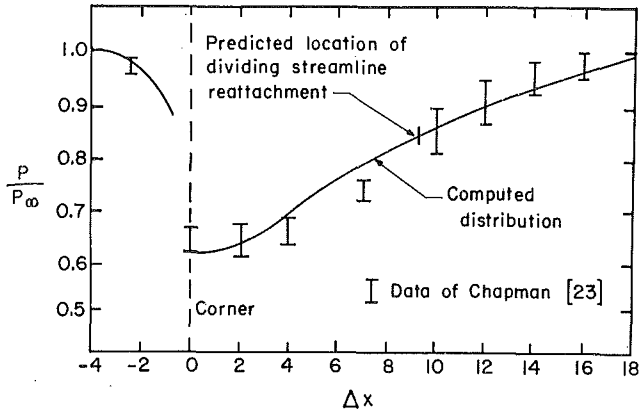

Fig. 5 Wall pressure distribution downstream of backstep with $M_\infty = 2$

of the dividing streamline reattachment point. Although the latter information was not obtained experimentally the predicted pressure distribution is in general agreement with measurements.

The present computations also predicted the characteristic corner flow behavior which occurs because the approach boundary layer overexpands and clings to the vertical backstep surface before breaking away to form a free shear layer [12]. For the $M_\infty = 2$ case of Figs. 3, 4 with a step height of $10\Delta y$ the breakaway point is approximately $2.5\Delta y$ below the corner (point C of Fig. 1).

Concluding Remarks

The complete time-dependent Navier-Stokes equations have been solved numerically using the explicit method of Rusanov. This study represents one of the few successful applications of this differencing technique to flows involving both natural viscosity and associated non-slip boundary conditions. Through numerical experimentation, an extended range for the explicit stability parameter was established. An additional convergence parameter, which relates the incremental spatial steps, was also defined.

Computation times for the two geometries investigated were quite different. For the flat plate, which utilized 2500 nodal points, practical steady state was achieved in approximately 50 minutes of UNIVAC 1108 computer time which is considered quite satisfactory for a numerical solution of this nature. Flow fields for the backstep and near wake geometry required approximately 200 minutes to reach a steady state solution; this is due primarily to the low velocities in the recirculating region of the field. The generally favorable agreement of the predicted flow parameters with experimental results suggests that the current numerical technique can be confidently applied to a wide spectrum of laminar flows which include imbedded shock waves and/or separated flow regions.

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DISCUSSION

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The authors have attempted to solve a very complex problem, and the result in terms of the shock wave resolution is excellent. It is interesting to see that even with the use of artificial viscosity, a very thin shock layer can be obtained as part of the result. However, when the real boundary layer is also computed, the question arises as to the effect of the artificial viscosity on the real fluid behavior near the walls. Since the paper does not contain any specific statement with regard to this effect, I am suggesting the following analysis of the properties of the artificial viscosity device used in the paper.

A classic method for computing the accuracy of finite difference schemes applied to nonlinear partial differential equations is to linearize the governing equations. Two dimensional problems are even then intractable, and hence in the following analysis, the flow is assumed to be one dimensional. The results of such an analysis can be believed to be qualitatively correct.

The governing equation (2) is simplified to:

$$\frac{\partial f}{\partial t} + \frac{\partial F}{\partial x} = \frac{\partial}{\partial x}\left[A(x, t)\,\frac{\partial f}{\partial x}\right] \equiv D(x, t) \tag{7}$$

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where $$f = \begin{bmatrix} \rho \\ \rho u \\ e \end{bmatrix}$$

and $$F = \begin{bmatrix} \rho u \\ \rho u^2 + p + \tau_{xx} \\ (e + p)u + q_x + \tau_{xx}u \end{bmatrix}$$

The finite difference equation (3) applied to (7) gives

$$f_j^{n+1} = f_j^n - \frac{\Delta t}{2\Delta x} [F_{j+1}^n - F_{j-1}^n] + \Delta t\bar{D} \tag{8}$$

$\bar{D}$ represents the finite difference representation for D.

Define J to be the Jacobian matrix of F with respect to f.

$$\frac{\partial F}{\partial x} = J \frac{\partial f}{\partial x}$$

When we assume J to be a constant, the left-hand side of (7) is linearized. The finite difference equation is then:

$$f_j^{n+1} = f_j^n - J \frac{\Delta t}{2\Delta x} [f_{j+1}^n - f_{j-1}^n] + \Delta t\bar{D} \tag{9}$$

Quantities f_{j+1}^n, f_{j-1}^n, and f_j^{n+1} can be expanded in a Taylor series about the value of f_j^n and truncated consistently. To evaluate terms involving time derivative, we can use the governing equation repeatedly.

i.e. $$\frac{\partial f}{\partial t} = D - J \frac{\partial f}{\partial x}$$

$$\frac{\partial^2 f}{\partial t^2} = \frac{\partial D}{\partial t} - J \frac{\partial D}{\partial x} + J^2 \frac{\partial^2 f}{\partial x^2}$$

etc.

Substituting these expressions into (9), we can get:

$$\frac{\partial f}{\partial t} + J \frac{\partial f}{\partial x} = \bar{D} - \frac{\Delta t}{2} \left[\frac{\partial D}{\partial t} - J \frac{\partial D}{\partial x} + J^2 \frac{\partial^2 f}{\partial x^2} \right]$$

Since D is of order Δ [equation (4)], we may simplify the right-hand side (RHS) to:

$$\text{RHS} = \bar{D} - \frac{\Delta t}{2} J^2 \frac{\partial^2 f}{\partial x^2}$$

If we neglect the errors occurring in the term $\bar{D}$ due to finite difference representation (as this will be of order higher than Δ),

$$\bar{D} \simeq \frac{\partial}{\partial x} A \frac{\partial f}{\partial x} = \frac{\partial}{\partial x} \left[\frac{\Delta x^2}{2\Delta t} C_1\sigma \frac{\partial f}{\partial x} \right]$$

For C_1 independent of x,

$$\bar{D} = \frac{\Delta x^2}{2\Delta t} C_1\sigma \frac{\partial^2 f}{\partial x^2}$$

Hence the finite difference system corresponds to an equivalent differential equation, which is:

$$\frac{\partial f}{\partial t} + J \frac{\partial f}{\partial x} = \frac{\partial^2 f}{\partial x^2} \left[C_1\sigma \frac{\Delta x^2}{2\Delta t} - J^2 \frac{\Delta t}{2} \right] \tag{10}$$

The right-hand side can be shown to be dissipative, and therefore we can define the artificial viscosity to be:

$$\nu_{\text{art}} = C_1\sigma \frac{\Delta x^2}{2\Delta t} - J^2 \frac{\Delta t}{2}$$

It is convenient to write a corresponding Reynolds number

$$\text{Re}_{\text{art}} = uL/\nu_{\text{art}}$$

where L is a characteristic length dimension.

$$\frac{\Delta t}{\Delta x} = \frac{1}{a(1 + M)}$$

by the Courant Friedrichs Lewy criterion with a = local sound speed.

Finally we get:

$$\text{Re}_{\text{art}} = \frac{L}{\Delta x} \frac{2M}{1 + M} \frac{1}{C_1\sigma - \left(J \frac{\Delta t}{\Delta x}\right)^2}$$

$L/\Delta x$ represents spatial resolution of the finite difference mesh and would be approximately 100. The quantities $C_1\sigma$ and $J \frac{\Delta t}{\Delta x}$ are of order 1. If we assume that their difference is also order (1), the Reynold's number is about 100.

In the two-dimensional computation with nonlinearities, the artificial Reynolds number may be somewhat higher. However, it will be far smaller than the true Reynold's number for high-speed flows which would be about 10^6.

In conclusion, it appears that the artificial viscosity used for adequate thickening of the shock wave is very large compared to real viscosity. Therefore the real boundary layer effects would be completely masked by the artificial viscosity. The same conclusion has also been stated by Moretti.[5]

I would appreciate if the authors can indicate whether the computed boundary layer parameters compare reasonably with experimental values.

Since the dissipative term is proportional to the second derivative of the dependent variable [equation (10)], it appears that the computed shock thickness will be inversely proportional to the shock strength.[6] I would like to know if the authors have found this to happen in their computations.

Authors' Closure

The previous one-dimensional, linear analysis, which attempts to assess the effects of numerical damping, or "artifical viscosity," on the real fluid behavior near the walls, does indeed provide some qualitative information concerning this numerical effect. However, I feel that it is necessary to supplement this analysis with some representative data from the present investigation. These data should also answer the question concerning comparisons of computed boundary layer parameters with experimental values of these parameters.

Computations from this numerical investigation were compared to compressible, isothermal ($M_\infty = 2.0$, $T_w = T$) flat plate boundary layer thickness computations reported in the open literature (viz. Schlichting's *Boundary Layer Theory*). The results of several comparisons indicated that the numerically calculated boundary layer thicknesses obtained in this investigation approached the values in the literature as the computational grid size across the boundary layer approached the magnitude of the local fluid mean free path. For example, the ratios of the numerically calculated boundary layer thicknesses to the thickness obtained in the literature at the same Mach number and Reynolds number ($M_\infty = 2.0$, $\text{Re}_\infty/\text{ft} = 10^4$) were 1.89 and 1.22 when the normal grid spacings were, respectively, 5 and 2.5 times the local mean free path. This indicates that the present error in the calculation of the boundary layer thickness varies as the square of the grid size.

Numerical data pertinent to the question concerning shock wave thickness are not available in a form suitable for assessing the influence of shock strength on shock thickness. However, I would speculate that the shock wave thickness will be inversely proportional to the shock strength.

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