Breakdown of Chaos Symmetry and Intermittency in Band-Splitting Phenomena

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A band-splitting phenomenon is studied from the viewpoint of the breakdown of the chaos symmetry and the development of the intermittency. The analyses are carried out for two kinds of coarse-grained variables relevant to the symmetry argument and the burst dynamics, mainly by utilizing similarity exponents introduced for analyzing one-dimensional time series. The development of the intermittency near the band-splitting point is shown to be characterized as the scaling laws for the similarity exponents.

§ 1. Introduction

A band-splitting phenomenon is a prototype of the chaos-chaos transition exhibiting the symmetry change. The famous logistic parabola

\[ x_{t+1} = ax_t (1 - x_t), \quad (t = 0, 1, 2, 3, \cdots) \tag{1.1} \]

shows a band-splitting phenomenon at \( a = a_B (\approx 3.67856) \) as \( a \) is decreased from above \( a_B \). Many studies on band-splitting phenomena have been carried out especially in connection with spectral structures in the course of band-splitting bifurcations of periodic chaos.

One typical example of the chaos-chaos transition, associated with the symmetry change, observed in the differential equation systems is the particle motion in the double-well potential under the periodic external excitation with the amplitude \( a \) and the angular frequency \( \omega \):

\[ \ddot{x} + \gamma \dot{x} - dU(x)/dx + a \cos(\omega t). \tag{1.2} \]

Here \( x(t) \) is the particle position at time \( t \), \( \gamma \) is the damping rate and \( U(x) = ax^2 + \beta x^4 (a < 0, \beta > 0) \). In this model the symmetry-breaking transition associates the localization of the particle position in one of wells as the strength of the external excitation is changed.

In these examples, at the transition point the strange attractor begins to split into two statistically equivalent attractors. After the splitting, the motion is limited to one of them, and the selection of the attractor is determined by the initial condition. Just before the splitting, the channel region in the state space, which renders the migration processes between two subregions of the attractor possible, is narrow. Therefore it takes long time to migrate for one subregion to the other as the system is close to the transition point. If one regards migration processes as bursts and the motion in one subregion of the attractor as laminar states, then the statistical dynamics just before the splitting of the strange attractor can be described as the

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intermittency. This characteristic develops as the system approaches the transition point.

The main aim of the present paper is to study the band-splitting phenomena in one-dimensional mapping systems from the viewpoint of the symmetry-breaking and the development of the intermittency. This will be done especially by using similarity exponents introduced so as to analyze time series. In § 2, we give models and discuss their general characteristics near the transition point. In §§ 3 and 4, the statistical dynamics of symmetry variable (see § 2) and bursts are studied, respectively. We will find that the symmetry change and the development of the intermittency are closely related to each other. This interrelation will turn out to be expressed as the scaling laws of similarity exponents. A summary and remarks are given in § 5.

§ 2. Breakdown of chaos symmetry and intermittency

As illustrative examples exhibiting the band-splitting phenomena, we hereafter consider two one-dimensional mapping models, A and B defined as follows.

**Model A**: The dynamics is generated by the one-dimensional map $x_{t+1} = f(x_t)$ ($-R \leq x_t \leq 1$, $t=0, 1, 2, 3, \cdots$, $R=O(1)>0$) with

$$f(x) = \begin{cases} f^+(-x/R) & \text{for } x \in \Omega_-=[-R, 0), \\ f^+(x) & \text{for } x \in \Omega_+=(0, 1], \end{cases}$$

where $f^+(x)$ and $f^-(x)$ are similar, i.e., $f^-(x) = -R f^+(-x/R)$ ($f^+(0)=0$). $f(x)$ satisfies the conditions, (i) $f^+(x)$ has a semi-continuous invariant density in the subregion $\Omega_+$,

![Fig. 1. Schematic figures of mapping systems exhibiting the symmetry-breaking transition. Subregions $I_1$ and $l_2$ in Model B correspond to $\Omega_-$ and $\Omega_+$ in Model A, respectively. If the ratio between sizes of $I_1$ and $l_2$ is estimated by $R=2r_c/r^2_c=2/r_c$, one gets $R=2.4$.](https://www.oup.com)
(ii) \( f^+(x) \) has the channel \( \omega_+ = (1 - \varepsilon_+, 1] \) through which the phase point migrates from \( Q_+ \) into \( Q_- \) and (iii) \( f^+(x) \) has one maximum at \( x = x_+ \), around which \( f^+(x) \) can be expanded as \( f^+(x) = 1 - c(x - x_+)^2 \), \( c \) being a positive constant. This is a straightforward extension of the model introduced in Ref. 7). A typical example of such map is shown in Fig. 1(a). In carrying out numerical calculations we hereafter adopt

\[
f^+(x) = 4x(-x^2 - 1 - \varepsilon_+)/(1 - \varepsilon_+)^2,
\]

where \( \varepsilon_+ > 0 \) is the width of \( \omega_+ \) and is chosen as the control parameter. The transition point is given by \( \varepsilon_+ = 0 \).

**Model B:** The second model is the one-dimensional map \( x_{t+1} = f(x_t) \) constructed by the logistic parabola as follows:

\[
f(x) = f_r^2(x) = f_r(f_r(x)), \quad f_r(x) = -x^2 - 2rx.
\]

This system has the same conditions as those (i) \( \sim \) (iii) for Model A. The band-splitting occurs at

\[
r_c = 0.83928\ldots,
\]

when \( r \) is decreased from above \( r_c \). In Fig. 1(b), \( f(x) \) at \( r = r_c \) is drawn. In this model the control parameter is chosen as \( \delta = r - r_c \).

Typical temporal evolutions of \( x_t \) near the transition point are shown in Figs. 2(a) \( \sim \) (c). When the system is sufficiently above the transition point, the phase point frequently moves between subregions \( Q_+ \) and \( Q_- \) through wide channels (Fig. 2(a)). As \( \varepsilon_+ \) is decreased, the channels become narrow and the intermittency characteristic develops (Fig. 2(b)), the chaos symmetry being still restored. Finally for \( \varepsilon_+ \leq 0 \), the phase point is confined in either \( Q_+ \) or \( Q_- \), depending on the initial condition (Fig. 2(c)). In this sense the present transition at \( \varepsilon_+ = 0 \) is associated with the change of the chaos symmetry.

Hereafter theoretical results will be given for Model A. Results for Model B can be easily obtained, except numerical factors, by replacing the control parameter \( \varepsilon_+ \) by \( \delta \).

Let \( p_- (p_+) \) be the probability that
the phase point migrates from $Q_+(Q_-)$ into $Q_-(Q_+)$ after a unit step. Solving the Frobenius-Perron equation $\rho(x) = \int \delta(f(y) - x) \rho(y) dy$, $\rho(x)$ being the invariant density at the position $x$, one obtains $\rho(x) \approx c^{-1/2} \rho(x_m^+) (1 - x)^{-1/2}$ for $x \in \omega_+$. Since $f^+(x)$ and $f^-(x)$ are similar to each other, one gets

$$p_+ = \int_{\omega_+} \rho(x) dx \approx 2 \rho(x_m^+) \varepsilon_+ c^{-1/2}, \quad (2.5a)$$

$$p_- = \int_{\omega_-} \rho(x) dx \approx 2 \rho(x_m^-) \varepsilon_- R^{-1/2}. \quad (2.5b)$$

Noting that $\rho(x_m^+)/\rho(x_m^-) = R$ at the transition point, we obtain

$$p_+ = p_- = p/2 \propto \varepsilon_+^{-1/2} \quad (2.6)$$

to the lowest order in $\varepsilon_+$. The residence time $\tau_e$, i.e., the average duration between two neighboring bursts, is therefore evaluated as $\tau_e \approx p^{-1} \propto \varepsilon_+^{-1/2}$. This critical behavior was first obtained in Ref. 2).

In order to analyze the symmetry dynamics and the development of the intermittency near the transition point, we introduce two kinds of coarse-grained variables by

$$G(x) = \begin{cases} 1 & (x \in Q_+) \\ -1 & (x \in Q_-) \end{cases}, \quad (2.7)$$

$$g(x) = \begin{cases} 1 & (x \in \omega_{\pm}) \\ 0 & \text{(otherwise)} \end{cases}, \quad (2.8)$$

where $g[f(x)] = |G(f(x)) - G(x)|/2$ (Fig. 2). The former is relevant to the symmetry dynamics, and the latter describes the migration process. Hereafter $G(x)$ and $g(x)$ are called the symmetry variable and the burst variable, respectively. In the following sections, we will discuss the statistical dynamics of these variables near the transition point $(\varepsilon_+(\delta) \approx 0)$.

### § 3. Statistical dynamics of the symmetry variable

For the present we assume that bursts are independent of each other. This leads to the double time correlation function $C_G(t)$ of the form

$$C_G(t) \approx \exp(-t/\tau_G), \quad (3.1)$$

where

$$\tau_G = 1/\ln(1 - 2p)^{-1} \propto \varepsilon_+^{-1/2} \quad (3.2)$$

is the correlation time. The diffusion coefficient** $D_G$ is therefore evaluated as

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* The double time correlation function $C_G(t)$ for the variable $u(x,t)$ is defined as $C_G(t) = \langle \delta u(x,t) \delta u(x_0) \rangle / \langle \delta u \rangle$, $\langle ... \rangle$ being the ensemble average.

** The diffusion coefficient $D_u$ for the steady time series $u(x_0)$ is defined through $\sigma_u(t) = \langle (Y_i - \langle Y_i \rangle)^2 \rangle = 2D_u t$ for a large $t$, where $Y_i = \sum_{i=1}^t \delta u(x_i)$. Namely, $D_u = C_G(0)/2 + \sum_{s=1}^t C_G(s)$, if it exists.††
\[ D_c = \frac{1}{2} \coth \left( \frac{1}{2} \tau_c \right) = \tau_c \propto \varepsilon^{1/2}. \]  

\( (3.3) \)

The anomalous enhancement of the diffusion coefficient is due to the elongation of the correlation time for \( \varepsilon \rightarrow 0 \). Numerical results are in agreement with the power law (3.3) (Fig. 3(a)).

Hereafter in §§ 3 and 4 numerical calculations are carried out mainly for three parameter values \( \varepsilon_{+a}, \varepsilon_{+b} \) and \( \varepsilon_{+c} \) for Model A (\( R=1 \)) (\( \delta_a, \delta_b \) and \( \delta_c \) for Model B), (Table I). Letters \( a, b \) and \( c \) in Figs. 5, 7 and 8 denote numerical results for the above parameter values \( \varepsilon_{+a}, \varepsilon_{+b} \) and \( \varepsilon_{+c} \) for Model A with \( R=1 \) (\( \delta_a, \delta_b \) and \( \delta_c \) for Model B), respectively. Furthermore, symbols \( \circ, \triangle \) and \( + \) in Figs. 4, 6 and 9 indicate numerical results for \( a, b \) and \( c \), respectively.

The comparison of numerical power spectra for \( G(x_t) \) with

\[ S(\omega) \sim \frac{\sinh(1/\tau_c)}{\cosh(1/\tau_c) - \cos \omega} \]

\[ \sim \frac{2D_c}{(\omega D_c)^2 + 1} \]

\( (3.4) \)

for \( \omega \rightarrow 0 \) and \( \tau_c \rightarrow \infty \), derived with (3.1), is given in Fig. 4, where \( \tau_c \) has been replaced by \( D_c \). Excellent agreement between (3.4) and numerical results supports the independence assumption on

<table>
<thead>
<tr>
<th>Model A (( R=1 ))</th>
<th>Model B</th>
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<tbody>
<tr>
<td>( \varepsilon_a )</td>
<td>( \varepsilon_b )</td>
</tr>
<tr>
<td>( 4 \times 10^{-4} )</td>
<td>( 1.9 \times 10^{-2} )</td>
</tr>
<tr>
<td>( 10^{-4} )</td>
<td>( 6.6 \times 10^{-1} )</td>
</tr>
<tr>
<td>( 1.225 \times 10^{-5} )</td>
<td>( 1.7 \times 10^{-2} )</td>
</tr>
</tbody>
</table>

\( (\equiv \varepsilon_{+a}) \)

\( (\equiv \varepsilon_{+b}) \)

\( (\equiv \varepsilon_{+c}) \)
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Fig. 4. Comparisons of numerical power spectra $S(\omega)$ for $G(x_i)$ with the Lorentzian (3.4) (solid lines). Power spectra were calculated by averaging $2 \times 10^4$ spectra each of which was obtained by FFT with a sample length $2^{10}$.

Model A

Model B

Fig. 5. Numerical results of similarity exponents for $G(x_i)$. In both models the $\lambda_q$'s have been calculated by $\lambda_q=\langle q T \rangle^{-1} \ln \langle \exp[q \sum_{f=0}^\infty f G(x_i)] \rangle$ with $T=10^3$. The averaging numbers are $9 \times 10^4$ (Model A) and $2 \times 10^5$ (Model B). As $\epsilon_+\epsilon_-=0$, the staircase structure in $\lambda_q$ becomes steep.

Fig. 6. Comparison of the scaling function (3.6) (solid line) with numerical results given in Fig. 5. $q_+$ was defined in (3.5), where the $D_c$'s have been replaced by the slopes of the $\lambda_q$'s at $q=0$.

the generation of bursts. The independence of bursts is due to the washing-out of the memory of the initial state because of the strong trajectory instability still existing near the transition point.7)

We turn to the similarity exponent analysis8) of $G(x_i)$. First one should remark that the similarity exponent $\lambda_q$ satisfies $\lambda_q=-\lambda_q$ for $\epsilon_+\epsilon_-=0$. Assuming that slightly above the transition point there exists an arbitary long subinterval in a time series, being in the region $\Omega_+$ or $\Omega_-$, we may put $\lambda_\omega(\lambda_-\omega)=-1(-1)$ near the transition point. Since $D_c$ is magnified for $\epsilon_+\epsilon_-=0$, the staircase structure of $\lambda_q$ develops, i.e.,

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8) The order-$q$ similarity exponent for the time series $\{x(t)\}; t=0,1,2,3,\ldots$ is defined by $\lambda_q = q^{-1} \lim_{\epsilon_+\epsilon_-=0} t^{-1} \ln \langle \exp[q Y_t] \rangle$, $(Y_t=\sum_{0}^{t-1} x(t))(11)$
the range $x_c$ of the diffusion branch* describes two different kinds of statistical characteristics, the diffusion and the intermittency, depending on the strength $|q|$. The former is seen for $|q| \ll x$ (diffusion branch) and the latter for $|q| \gg x$ (intermittency branches). Here $x$ is the radius of the convergence of the cumulant expansion of $\lambda_q$. These characteristics cannot be perturbationally connected. For details, see Ref. 11).

*) $A_q$ describes two different kinds of statistical characteristics, the diffusion and the intermittency, depending on the strength $|q|$. The former is seen for $|q| \ll x$ (diffusion branch) and the latter for $|q| \gg x$ (intermittency branches). Here $x$ is the radius of the convergence of the cumulant expansion of $\lambda_q$. These characteristics cannot be perturbationally connected. For details, see Ref. 11).

§ 4. Statistical dynamics for bursts

In this section we will discuss the burst dynamics, starting with the calculation of $C_g(t)$. Using the approximation to neglect the so-called memory term in the Mori equation of motion for $C_g(t)$, one gets

$$C_g(t) \approx C_g(0)(-1)^t \exp(-t/\tau_g),$$

(4.1)

where $\tau_g = -1/\ln|C_g(1)/C_g(0)|$. The assumption of the independence of bursts leads to

$$\tau_g \propto (\ln \varepsilon_+)^{-1},$$

(4.2)

$$D_g \approx C_g(0) \tanh(1/2 \tau_g)/2 \propto \varepsilon_+^{1/2}$$

(4.3)

to the lowest order in $\varepsilon_+$. In contrast to the $G(x)$ case, the correlation time $\tau_g$ and the diffusion coefficient $D_g$ tend to diminish for $\varepsilon_+ \to 0$ (Fig. 3(b)). Consequently the power spectrum for $g(x_t)$ tends to be approximated as the white noise,

$$S(\omega) \propto \sinh(1/\tau_g) \cosh(1/\tau_g) + \cos \omega \to 1$$

(4.4)

for $\varepsilon_+ \to 0$ (Fig. 7).

Since $\tau_g$ becomes short as $\varepsilon_+ \to 0$, the similarity exponent for $\{g(x_t)\}$ may be evaluated with the random insertion assumption on bursts. Thus we obtain

$$q_* = 2(\pi D_c)^{-1} \varepsilon_+^{1/2}$$

(3.5)

estimates the boundary between the diffusion- and the intermittency-branch, and the solid line in Fig. 6 is the function

$$\lambda_q = \frac{2}{\pi} \arctan \left( \frac{q}{q_*} \right).$$

(3.6)

Although this function explains numerical results satisfactorily well, its derivation is rather empirical. Another scaling form derived with a physically more plausible approximation is proposed in § 5. In the sense that the similarity exponent satisfies the scaling relation (Fig. 6), the diffusion- and the intermittency-characteristic are closely connected to each other near the transition point.
Fig. 7. Comparison of numerical power spectra $S(\omega)$ for $g(x_t)$ with the approximate result (4.4), where we have used $C_0(0) \approx 2D_g$ (Eq. (4.3)). Methods of computation are the same as in Fig. 4. Scales in horizontal and vertical lines are the same in all figures.

$$\lambda_q \approx 2D_g \frac{e^q - 1}{q}, \quad (q \ll q_c)$$

$$\approx 1 - \frac{q}{q_c}, \quad (q \gg q_c)$$

For $q \ll q_c$, $\lambda_q$ tends to diminish in the same way as $D_g$ as $\varepsilon_+ \to 0$, the form $(e^q - 1)/q$ being the characteristic of the Poisson process. On the other hand, for $q \gg q_c$, (4.5) approaches 1 as $q \to \infty$. The result $\lambda_{\infty} = 1$ is clearly contradictory to the experimental results (Fig. 8), and numerical $\lambda_{\infty}$ seems to also diminish as $\varepsilon_+(\delta) \to 0$.

The incorrectness of (4.5) for $q \gg q_c$ stems obviously from the correlation effect of bursts neglected in deriving (4.5). The correlation effect can be included as follows. If the phase point $x_0$ is in the channel $\omega_-(\omega_+), x_1$ is in $\Omega_-(\Omega_+)$. Let $M$ be a step number for the phase point to jump into $\Omega_-(\Omega_+)$ again. Namely, $x_0 \notin \Omega_-(\Omega_+)$ for
The approximation used in (4.5) permits any positive value for \( M \). It is obviously wrong. The temporal correlation effect of bursts gives a lower bound of \( M \). This lower bound, denoted hereafter by \( M \), is estimated as follows. Since \( R\varepsilon_- (R\varepsilon_+) \ll 1 \), almost all parts of the trajectory are close to the origin \( x = 0 \). By assuming that \( x_j \sim f'(0)x_{j-1} \) for \( j < M \), i.e., \( x_j \sim (f'(0))^{j-1} A_- ((f'(0))^{j-1} A_+ \), where \( A_- = f'(-R)\varepsilon_- = Rf'(1)\varepsilon_+ \) (\( A_+ = f'(1)\varepsilon_+ \)), \( M \) is estimated as

\[
M \sim \ln A_-^{-1}/\ln f'(0) \propto \ln \varepsilon_-^{-1}. \tag{4.8}
\]

On the other hand, since the next burst occurs at least after \( M \) steps, we can put \( \lambda_\infty \leq M^{-1} \). Thus \( \lambda_\infty \) is less than

\[
\lambda_\infty \propto (\ln \varepsilon_+^{-1})^{-1}. \tag{4.9}
\]

This tends to diminish as \( \varepsilon_+ \to 0 \), which is consistent with the observation in Fig. 8. If the width \( x_g \) of the diffusion branch is estimated by \( x_g = \lambda_\infty/D_g \), it diverges as \( \propto \varepsilon_+^{-1/2} \) by neglecting the logarithmic \( \varepsilon_+ \)-dependence.

For \( q \ll q_c \), Eq. (4.7a) provides us with the scaling behavior \( \lambda_q/2D_g = (e^q - 1)/q \), whose numerical evidence is shown in Fig. 9. The boundary of the scaling behavior is evaluated with \( q_c \). Since \( q_c \) becomes long as the system approaches the transition point, the \( g \) dynamics tends to be well approximated by the Poisson process as is expected.

In closing this section we make the following remark. Since (4.5) is derived on the assumption of the random bursts, the dynamics of \( g(x) \) for \( q \ll q_c \) is insensitive to the correlation effect among bursts. On the contrary, its dynamics for \( q \gg q_c \) is strongly affected by the correlation effect. This is different from the \( G(x) \) case.

§ 5. Summary and remarks

In the present paper we studied the band-splitting phenomena as the prototype of the chaos-chaos transition, from the viewpoint of the breakdown of the chaos symmetry and the development of the intermittency. The study was carried out mainly with the use of the similarity exponents for two coarse-grained variables \( G \) and \( g \). The former is relevant to the chaos symmetry and the latter to the burst dynamics. The statistical dynamics associated with the symmetry-breaking transition turned out to be characterized by the development of the intermittency in the sense explained in § 2. We have shown that the development of the intermittency can
Table II. Numerical results of $\Lambda_q$ for $G(x_t)$ at $q = D_c^{-1}$. Theoretical results (3·6) and (5·2) give 0.6391 and 0.6625, respectively.

<table>
<thead>
<tr>
<th>$\varepsilon$</th>
<th>$R=1$</th>
<th>$R=2$</th>
<th>$\delta$</th>
<th>$\varepsilon$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\varepsilon_{+a}$</td>
<td>0.5959</td>
<td>0.5868</td>
<td>$\delta_a$</td>
<td>0.6070</td>
</tr>
<tr>
<td>$\varepsilon_{+b}$</td>
<td>0.6092</td>
<td>0.6112</td>
<td>$\delta_b$</td>
<td>0.6188</td>
</tr>
<tr>
<td>$\varepsilon_{+c}$</td>
<td>0.6274</td>
<td>0.6241</td>
<td>$\delta_c$</td>
<td>0.6244</td>
</tr>
</tbody>
</table>

developed with the approximation in the Appendix. Numerical values of (5·1) are very close to those of (3·6). We compared values of $\Lambda_q$ for (3·6) and (5·1) for $q = D_c^{-1}$ in Table II. Although numerical results are closer to $(2/\pi)\arctan(\pi/2) = 0.6391$ from (3·6) than $F(2) = 0.6625$ from (5·2), the physical meaning used in deriving (5·2) may be clearer than that in (3·6) (the Appendix).

Furthermore, although similarity exponents for $G(x)$ is well approximated by the independence assumption on the generation of bursts, $\Lambda_q$ for $g(x)$ preserves the correlations among bursts. Namely, $\Lambda_q$ for $g(x)$ for $q \ll q_c$ is insensitive to the correlation effect, but that for $q \gg q_c$ reflects a strongly correlated temporal region. If $x$ is regarded as a continuous time, then $g(x_t) \propto \frac{dG(x_t)}{dt}$. So the $g$ dynamics contains a more mechanical property of the basic law $x_{t+1} = f(x_t)$ than $G(x_t)$ in the relatively high frequency region. The invalidity of the independence assumption on the generation of bursts for the $q \gg q_c$ region of $\Lambda_q$ for $g(x_t)$, in comparison with that for $G(x_t)$, may be due to such qualitative difference of the $g$ and $G$ dynamics. One should remark that different statistical aspects in time series, e.g., the temporal correlation effect described above can be singled out with $\Lambda_q$ by controlling the parameter $q$.

Recently we have developed a thermodynamics formalism for similarity exponents,\(^{14}\) where $q$ and $\Lambda_q$ play a role similar to the inverse temperature and the Helmholtz free energy in the equilibrium statistical mechanics, respectively. The corresponding relation between other "thermodynamic" variables constitutes a fluctuation spectrum. The study of the band-splitting phenomena from this new point of view in connection with the scaling laws of thermodynamic variables will be reported elsewhere.

**Acknowledgements**

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Appendix
—Derivation of (5.1)—

The assumption that $G(x_j)$ and $G(x_l)$ are independent of each other, provided that $|j - l| > \tau_\text{re}(=2\tau_c)$, yields

$$
\langle \exp[q \sum_{j=0}^{n\tau_\text{re}-1} G(x_j)] \rangle = \prod_{l=1}^{n} \exp[q \sum_{j=m_l+1}^{m_{l+1}} G(x_j)]
$$

$$
\approx \prod_{l=1}^{n} \langle \exp[q \sum_{j=m_l+1}^{m_{l+1}} G(x_j)] \rangle,
$$

(A·1)

where $m_l = (l-1)\tau_\text{re} - 1$. So $\lambda_q$ is expressed as

$$
\lambda_q = \frac{1}{q} \frac{1}{\tau_\text{re}} \ln \langle \exp(q\tau_\text{re}a_l) \rangle,
$$

(A·2)

where the local average $a_l$ introduced by

$$
a_l = \frac{1}{\tau_\text{re}} \sum_{j=m_l+1}^{m_{l+1}} G(x_j)
$$

(A·3)

has been assumed to be statistically independent of $l$. Let us furthermore assume that $a_l$ takes only 1 or $-1$ as $G(x_j)$ does, with the equal probability 1/2. This leads to the approximate result

$$
\langle \exp(q\tau_\text{re}a_l) \rangle = \cosh(\tau_\text{re}q).
$$

(A·4)

The insertion of (A·4) into (A·2) gives (5·1) with (5·2), where we have used $\tau_\text{re} = 2\tau_c \approx 2D_c$.

References

10) On studies about the double-well potential system, see references cited in Ref. 9).