Pauli-Villars Regularization and
One-Loop Finiteness of Type-I Superstring Theory

Naohito NAKAZAWA and Hisashi YAMAMOTO

Research Institute for Theoretical Physics
Hiroshima University, Takehara, Hiroshima 725

(Received September 22, 1986)

In a previous paper we reanalyzed the issue of one-loop finiteness of the light-cone 4-point amplitude of the type-I SO(N) superstring theory in terms of the Pauli-Villars regularization. The same analysis is applied, in this paper, to the 5-point amplitude in the light-cone formalism as well as to the parity-conserving covariant M-point amplitude and the generality of the previous results is confirmed. Namely, the assignment of the regulator masses to the planar and nonorientable diagrams is essentially arbitrary in the present scheme. However, once a particular mass assignment is fixed, by one way or another, it is firmly established that the one-loop finiteness condition picks up the same $N$ for an arbitrary number $M$ of external lines. In particular, the assignment $2m_{\varphi} = m_N$ renders all the one-loop amplitude finite for $N=32$.

§ 1. Introduction

Up to now, several string theories have been constructed. Among them, for its simplicity, the SO(N) type-I superstring theory (SST-I) is interesting as a theoretical laboratory for more complex and realistic theories. But since it includes both open and closed strings, there occur unique problems not seen in the pure closed superstrings. A typical and interesting example is the one-loop finiteness issue, to which we address ourselves in the present paper. If we restrict ourselves to the open superstrings, one-loop finiteness, even if it really holds, may hold by means of the cancellation of divergences arising from separate diagrams, i.e., planar and nonorientable diagrams. Hence one might think that at the one-loop level some regularization procedures are needed before we combine the contributions from the two diagrams. Although some important analyses are so far made as to the one-loop finiteness of open superstrings,\(^1\)–\(^3\) none of them utilized explicit regularization.

Thus in a previous paper,\(^5\) by introducing the regulators from the start, we reanalyzed the finiteness of the 4-point light-cone amplitude of Green and Schwarz. Our regularization method is the string version of the Pauli-Villars method employed by Green and Schwarz.\(^6\) This method can render each amplitude finite and makes it possible to evaluate the strength of the divergence included. We found that in the Pauli-Villars scheme the one-loop finiteness essentially depends on how the masses of regulators are assigned between the planar and nonorientable diagrams. If and only if the masses of the regulators in the planar diagram ($m_{\varphi}^j$; $j=1, 2, \cdots$) are half of those in the nonorientable diagram ($m_N$'s), i.e., $4m_{\varphi}^2 = m_N^2$, the requirement of the infinity cancellations singles out SO($N=32$) gauge group and the regularized amplitude has the form of the principal-value integral at $N=32$. In any other mass assignments, there remain the divergences proportional to the mass square of the regulators at $N=32$. For example, if we naively work in the equal mass assignment $m_{\varphi}^2 = m_N^2$, 


the divergences cancel only at \( N=8 \).

Are there any criteria to choose a particular mass assignment? If they exist, what are their physical or geometrical meanings? Since we change the value of the intercept by introducing the regulator masses \( m_i \) in the string version of the Pauli-Villars regularization, it is expected that either the Lorentz invariance or the unitarity is broken in the regularized amplitudes. These invariance properties related to the value of the intercept, i.e., the Lorentz invariance and the unitarity of the regularized amplitudes may give a physical reason to choose a particular mass assignment for the regularization. As the Lorentz invariance is not manifest in the light-cone formalism, this point should be investigated in the covariant formalism. It is also important to check whether the light-cone formalism and the covariant formalism give the same answer for the one-loop finiteness in terms of the same regularization procedure, since there is no explicit proof of the equivalence between the light-cone and covariant formalisms.

The aim of this paper is to show first that the above observations obtained in the previous reanalysis of the light-cone 4-point amplitude continue to hold also for the light-cone 5-point amplitude. Based on a physical argument for the divergences in SST-I, we easily extend the results to the cases of the general number \( (M \geq 6) \) of external lines (massless vector emission) in the light-cone formalism. Second, examining the parity-conserving \( M \)-point amplitude in the covariant formalism, we confirm that the situation in the light-cone formalism is not changed also in the covariant formalism. That is, the mass assignment conditions for the regularization and for the finiteness at \( N=32 \) obtained in the light-cone 4-point case are unchanged by the number of external massless vector bosons in both the light-cone and covariant formalisms. But the mass assignment of regulators and therefore the finiteness at \( N=32 \) as well are essentially arbitrary in the present regularization scheme.

In the following the reanalyses of the light-cone 5-point amplitude and the parity-conserving \( M \)-point amplitude in the covariant formalism are assigned to §§ 2 and 3, respectively. Both of the amplitudes have already been investigated without explicit regularization by Frampton et al.\(^2\) and Clavelli.\(^3\) Section 4 is devoted to a summary and discussion.

§ 2. Light-cone formalism

In this section, we discuss the cancellation of divergences in SST-I based on the light-cone formalism.\(^4\) The finiteness of \( M \)-point one-loop amplitudes was first discussed by Green and Schwarz\(^7\) for \( M=4 \). Frampton et al. made the same observation as that of Green and Schwarz for \( M \geq 5 \).\(^2\) Based upon the principal part (PP) prescription as a regularization procedure, they claimed that the requirement of the infinity cancellations singled out \( SO(32) \) gauge group. Our present viewpoint is the same as in Ref. 5), that is, we study the issue of the one-loop finiteness by using an explicit regularization procedure (we employ the Pauli-Villars regularization in this paper) to avoid ambiguities which may be caused by handling the ill-defined divergent integrals. It will be shown below that our previous observation in Ref. 5) is true not only for \( M=4 \) but also for \( M \geq 5 \) in the light-cone formalism. For simplicity, we first
consider the 5-point one-loop amplitude with external massless gauge bosons. Following Ref. 2), its planar part is given by

\[ A_{sp}(0) = \int d^n \text{tr}[\Delta V(1)\Delta V(2)\Delta V(3)\Delta V(4)\Delta V(5)] \]

\[ = ig^5 \pi^5 NG \int_0^1 \frac{dq}{q} \left( \frac{1}{-\ln q} \right) \int_0^1 \prod_{i=1}^4 \theta(\nu_{i+1} - \nu_i) d\nu_i \]

\[ \times \left[ I_a + I_b + I_c + I_d \prod_{i<j} A_{ij} \sin(\pi \nu_{ij}) \right]^{2\alpha_{k_1 \cdots k_5}}, \]

where

\[ G = \text{tr} \left[ \Lambda^a \Lambda^b \Lambda^c \Lambda^d \Lambda^e \right], \]

\[ A_{ij} = \prod_{n=1}^m \left[ 1 - 2q^{2n} \cos(2\pi \nu_{ij}) + q^{4n} \right]. \]

The detailed expressions of \( I_a, I_b, I_c \) and \( I_d \) are

\[ I_a = \xi_1 \xi_2 \xi_3 \xi_4 \xi_5 \sum_{k_1} \sum_{k_2} \sum_{k_3} \sum_{k_4} \sum_{k_5} \text{tr} \left[ R_{0,1}^{(i)} R_{0,1}^{(j)} R_{0,1}^{(k)} R_{0,1}^{(l)} R_{0,1}^{(m)} \right], \]

\[ I_b = \xi_1 \xi_2 \cdots \xi_5 \sum_{k_0} \sum_{k_1} \sum_{k_2} \sum_{k_3} \sum_{k_4} \sum_{k_5} \frac{1}{64} \left( \frac{w}{x_2 x_3} \right) \text{tr} \left[ \gamma^{i,j} \gamma^{i,j} \gamma^{i,j} \right] \]

\[ \times \text{tr} \left[ R_{0,1}^{(i)} R_{0,1}^{(j)} R_{0,1}^{(k)} R_{0,1}^{(l)} R_{0,1}^{(m)} \right] \]

\[ + \left( \frac{w}{x_2 x_3} \right) \text{tr} \left[ \gamma^{i,j} \gamma^{i,j} \gamma^{i,j} \right] \text{tr} \left[ R_{0,1}^{(i)} R_{0,1}^{(j)} R_{0,1}^{(k)} R_{0,1}^{(l)} R_{0,1}^{(m)} \right] \]

\[ + \text{cyclic permutations}, \]

\[ I_c = \xi_1 \xi_2 \xi_3 \xi_4 \xi_5 \sum_{k_1} \sum_{k_2} \sum_{k_3} \sum_{k_4} \sum_{k_5} \frac{\ln(x_1 \cdots x_5)}{(-\ln w)} K_1(2, 3, 4, 5) \]

\[ + \xi_2 \xi_3 \xi_4 \xi_5 \sum_{k_1} \sum_{k_2} \sum_{k_3} \sum_{k_4} \sum_{k_5} \frac{\ln(x_2 \cdots x_5)}{(-\ln w)} K_4(3, 4, 5, 1) + \text{cyclic permutations}, \]

where

\[ K_1(2, 3, 4, 5) = \xi_1 \xi_2 \xi_3 \xi_4 \xi_5 \sum_{k_1} \sum_{k_2} \sum_{k_3} \sum_{k_4} \sum_{k_5} \text{tr} \left[ R_{0,1}^{(i)} R_{0,1}^{(j)} R_{0,1}^{(k)} R_{0,1}^{(l)} R_{0,1}^{(m)} \right], \]

\[ I_d = K_1(2, 3, 4, 5) \xi_1 \xi_2 \xi_3 \xi_4 \xi_5 \sum_{k_1} \sum_{k_2} \sum_{k_3} \sum_{k_4} \sum_{k_5} \frac{1}{1 - \frac{w}{x_1}} \left( k_{2,i} \frac{x_2}{x_1} - \left( \frac{w}{x_3} \right) \right) \]

\[ + k_{3,i} \left( \frac{x_3}{x_2} - \frac{w}{x_3} \right) \left( \frac{x_2}{x_1} \right) + k_{4,i} \left( \frac{w}{x_3} - \left( \frac{x_3}{x_1} \right) \right) + \text{cyclic permutations}. \]

The explicit evaluation of the trace part shows that the integrand is meromorphic in \( q \). This fact is consistent with the physical picture that the pole corresponds to dilaton emission. This means that \( I_b \) and \( I_d \) only contribute to the amplitude. By using the following formula for the planar mode sum,

\[ f_p(x, w) = \sum_{i=1}^\infty \frac{1}{1 - \frac{w}{x}} \left( x^i - (w/x)^i \right), \]
Pauli-Villars Regularization and One-Loop Finiteness

\[ = -\frac{1}{2} + \nu - \frac{\ln q}{2\pi} \left( \cot(\pi\nu) + 4 \sum_{n=1}^{\infty} \frac{q^{2n}}{1 - q^{2n}} \sin(2n\pi\nu) \right), \]  

(2.9)

where \( \nu \) is defined by \( x = w^\nu \), we have

\[ A_{\delta\tau}(0) = i \frac{G^5}{2} \pi^3 \kappa G \int_0^1 \frac{dq}{q} \mathcal{F}_5(q^2), \]  

(2.10)

where

\[ \mathcal{F}_5(q^2) = \pi \int_0^1 \left( \prod_{i=1}^{4} \theta(\nu_i + 1 - \nu_i) d\nu_i \right) \left[ \prod_{\langle i,j \rangle} [A_{ij} \sin(\pi\nu_{ij})]^{2a_{ij} k_i \cdot k_j} \right] \]

\[ \times \sum_{\langle i,j \rangle} \left[ \cot(\pi\nu_{ij}) + 4 \sum_{n=1}^{\infty} \frac{q^{2n}}{1 - q^{2n}} \sin(2n\pi\nu_{ij}) \right] \]

\[ \times \left( -\frac{1}{64} R^4 \gamma^{i_1} \cdots \gamma^{i_4} \tr[\gamma^{i_1} \gamma^{i_2} \gamma^{i_3} \gamma^{i_4}] \tr[R_0^{i_1} \gamma^{i_1} R_0^{i_2} \gamma^{i_2} \gamma^{i_3} R_0^{i_4} \gamma^{i_4}] \right) \]

\[ + K_4(I+1, I+2, \cdots, I-1) \xi \cdot k_I - K_4(J+1, \cdots, J-1) \xi \cdot k_J \]  

(2.11)

with \( \ast \neq I, J \). We note that those parts of \( I_5 + I_d \) which come from the factor \( -\frac{1}{2} + \nu \) in Eq. (2.9) is cancelled out.

Similarly, for the 5-point nonorientable diagram, we have

\[ A_{5N}(0) = -i \frac{G^5}{2} \pi^{3 - 2} G \int_0^1 \frac{dq}{q} \mathcal{F}_5(-\sqrt{q}). \]  

(2.12)

Here \( 2^4 \) in the factor \( 2^{4-1} \) is generated by the substitution \( \nu_i \rightarrow 2\nu_i \) in the nonorientable amplitude. Additional \( 2^{-1} \) in the factor \( 2^{4-1} \) comes from the nonorientable mode sum,

\[ f_N(x, w) = \sum_{l=1}^{\infty} \frac{1}{1 - \left( -w \right)^l} \left[ x^l - \left( -w / x \right)^l \right], \]

\[ = -\frac{1}{2} + \nu - \frac{1}{2} \frac{\ln q}{2\pi} \left( \cot\left( \frac{\pi\nu}{2} \right) + 4 \sum_{n=1}^{\infty} \frac{(-\sqrt{q})^n}{1 - (-\sqrt{q})^n} \sin(n\pi\nu) \right). \]  

(2.13)

Now we are in a position to regularize the one-loop amplitudes in Eqs. (2.10) and (2.12) with a string version of the Pauli-Villars regularization. Following Ref. 5, we define a regularized amplitudes

\[ A_\text{reg} = A(0) + \sum_{j=1}^{\infty} C_j A(m_j^2) \]

\[ = \sum_{j=0}^{\infty} C_j A(m_j^2), \]  

(2.14)

where \( C_0 = 1, m_0 = 0 \) and the \( C_j \)'s \( (j = 1, 2, \cdots) \) are arbitrary parameters that should be chosen so that \( A_\text{reg} \) is finite. \( A(m_j^2) \) is obtained from Eq. (2.1) by replacing the propagator with a massive one, i.e.,

\[ A(0) = \frac{\alpha'}{L_0} \rightarrow A(m_j^2) = \frac{\alpha'}{L_0 + m_j^2}. \]  

(2.15)
The regularized amplitudes Eq. (2.14) have the forms

\begin{equation}
A_{\text{reg}}^{\text{eff}}(m_{P}^{2}) = -i \frac{g^{5}}{2} \pi^{3} N G \int_{0}^{1} \frac{dq}{q} \left( \sum_{j=0}^{N} C_{j} e^{(2\pi^{2}/\ln q)^{m_{P}^{2}}} \right) F_{6}(q^{2}),
\end{equation}

(2.16)

\begin{equation}
A_{\text{reg}}^{\text{eff}}(m_{3N}^{2}) = -i \frac{g^{3}}{2} \pi^{3} 8 G \int_{0}^{1} \frac{dq}{q} \left( \sum_{j=0}^{N} C_{j} e^{(2\pi^{2}/\ln q)^{m_{3N}^{2}}} \right) F_{6}(-q^{2}).
\end{equation}

(2.17)

Here we have made the possible distinction of the masses of regulators between the planar and nonorientable diagrams. In order to find the regularization condition, it is appropriate to write the regularized amplitudes in Eqs. (2.16) and (2.17) in terms of the variable \( w = x_{1} x_{2} x_{3} x_{4} x_{5} = \exp(2\pi^{2}/\ln q) \). Since the “ultraviolet” divergence comes from \( w \rightarrow 1 \) in the \( w \)-integrand (which corresponds to \( q \rightarrow 0 \) in the \( q \)-integrand), we have the following regularization conditions by expanding the \( w \)-integrand near \( w = 1 \):

\begin{equation}
\sum_{j=0}^{N} C_{j} = 0, \quad \sum_{j=0}^{N} C_{j} m_{j}^{2} = 0.
\end{equation}

(2.18)

As we showed in the previous paper,\(^{5}\) the PP prescription corresponds to a different regulator mass assignment for the planar and nonorientable diagrams, namely \( m_{NP}^{2} = 4 m_{P}^{2} = 4 m_{j}^{2} \). Changing the variables as \( \lambda = q^{2} \) in Eq. (2.16) and \( \lambda = q^{2} \) in Eq. (2.17), we obtain

\begin{equation}
A_{5}^{\text{reg}} = A_{5}^{\text{eff}}(m_{j}^{2}) + A_{5}^{\text{eff}}(4 m_{P}^{2})
\end{equation}

\begin{equation}
= -i \frac{g^{3}}{4} \pi^{3} G \int_{0}^{1} \frac{d\lambda}{\lambda} \left( \sum_{j=1}^{N} C_{j} e^{(8\pi^{2}/\ln \lambda)^{m_{j}^{2}}} \right) \left[ N F_{6}(\lambda) - 32 F_{6}(-\lambda) \right].
\end{equation}

(2.19)

Equation (2.19) can be reduced to a principal-value integral if and only if \( N = 32 \). Also, following Ref. \( 5 \), we can easily evaluate the divergent part \( (m_{j}^{2} \rightarrow \infty) \) of Eqs. (2.16) and (2.17),

\begin{align}
A_{5}^{\text{reg}} &= A_{5}^{\text{eff}}(m_{j}^{2}) + A_{5}^{\text{eff}}(m_{P}^{2}) \\
&= -i \frac{g^{3}}{4} \pi^{3} G \left( (N-8) C m_{P}^{2} F_{6}(0) \\
&\quad + \int_{0}^{1} \frac{d\lambda}{\lambda} \left[ N [F_{6}(\lambda) - F_{6}(0)] - 32 [F_{6}(-\lambda) - F_{6}(0)] \right] \right),
\end{align}

(2.20)

where

\begin{equation}
m_{j}^{2} = \xi_{j}^{2} m_{P}^{2}, \quad C = -4 \pi^{2} \sum_{j=1}^{N} C_{j} \xi_{j}^{2} \ln \xi_{j}^{2} \quad (j=1, 2, \ldots)
\end{equation}

(2.21)

We control the limit \( m_{j}^{2} \rightarrow \infty \) by introducing a parameter \( m^{2} \). The \( \xi_{j} \)'s are constant parameters \( (j=1, 2, \ldots) \). In this case, that is, if we assign the equal mass to regulators for \( A_{5P} \) and \( A_{5N} \), the divergence proportional to \( m^{2} \) does not completely cancel for \( N = 32 \).

Finally we comment on the cases for \( M \geq 6 \). For general numbers of external lines \( M \), the Jacobian for the change of variables appearing in Eqs. (2.1) and (2.2) gives the following,
\[ \left( \prod_{i=1}^{M} dx_i \right) w^{-1} \left( -\frac{\pi}{\ln w} \right)^{3} \left( -\frac{2\pi^2}{q} \right)^{M-4} \prod_{i=1}^{M-1} \theta(\nu_{i+1} - \nu_i) d\nu_i. \]  \hspace{1cm} (2.22)

Since the amplitudes must be meromorphic in \( q \) from the physical picture of soft dilaton emission, the \( M \)-point amplitude contains \( M-4 \) mode sums in Eqs. (2.9) and (2.13). We may expect the \( M \)-point amplitude to be given by

\[ A_{M^p} - A_{M^N} = \text{const} \cdot \int_0^1 dq \left( NF_M(q^2) - (1/2)^{M-4} 2^{M-1} F_M(-\sqrt{q}) \right). \]  \hspace{1cm} (2.23)

Here the factor \( 2^{M-1} \) comes from the substitution \( \nu_i \to 2\nu_i \) in the nonorientable diagram and \( (1/2)^{M-4} \) from the \( M-4 \) nonorientable mode sums in Eq. (2.13). Based on the structure of the integrand in Eq. (2.23), it is easy to show that the results in Eqs. (2.16) \sim (2.21) are not changed even for \( M \gg 6 \). In the following section, we also show that the situation is not changed in the covariant formalism.

\section*{§ 3. Covariant formalism}

We apply here the method of the Pauli-Villars regularization to the Lorentz covariant formalism and confirm the previous results in the parity-conserving \( M \)-point \( (M \) denotes the number of external massless bosons) amplitudes. (Parity-violating amplitudes are known to be finite in each diagram and need not be considered here.) Clavelli has already examined the parity-conserving \( M \)-point amplitudes in the covariant formalism without regularization. Although he claims that the theory is one-loop finite at \( N=32 \), his results are crucially dependent on the change of the Jacobi-transformed variable he made in the expression of the nonorientable amplitude. Potting and Shapiro recently pointed out this fact and showed that if the amplitudes are expressed by the same variable in the planar and nonorientable diagrams, the combined amplitude becomes finite for \( N=8 \) and not for \( 32 \). They also explained the connection between their results and our previous work. From these it is expected also in the following covariant Pauli-Villars formalism that the finiteness at \( N=32 \) depends on a particular mass assignment of the regulators between the planar and nonorientable diagrams as observed in the light-cone 4-point case.

In the following we will mainly employ the notation of Ref. 3. A covariant one-loop \( M \)-point amplitude in \( D \)-dimensions regularized by the Pauli-Villars method is given by

\[ A_{M}^{\text{reg}} = A_{M}(0) + \sum_{j=1}^{\infty} C_j A_{M}(m_j^2) \]

\[ = \sum_{j=0}^{\infty} C_j A_{M}(m_j^2) \]  \hspace{1cm} (3.1)

with \( C_0 = 1 \) and \( m_0 = 0 \). Here the \( C_j \)'s \( (j \geq 1) \) are parameters that should be chosen so that \( A_{M}^{\text{reg}} \) is finite. Each amplitude \( A_{M}(m_j^2) \) is of the form

\[ A_{M}(m_j^2) = g^M \int d^Dp \text{Tr} P \prod_{i=1}^{M} \left[ V(k_i, 1) \Delta(m_j^2) \right], \]  \hspace{1cm} (3.2)

where \( P \) is the Blink-Olive projection operator onto positive norm states and \( \text{Tr} \)
includes the trace over $\gamma$ matrices. The boson sector of SST-I is the Neveu-Schwarz model projected onto the even $G$-parity sector. The vertex for a gauge boson emission is

$$V_b(k_i, \rho_i) e^{ik \cdot q(\rho_i)} [\zeta_i \cdot P(\rho_i) + k_i \cdot H(\rho_i)] \zeta_i \cdot H(\rho_i)],$$

and the massive propagator in the $(F2)$ formulation is

$$\Delta_b(m_j) = \frac{1}{L_0^{N-S} - \frac{1}{2} + m_j^2} (1 + \Omega)(1 + G)/2$$

$$= \frac{1}{2} \int_0^1 \frac{dx}{x} x^{L_0^{N-S} - 1/2 + m_j^2} (1 + \Omega)(1 + G). \quad (3.4)$$

The fermionic sector is the Ramond model with the Majorana-Weyl spinors. The vertex and the parity-conserving massive propagator in $(F2)$ formulation are

$$V_{F2}(k_i, \rho_i) = e^{ik \cdot q(\rho_i)} \frac{\zeta_i \cdot \Gamma(\rho_i)}{\sqrt{2i}} ,$$

$$\Delta_{F2} = \frac{1}{F_0 - im_j} \frac{1 + \Omega}{2}$$

$$= \frac{F_0 + im_j \frac{1 + \Omega}{2}}{L_0^{N} + m_j^2}$$

$$= \frac{F_0 + im_j}{2} \int_0^1 \frac{dx}{x} x^{L_0^{N} + m_j^2} (1 + \Omega). \quad (3.6)$$

We note that the massive propagators Eqs. $(3.4)$ and $(3.6)$ maintain the supersymmetry of the mass spectrum. $\Omega$ is the twist operator defined by

$$\Omega = -(-)^{N_{\bar{N}} \cdot s - (1/2)}$$

$$= - e^{i\pi (L_0^{N-S} - (p^2/2) - (1/2))} \quad (3.7)$$

for the Neveu-Schwarz model and

$$\Omega = -(-)^{N_{\bar{N}}}$$

$$= - e^{i\pi (L_0^{N} - (p^2/2))} \quad (3.8)$$

for the Ramond model. The $G$-parity operator is

$$G = -(-)^{\sum_{n=1/2}^{N_{\bar{N}}}} b^{n+1, b^n}$$

$$= e^{2\pi i (L_0^{N-S} - (p^2/2) - (1/2))} \quad (3.9)$$

$L_0$ and $N$ operators are given by

$$L_0^{N-S} = \sum_{n=0}^{N} \left[ (n + \epsilon) a^{n+1, \alpha} a^n_{\alpha} + \left(n + \frac{1}{2}\right) b^{n+1, \alpha} b^n_{\alpha} \right]$$
\[ \frac{p^2}{2} + N_{N_s}, \quad (3.10) \]

\[ L_0^R = \sum_{n=0}^\infty [(n+\epsilon)\alpha^{n}_\mu \alpha^{\dagger n}_\mu + nd^{n}_\mu d^{\dagger n}_\mu] \]

\[ = \frac{p^2}{2} + N_\epsilon. \quad (3.11) \]

We use the limiting procedure of Ref. 13) and the loop momentum integral in Eq. (3.2) is replaced by a trace over canonical zeroth-mode oscillators. Then the regularized bosonic loop amplitude takes the form

\[ A_{m}^{\text{regB}} = \frac{g^M}{2} \int d\omega \sum_{j=0}^M C_j w^{n_j} \text{Tr} \left[ \omega \alpha^{n_j} - (1/2)P(1+\Omega)(1+G) \prod_{i=1}^M V \rho_i \right], \quad (3.12) \]

where \( w = x_1 x_2 \cdots x_M \), \( \rho_0 = 1 \), \( \rho_i = x_i \cdots x_1 \) and

\[ \int d\omega = \prod_{i=1}^M \int_0^{\rho_{i-1}} d\rho_i \int_0^{\rho_{i-1}} \frac{d\omega}{\omega}. \quad (3.13) \]

The regularized parity-conserving fermionic loop amplitude is given by

\[ A_{m}^{\text{regF}} = -\frac{g^M}{2} \sum_{j=0}^M C_j \text{Tr} \left[ P(1+\Omega) \prod_{i=1}^M \frac{F_0 + im_j}{L_0^R + m_j^2} V_{R_2}(k_i, 1) \right]. \quad (3.14) \]

We note here that the operator \( F_0 \) satisfies

\[ (F_0 + im) V_{R_2} = (F_0, V_{R_2}) - V_{R_2}(F_0 - im) \]

\[ = V_{R_1} - V_{R_2}(F_0 - im), \quad (3.15) \]

\[ (F_0 - im)(F_0 + im) = L_0^R + m^2, \quad (3.16) \]

where

\[ V_{R_1}(k, \rho) = e^{i\rho \cdot \xi(k)} \left[ \xi \cdot P(\rho) - \frac{1}{2} k \cdot \Gamma(\rho) \xi \cdot \Gamma(\rho) \right]. \quad (3.17) \]

Using Eqs. (3.15) and (3.16), the \((F_0 + im)\)'s in Eq. (3.14) can be pulled to the right beginning with the second and discarding the terms which vanish due to the cancelled propagator argument. Thus

\[ A_{m}^{\text{regF}} = -\frac{g^M}{2} \sum_{j=0}^M C_j \text{Tr} \left[ P(1+\Omega) \frac{F_0 + im_j}{L_0^R + m_j^2} V_{R_2}(k_i, 1) \right. \]

\[ \times \prod_{i=2}^M \frac{1}{L_0^R + m_j^2} V_{R_1}(k_i, 1) \left]. \quad (3.18) \]

Another expression is obtained from Eq. (3.14) by pulling the \((F_0 + im)\)'s left to the second position. Averaging the resultant expression with Eq. (3.18), we get

\[ A_{m}^{\text{regF}} = -\frac{g^M}{4} \sum_{j=0}^M C_j \text{Tr} \left[ P(1+\Omega) \frac{1}{L_0^R + m_j^2} (F_0 + im_j, V_{R_2}(k_i, 1)) \right] \]
× \prod_{i=2}^{M} \frac{1}{L_0 + m_j^2} V_{F_1}(k_i, 1)
\right]
- \frac{g^M}{4} \sum_{j=0}^{N} C_j \text{Tr} \left[ P(1 + \Omega) \frac{1}{L_0 + m_j^2} [V_{F_1}(k_i, 1) + 2 i m_j V_{F_2}(k_i, 1)] \right]
\times \prod_{i=2}^{M} \frac{1}{L_0 + m_j^2} V_{F_1}(k_i, 1)
\right]
- \frac{g^M}{4} \int d\omega \sum_{j=0}^{N} C_j w^{m_j^2} \text{Tr} \left[ w^{l_{0}^2} P(1 + \Omega) V_{F_1}(k_i, \rho_i) \right]
+ 2 i m_j V_{F_2}(k_i, \rho_i) \right]\prod_{i=2}^{M} V_{F_1}(k_i, \rho_i) . \tag{3.19}

Using the trace reduction technique of Clavelli and Shapiro\textsuperscript{12} with Eqs. (3.7) and (3.9), the regularized bosonic amplitude Eq. (3.12) can be written as

\begin{align*}
A_{M}^{\text{regB}} &= \frac{g^M}{2} \int d\omega \sum_{j=0}^{N} C_j w^{m_j^2} \left[ N \left( T_{NS}(\rho, w) + F_{NS}(w^2 \epsilon) T_{NS}(\rho, \epsilon w^2) \right) \right. \\
& \left. - \left[ F_{NS}(-w) T_{NS}(\rho, -w) + F_{NS}(-\epsilon w^2) T_{NS}(\rho, -\epsilon w^2) \right] \right] , \tag{3.20}
\end{align*}

where

\begin{align*}
F_{NS}(w) &= w^{-1/2} \prod_{n=1}^{m} \left( \frac{1 + w^{n-(1/2)}}{1 - w^n} \right)^{p-2} , \\
T_{NS}(\rho, w) &= (-\epsilon \ln|w|)^{-\rho} \prod_{i=1}^{M} V_b(\rho_i, d(w), \bar{d}(w))|0\rangle , \tag{3.22}
\end{align*}

with

\begin{align*}
d^n(w) &= \frac{a^n}{1 \pm w^{n-j}} + a^{n^+} , \tag{3.23}
\bar{d}^n(w) &= a^{n^+} + \frac{a^n w^{n-j}}{1 \pm w^{n-j}} , \tag{3.24}
\end{align*}

the plus and minus signs referring, respectively, to the fermionic and bosonic oscillators which have, respectively, $J = -\frac{1}{2}$ and $-\epsilon$.

Similarly the parity-conserving fermionic loop amplitude, Eq.(3.19), can be expressed by

\begin{align*}
A_{M}^{\text{regF}} &= -\frac{g^M}{2} \int d\omega \sum_{j=0}^{N} C_j w^{m_j^2} \left[ N F_{\bar{r}}(w) \left[ T_{\bar{r}}(\rho, w) + 2 i m_j T_{R_2}(\rho, w) \right] \right. \\
& \left. - F_{\bar{r}}(-w) \left[ T_{\bar{r}}(\rho, -w) + 2 i m_j T_{R_2}(\rho, -w) \right] \right] , \tag{3.25}
\end{align*}

where

\begin{align*}
F_{\bar{r}}(w) &= 2^{(\bar{r}/2) - 1} \prod_{n=1}^{m} \left( \frac{1 + w^n}{1 - w^n} \right)^{p-2} , \tag{3.26}
\end{align*}
\[ T_\rho(\rho, w) = (-\varepsilon \ln |w|)^{-D} \langle 0 | \prod_{i=1}^{M} V_{F_1}(k_i, \rho_i, d(w), \bar{d}(w)) | 0 \rangle, \quad (3.27) \]

\[ T_{R_2}(\rho, w) = (-\varepsilon \ln |w|)^{-D} \langle 0 | V_{F_2}(k_1, \rho_1, d(w), \bar{d}(w)) \times \prod_{i=2}^{M} V_{F_1}(k_i, \rho_i, d(w), \bar{d}(w)) | 0 \rangle. \quad (3.28) \]

The vertices \( V_{F_1} \) and \( V_{F_2} \) of Eqs. (3.17) and (3.5) appear in Eqs. (3.27) and (3.28) with their fermionic and bosonic oscillators transformed by Eqs. (3.23) and (3.24) with \( J = 0 \) and \(-\varepsilon\), respectively. From the definitions of \( V_{F_2} \) and \( V_{F_1} \) we see that \( T_{R_2} \), Eq. (3.28), is proportional to the vacuum expectation value (v.e.v.) of the product of an odd number of the \( \Gamma_i \)'s defined by

\[ \Gamma_i(\rho_i) = \gamma_i + i\sqrt{2} \gamma_1 \sum_{\alpha=1}^{2N} (c^{\alpha} \rho_i^{\alpha} + c^{\alpha*} \rho_i^{-\alpha}), \quad i = 1, \ldots, M, \quad (3.29) \]

which is then reduced to the trace over an odd number of \( \gamma \)-matrices or the v.e.v. of the product of an odd number of non-zeroth mode oscillators. As both are zero, \( T_{R_2} = 0 \).

Thus, up to the group factor, the full \( M \)-point regularized one-loop amplitude is given by

\[ A_{\text{reg}}^M = \frac{g^M}{2} \int dw \sum_{\mu=0}^{M-1} C_{\mu} w^{m_\mu} \left[ N [F_{\text{NS}}(w) T_{\text{NS}}(\rho, w) + F_{\text{NS}}(w e^{2\pi i}) T_{\text{NS}}(\rho, w e^{2\pi i})
\right.
\]
\[ - F_{\rho}(w) T_{\rho}(\rho, w)] - [F_{\text{NS}}(-w) T_{\text{NS}}(\rho, -w)
\]
\[ + F_{\text{NS}}(-w e^{2\pi i}) T_{\text{NS}}(\rho, -w e^{2\pi i}) - F_{\rho}(-w) T_{\rho}(\rho, -w)] \right]. \quad (3.30) \]

After all, as seen from Eq. (3.30), the effect of the Pauli-Villars regulators is only introducing the factor \( \sum_{\mu=0}^{M-1} C_{\mu} w^{m_\mu} \) in the \( w \)-integral. Other parts are unchanged. Hence the rest of the work, the Jacobi transformation, can be done in just the same way as that done by Clavelli.\(^3\) The resultant Jacobi-transformed regularized amplitude can be written by

\[ A_{\text{reg}}^{\text{JAC}} = (-2\pi)^{M-1} i^{M-2} g^M \int_0^1 \frac{dq}{q} \sum_{\mu=0}^{M-1} C_{\mu} \exp\left(\frac{4\pi^2 m_{\mu}^2}{\ln q^2}\right) N F(q^2) \quad (3.31) \]

for the planar diagram and

\[ A_{\text{reg}}^{\text{JAC}} = (-2\pi)^{M-1} i^{M-2} g^M \int_0^1 \frac{dq}{q} \sum_{\mu=0}^{M-1} C_{\mu} \exp\left(\frac{4\pi^2 m_{\mu}^2}{\ln q^2}\right) 2^{d/2-2} F(-q^{1/2}) \quad (3.32) \]

for the nonorientable diagram, where

\[ F(\lambda) = \int_0^1 \prod_{i=1}^{M-1} \theta(\nu_{i+1} - \nu_i) d\nu_i [F_{\text{NS}}(\lambda) [\bar{T}_{\text{NS}}(e^{2\pi i \nu}, \lambda) - \bar{T}_{\rho}(e^{2\pi i \nu}, \lambda)]
\]
\[ + F_{\text{NS}}(\lambda e^{2\pi i}) [\bar{T}_{\text{NS}}(e^{2\pi i \nu}, \lambda e^{2\pi i}) - \bar{T}_{\rho}(e^{2\pi i \nu}, \lambda)]] \quad (3.33) \]

with
\[ q = e^{2\pi i n w}, \quad \nu_i = \frac{\ln \rho^L}{\ln w}, \quad i = 1, 2, \ldots, M - 1, \] (3.34)

and \( \hat{T}_{NS} \) and \( \hat{T}_k \) are defined in Ref. 3). As is shown by Clavelli, the function \( F(\lambda) \) is not singular at \( \lambda = 0 \) where the "ultraviolet" divergence can occur, so that the potential divergences in Eqs. (3.31) and (3.32) can arise only from the \( dq/q \) behavior of the measure. Hence the conditions for the regularization are determined from the \( q \sim 0 \) behavior of the integrand

\[ \frac{dq}{q} \sum_{j=0} C_j \exp\left(\frac{4\pi^2 m_j^2}{\ln q^2}\right) = \frac{dw}{w \ln^2 w} \sum_{j=0} C_j w^{m_j^2}, \] (3.35)

giving the same condition Eq. (2.18) as in the light-cone case. As seen from Eqs. (3.31) and (3.32), the finiteness condition is also the same as that obtained in the light-cone 4- and 5-point cases. If and only if \( m_{NS}^2 = 4m_{NS}^2 = 4m_{NI}^2 \), the combined amplitude of Eqs. (3.31) and (3.32) takes the form of the principal-value integral at \( N = 32(D = 10) \),

\[ A_{reg}^{NS} + A_{reg}^{NS} = (-2\pi)^{M-1} i^{M-2} g^M 16PP \int_{-1}^{1} \frac{d\lambda}{\lambda} \sum_{j=0} C_j \exp\left(\frac{8\pi^2 m_j^2}{\ln(\lambda^2)}\right) F(\lambda), \] (3.36)

where we have changed the variables as \( q^2 = \lambda \) for \( A_{reg}^{NS} \) and \( q^{1/2} = \lambda \) for \( A_{reg}^{NS} \) and PP means principal-value. All the \( M \)-point amplitudes are finite in the \( m_j \to \infty \) limit if we take the PP prescription. This connection between the finiteness and the mass assignment of regulators is also seen by expanding \( F_{NS}(\lambda), \hat{T}_{NS}(e^{2\pi i \nu}, \lambda) \) and \( \hat{T}_k(e^{2\pi i \nu}, \lambda) \) around \( \lambda = 0 \) and taking out the divergent parts of the amplitudes. Anyhow if we assign the equal mass to the Pauli-Villars regulators appearing in the planar and nonorientable diagrams, the one-loop finiteness does not hold for \( N = 32 \) but for \( N = 8 \) in all the \( M \)-point amplitudes.

\[ \S \ 4. \ Discussion \]

In this paper, we reanalyzed the one-loop finiteness of SST-I with \( SO(32) \) gauge group in both the light-cone and the covariant formalism. We employed a string version of the Pauli-Villars regularization as an explicit regularization procedure. Examining the parity-conserving one-loop \( M \)-point amplitudes for the case \( M = 5 \) in the light-cone formalism and of general numbers of external lines \( M \) in the covariant formalism, we arrived at the following observations.

(1) The string version of the Pauli-Villars regularization can really work as an explicit regularization. The regularization condition (i.e., mass assignment condition which makes each diagram finite) is given by Eq. (2.18) for general numbers of external lines.

(2) PP prescription corresponds to a particular mass assignment for the planar and the nonorientable diagram, that is, \( m_j^2 \) for \( A_{reg}^{NS} \) and \( 4m_j^2 \) for \( A_{reg}^{NS} \). In this case, the divergences cancel at \( N = 32 \), and the regularized \( M \)-point amplitude takes the form of the principal-value integral at \( N = 32 \).

(3) In general, except for the above mass assignment, the divergences propor-
Pauli-Villars Regularization and One-Loop Finiteness

143

tional to the mass square of the regulator remain at \( N = 32 \). For example, if we assign the equal mass to the regulators for the planar and nonorientable diagrams, the divergences cancel if and only if \( N = 8 \) in all the \( M \)-point amplitudes.

We now give two comments. The first is on the relation between the light-cone formalism of Green and Schwarz and the covariant formalism. In the light-cone case, for example, we consider 5-point amplitude, the factor \((\ln q)^{-1}\) in Eq. (2·2) which comes from the Jacobian for the change of the variables in Eq. (2·22) cancels with \((\ln q)\) in the mode sums in Eqs. (2·9) and (2·13). This mechanism makes the 5-point amplitude meromorphic in \( q \) and the \( q \)-integrand have the same structure as that in the 4-point case. In the covariant formalism, the corresponding cancellation of the \((\ln q)\) factors can really occur\(^3\) where the Jacobi transformation properties of the two-point correlations, the partition functions and the measure, i.e., \( \langle 0|Q_{\mu}Q_\nu|0 \rangle \), \( \langle 0|\bar{Q}_{\mu}P_\nu|0 \rangle \), \( \langle 0|P_{\mu}P_\nu|0 \rangle \), \( \langle 0|H_{\mu}H_\nu|0 \rangle \), \( \langle 0|H_{\mu}F_\nu|0 \rangle \), \( F_{NS}(w) \) in Eq. (3·21), \( F_p(w) \) in Eq. (3·26) and \( dw \) in Eq. (3·13) play important roles. In both formalisms, the fact that the amplitudes are meromorphic in \( q \) and have the same pole structure (single pole of \( q \)) leads the same regularization condition and does not change the situation observed in a previous paper.\(^5\)

The second comment is on the string version of the Pauli-Villars regularization. As we noted in the Introduction, it may be expected that the change of the value of the intercept causes the violation of either the Lorentz invariance or the unitarity in the regularized amplitudes. If the light-cone formalism of Green and Schwarz and the covariant formalism are equivalent completely at the one-loop level, the only candidate of such unitarity-violating terms, if it exists, is the term which is proportional to \( \exp((2\pi^2/\ln q)m_f^2) \) in Eqs. (2·16), (2·17), (3·31) and (3·32). These terms may give the so-called unitarity-violating cut which should not be included in the amplitudes from the physical picture of the soft dilaton emission. However, we do not have the answer whether these terms may give a criterion to determine a particular mass assignment or not.

Since the anomaly cancellations single out \( N = 32 \) in SST-I, the desired result may be that the requirement of the one-loop finiteness also singles out \( N = 32 \). In the present viewpoint of our analysis, however, the one-loop finiteness crucially depends on the regularization procedure, that is, how we assign the mass to the regulators for the planar and nonorientable diagrams. As we noted in a previous paper,\(^5\) our analysis does not exclude the PP prescription as a correct regularization, but it definitely shows that, in order to treat the finiteness problem rigorously, we further need to search for a detailed regularization procedure consistent with the invariance properties of string theories such as the reparametrization invariance.

Acknowledgements

The authors would like to thank Professor R. Sasaki for valuable discussions and careful reading of the manuscript.
References

10) L. Clavelli, P. H. Cox and B. Harms, UA HEP 861.

Note added in proof: The authors would like to thank R. Potting and J. A. Shapiro for informing them of their work (Ref. 12) prior to the publication.