (*b*) the shank width, and (*c*) the shank length. Extrapolating the experimental results to the limiting case of zero fillet radius and infinite length and width of the shank, being the mathematical model of a beam clamped in an elastic half-space, each case gives a value which corresponds to the analytical result.

Authors' Closure

Mr. Lekkerkerker is quite correct in his statement that the analytical value presented in our paper is an upper bound. In addition, the lower bound he (accurately) determined, coupled with the upper bound, fixes the effective additive length L_0 between $0.295\ (1-\mu^2)d$ and $0.340\ (1-\mu^2)d$. Thus L_0 can be taken as $0.317\ (1-\mu^2)$ with an accuracy of ± 7 percent. This accuracy is somewhat in agreement with the intuitive claim of ± 10 percent accuracy given in our paper.

Experimental results of bending and torsional tests on a large number of specimens with various fillet radii, groove lengths, disk lengths, groove diameters, and disk diameters are presented in references [2] to [4] of our paper.

[1] By J. M. Brown and A. S. Hall, published in the March, 1962, issue of the JOURNAL OF APPLIED MECHANICS, vol. 29, TRANS. ASME, vol. 84, Series E, pp. 86–90.

[2] Laboratory of Technical Mechanics, Technological University, Delft, The Netherlands.

[3] J. G. Lekkerkerker, "The Effective Length of a Cantilever Beam With Circular Cross-Section" (in Dutch), *De Ingenieur*, vol. 73, 1961, p. O 110.

[4] C. B. Biezeno and J. J. Koch, "Some Experimental Data About the Reduced Length of Torqued Beams of Varying Cross-Section" (in Dutch), *De Ingenieur*, vol. 54, 1939, p. A433.