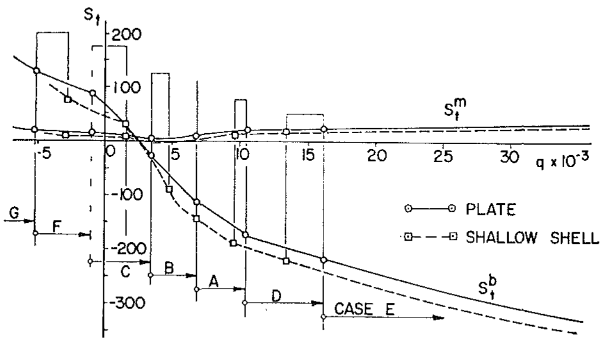

Fig. 8 Tangential bending and membrane stresses at ξ = 1.0

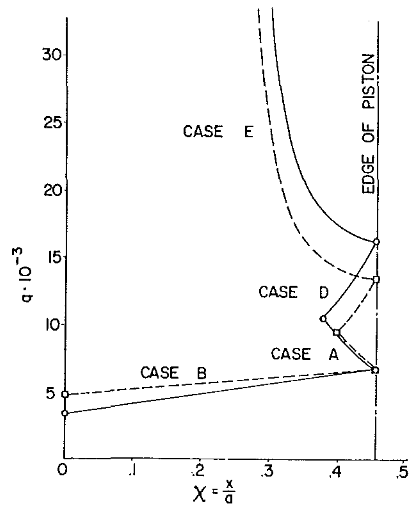

Fig. 9 Shell-piston contact radius

Summary and Conclusions

The problem of a shallow shell of revolution type diaphragm experiencing large deflections while being in contact with a centrally located piston was solved for some typical, technically important cases of pressure-displacement converter diaphragms.

It was assumed that shell deflections could be described by Marguerre's equations. These equations were, for the axisymmetric subset, combined into one equation and were solved by successive integration. It was found that convergence of the iteration depended on the use of an interpolation coefficient. The magnitude of this coefficient was a function of the quality of the first solution estimate.

The problem of the large deflection of a shallow spherical shell clamped at the boundary was used as a test case. Comparison with solutions obtained by Thurston showed good agreement.

The two types of pressure-displacement converter diaphragm treated in this report were a shallow shell of revolution whose radial slopes were described by a sine function and the circular plate.

The assumption that the contact cases shown in Fig. 2 were sufficient to describe the problem at hand was substantiated by experimental inspection.

Acknowledgment

The authors wish to thank the Alco Controls Corporation in St. Louis, Missouri, for sponsorship of the research work.

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1 Hofmann, R., "Über ein nichtlineares Problem der Plattenstatik," *Zeitschr. f. angew. Math. u. Mech.*, Band 18, Heft 4, Aug. 1938.

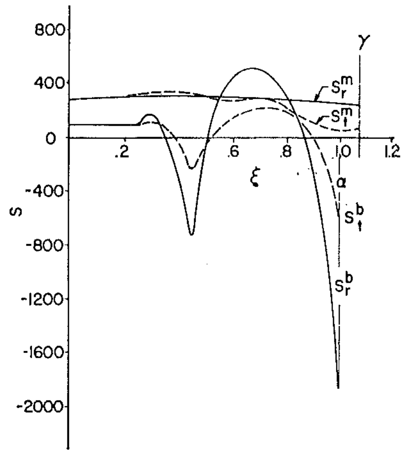

Fig. 10 Stress distribution across shell (q = 110 300)

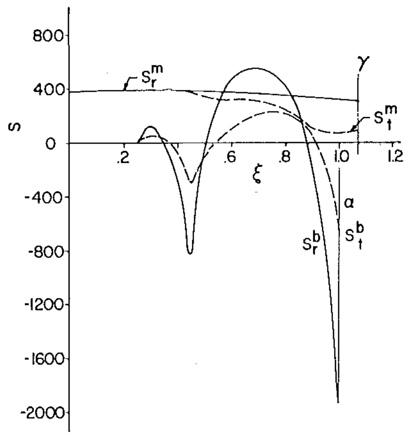

Fig. 11 Stress distribution across plate (q = 128 800)

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DISCUSSION

A. Iberall[2]

The authors are to be commended on an important addition to elastic analyses available to the designer. To amend the observation that they could not find sources treating large deflection of shallow shells of revolution in contact with force transmitters, the readers' attention is called to my 1954 ASME paper which was developed to fill that gap for instrument design ("Characteristics of Slack Diaphragms," ASME Paper 54-A-195). Our solution methods differ, but both are of interest. They both advance the art presented in two classic papers—Eaton and Buckingham (NACA Report 206, 1924), Way (ASME, 1934). Had my plate paper been published, I would have followed it up with a corrugated diaphragm (curved shell) paper. I have requested the opportunity to outline my method.

[2] General Technical Services, Inc., Upper Darby, Pa., Fellow, ASME.

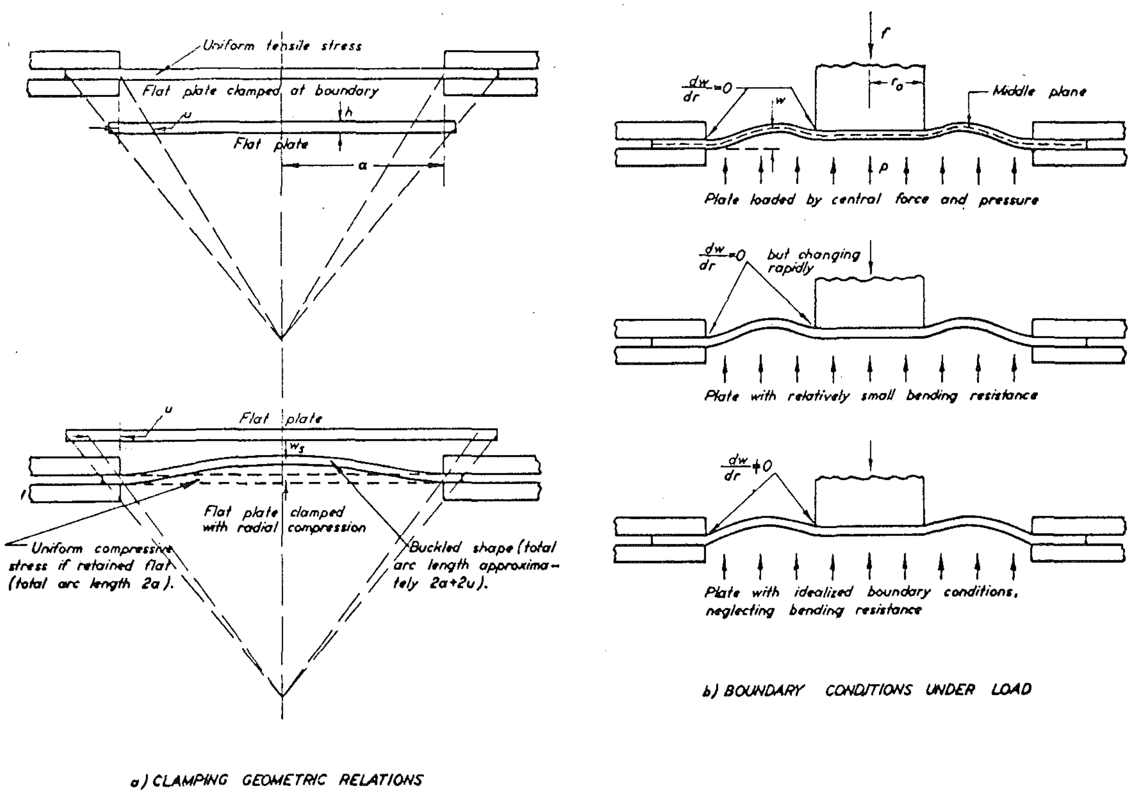

Fig. 1 Illustration of the boundary conditions for clamped circular plates

The Föppl-vonKármán equations for large deflection of axisymmetric plates are [1, 2, 3][3]

[3] Numbers in brackets designate Additional References at end of discussion.

$$\frac{1}{r}\frac{d}{dr}\left[r\frac{d}{dr}\left\{r\frac{d}{dr}\left(r\frac{ds}{dr}\right)\right\}\right] = -\frac{E}{2r}\frac{d}{dr}\left[\frac{dw}{dr}\right]^2 \quad (1)$$

$$\frac{D}{r}\frac{d}{dr}\left[r\frac{d}{dr}\left\{\frac{1}{r}\frac{d}{dr}\left(r\frac{dw}{dr}\right)\right\}\right] = \frac{h}{r}\frac{d}{dr}\left[\frac{ds}{dr}\frac{dw}{dr}\right] + p \quad (2)$$

Chien [4] has shown the major sources of error in these equations if the plate is markedly curved (i.e., a shell). Thus the initial development was restricted to large deflections of plates (i.e., flat shells).

The circumferential midplane tensile stress C, radial midplane tensile stress T, and the stress function s, are related by

$$\frac{d}{dr}(rT) = C = \frac{d^2s}{dr^2} \quad (3)$$

Since the boundary conditions of interest will be related to the radial stresses, equations (1) and (2) may be expressed more conveniently as

$$p = \frac{D}{r}\frac{d}{dr}\left[r\frac{d}{dr}\left\{\frac{1}{r}\frac{d}{dr}\left(r\frac{dw}{dr}\right)\right\}\right] - \frac{h}{r}\frac{d}{dr}\left[rT\frac{dw}{dr}\right] \quad (4)$$

$$r\frac{d}{dr}\left[\frac{1}{r}\frac{d}{dr}(r^2T)\right] = -\frac{E}{2}\left[\frac{dw}{dr}\right]^2 \quad (5)$$

Equation (5) is the first integral of equation (1).

Boundary Conditions. Boundary conditions of interest are for a clamped plate of radius a. Generally they include (Fig. 1):

$$\frac{dw}{dr} = 0 \quad \text{at} \quad r = 0 \quad (6)$$

valid for problems involving either bending or stretching under distributed loads, or bending under concentrated loads. For stretching under concentrated loads, equation (6) is true if there is a central support disc.

$$\frac{dw}{dr} = 0 \quad \text{at} \quad r = a, \quad \text{or} \quad r = r_o^{+} \quad (7)$$

where r_o = radius of a cemented flat rigid support disc.

This boundary condition is for a clamped plate, where the plate material is assumed to have some bending resistance. If the bending resistance may be neglected, then this condition may be neglected.

$$w = 0 \quad \text{at} \quad r = a \quad (8)$$

An arbitrary boundary condition that determines the plane of reference from which deflection is measured

$$r\frac{dT}{dr} + [1 - \nu]T = E\frac{u}{r} \quad \text{at} \quad r = a \quad \text{or} \quad r = r_o \quad (9)$$

If it is assumed that a constant radial displacement u was "built" into the plate when the central disc was cemented on, or the edge clamped, the boundary tension will not remain constant as the plate is subsequently loaded. Instead the tension and its derivative will remain connected. Two other forms for u may be derived. Suppose the diaphragm has been stretched to a uniform radial displacement at its edge. The application of the equations under the conditions $p = 0$, $w = 0$, finite tension, shows that the radial tension T_1 is uniform, when clamped flat. One alternate form for a loaded plate thus becomes

$$r\frac{dT}{dr} + [1 - \nu]T = [1 - \nu]T_1 \quad (10)$$

On the other hand, suppose the diaphragm has been subjected to uniform radial thrusts, while confined flat, and then clamped at its edge. An elegant theory of buckling derivable from the preceeding equation [5] provides the critical buckling load and the central deflection of the various buckling modes. However, with very little loss in accuracy, it is convenient to estimate the radial edge displacement, for the case when it is negative and appreciable, from a model of the buckled plate as a spherical segment, from which it is simple to show

$$\frac{u}{a} = -\frac{2}{3}\left[\frac{w_s}{a}\right]^2 \tag{11}$$

where w_s = the deflection of a buckled diaphragm from the plane of its rim (the "slackness").

While at first sight this might appear to be an approximation for a slack diaphragm in which bending resistance may be neglected, it turns out to be a good approximation for the first buckling mode of a radial symmetric plate. It provides a second useful form of u.

Reduction of the Equations to Solvable Form. With the substitutions

$$T = \frac{E}{12(1-\nu^2)}\left[\frac{h}{a}\right]^2 \tau \tag{12}$$

$$\frac{dw}{dr} = \frac{1}{\sqrt{12(1-\nu^2)}}\left[\frac{h}{a}\right] z \tag{13}$$

$$p = \frac{E}{[12(1-\nu^2)]^{3/2}}\left[\frac{h}{a}\right]^4 \varphi \tag{14}$$

$$r = a\rho \tag{15}$$

equations (4) and (5) become

$$\varphi = \frac{1}{\rho}\frac{d}{d\rho}\left[\rho\frac{d}{d\rho}\left\{\frac{1}{\rho}\frac{d}{d\rho}(\rho z)\right\}\right] - \frac{1}{\rho}\frac{d}{d\rho}[\rho z\tau] \tag{16}$$

$$\rho\frac{d}{d\rho}\left[\frac{1}{\rho}\frac{d}{d\rho}(\rho^2\tau)\right] = \frac{-1}{2}z^2 \tag{17}$$

To reduce these equations to solvable form for a diaphragm with boundary conditions at $\rho = 1$ ($r = a$), and at $\rho = \rho_0 < 1$ ($r = r_o$): doubly integrate equation (17), first outward to $\rho = 1$, then outward from $\rho = \rho_o$.

$$\rho^2\tau = \rho_o^2\left[(\tau)_{\rho_o} - \frac{1}{2}\frac{d}{d\rho}(\rho^2\tau)_1\right] + \frac{\rho^2}{2}\left[\frac{d}{d\rho}(\rho^2\tau)\right]_1 + \frac{1}{2}\int_{\rho_o}^{\rho}\rho\int_{\rho}^{1}\frac{z^2}{\rho}\,d\rho d\rho \tag{18}$$

Single integration of equation (16) results in

$$\rho z\tau = -\int_{\rho_0}^{\rho}\varphi\rho d\rho + [\rho z\tau]_{\rho_o} - \left[\rho\frac{d}{d\rho}\left\{\frac{1}{\rho}\frac{d}{d\rho}(\rho z)\right\}\right]_{\rho_o} + \left[\rho\frac{d}{d\rho}\left\{\frac{1}{\rho}\frac{d}{d\rho}(\rho z)\right\}\right] \tag{19}$$

The tension may then be eliminated. By means of the additional transformations

$$y = z/\rho \qquad x = \rho^2 - \rho_o^2 \qquad x_1 = 1 - \rho_o^2 \tag{20}$$

there results

$$\left[\rho z\tau - \rho\frac{d}{d\rho}\left\{\frac{1}{\rho}\frac{d}{d\rho}(\rho z)\right\}\right]_{\rho_o} - \frac{1}{2}\int_o^x \varphi dx = \left[(\rho^2\tau)_{\rho_o} + \frac{x}{2}\left\{\frac{d}{d\rho}(\rho^2\tau)\right\}_1 + \frac{1}{8}\int_o^x\int_x^{x_1} y^2 dx dx\right] y - 8[\rho_o^2 + x]\frac{dy}{dx} - 4[\rho_o^2 + x]^2\frac{d^2y}{dx^2} \tag{21}$$

an integro-differential equation of the original equation set.

By the following considerations, the problem falls apart for elementary approximations. It is reasonable to assume that the series expansion of the integral

$$\int_o^x\int_x^{x_1} y^2 dx dx = x\int_o^x y^2 dx - \frac{x^2}{2}[y^2]_o + \ldots \tag{22}$$

is adequately represented by the first term in the series, since a basic assumption made in deriving the original equations was that the square of the slope of the deflection curve was small. As a good approximation, therefore, equation (21) becomes

$$\left[\rho z\tau - \rho\frac{d}{d\rho}\left\{\frac{1}{\rho}\frac{d}{d\rho}(\rho z)\right\}\right]_{\rho_o} - \frac{1}{2}\varphi_o x = \left[(\rho^2\tau)_{\rho_o} + \frac{x}{2}\left\{\frac{d}{d\rho}(\rho^2\tau)\right\}_1 + \frac{x}{8}\int_o^{x_1} y^2 dx\right] y - 8[\rho_o^2 + x]\frac{dy}{dx} - 4[\rho_o^2 + x]^2\frac{d^2y}{dx^2} \tag{23}$$

where the additional assumption, of interest here, has been added so that the applied pressure is constant over the annulus (i.e., $\varphi = \varphi_o$ for $\rho_o < \rho < 1$). The equation is a second order equation in y, with variable coefficients of relatively simple form, whose coefficients are obtained by evaluating integrals.

It may also be noted that if the bending terms are neglected, equation (23) assumes the simpler form

$$y = \frac{[\rho z\tau]_{\rho_o} - \varphi_o x/2}{[\rho^2\tau]_{\rho_o} + \left[\frac{1}{2}\left\{\frac{d}{d\rho}(\rho^2\tau)\right\}_1 + \frac{1}{8}\int_o^{x_1} y^2 dx\right] x} \tag{24}$$

(i.e., the equation has been "solved" for those problems in which the denominator is not improper).

Uniform Pressure Loading, a Fundamental Example

An Elementary Solution. The most significant features of the deflection of thin plates may be discovered by consideration of a very simple example, namely, the deflection of a thin plate loaded by a constant pressure. The boundary conditions will be those for a thin circular plate clamped flat at the edge with a definite radial displacement u, built into the plates before clamping, namely

$$\begin{aligned} &\rho_o = 0 && z = 0 \text{ at } \rho = 0 \\ &x_1 = 1 && z = 0 \text{ at } \rho = 1 \\ &\varphi = \varphi_o && w = 0 \text{ at } \rho = 1 \\ &\rho\frac{d\tau}{d\rho} + [1-\nu]\tau = (1-\nu)\tau_1 \text{ at } \rho = 1 \end{aligned} \tag{25}$$

With the above boundary condition for τ together with equation (18), it may be shown that equation (23) becomes

$$\left[\tau_1 + \frac{1}{8}\int_o^1 y^2 dx + \frac{1+\nu}{8(1-\nu)}\int_o^1\int_x^1 y^2 dx dx\right] y - 8\frac{dy}{dx} - 4x\frac{d^2y}{dx^2} = -\frac{1}{2}\varphi_o \tag{26}$$

Approximate solutions for the integro-differential equations may be sought as series or rational fractional form. It may be noted that such forms satisfy the condition of the radial symmetry of the deflection curve.

Consider first a polynomial having the minimum number of terms required to satisfy the boundary conditions.

$$y = b_o + b_1 x \tag{27}$$

The vanishing slope of the edge requires that

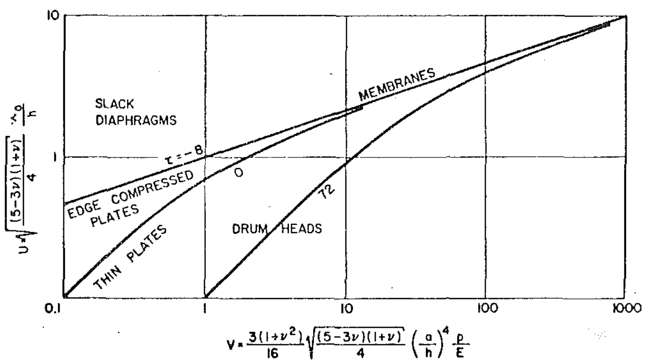

Fig. 2 Characteristic regimes of thin plates as determined by their initial edge tension

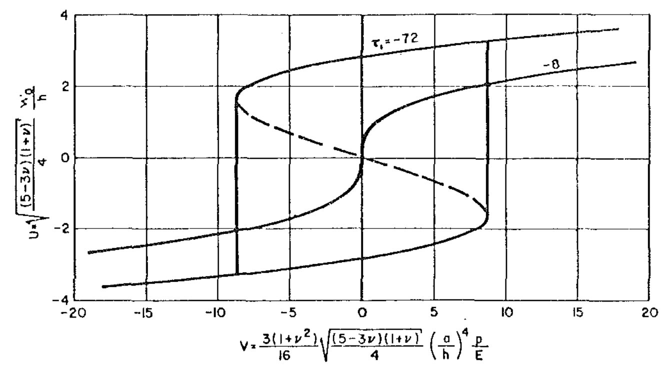

Fig. 3 Characteristic model of a snap action diaphragm

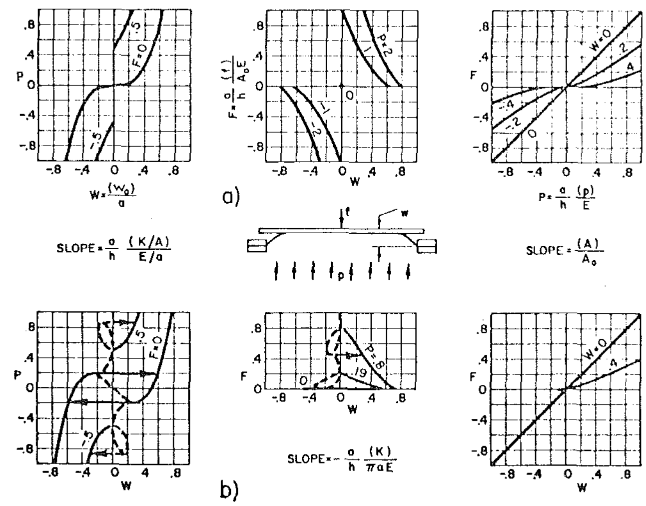

Fig. 4 Relations among dimensionless force F, pressure P, deflection W for diaphragm ($\nu = 1/3$) loaded through tangent plate: (a) with dimensionless slackness $W_s = 1/2$ ($W_s = w_s/a$).

$$b_o = -b_1 \tag{28}$$

and equation (26) requires, as an approximation, that

$$\frac{[5 - 3\nu]}{96} b_o^3 + [1 - \nu][8 + \tau_1]b_o + \frac{[1 - \nu]}{2} \varphi_o = 0 \tag{29}$$

The problem may be completed by integrating

$$z = b_o[\rho - \rho^3] = \frac{\sqrt{12(1 - \nu^2)}}{h} \frac{dw}{d\rho} \tag{30}$$

to give the deflection curve

$$\frac{w}{h} = \frac{b_o}{\sqrt{12(1 - \nu^2)}} \left[-\frac{1}{2} + \frac{\rho^2}{2} - \frac{\rho^4}{4} \right] \tag{31}$$

or a central deflection of

$$\frac{w_o}{h} = -\frac{b_o}{4\sqrt{12(1 - \nu^2)}} \tag{32}$$

Substitution gives the following relationship between central deflection and pressure:

$$\left[\frac{w_o}{h}\right]^3 + \frac{8 + \tau_1}{2[5 - 3\nu][1 + \nu]} \left[\frac{w_o}{h}\right] = \frac{3}{4} \frac{[1 - \nu]}{[5 - 3\nu]} \frac{\varphi_o}{[12(1 - \nu^2)]^{3/2}} \tag{33}$$

Equation (33) is plotted in Fig. 2 for different values of the parameter τ_1, including two negative values. For τ_1 less than -8, w_o is no longer a single valued function of p. In particular, in the vicinity of $p = 0$, since p is a cubic function of w_o, the curve—as shown by the dashed portion—reverses its direction. Since a diaphragm is driven by a monotonic pressure signal, not a displacement, the reversal in direction of p is not achieved physically. The diaphragm thus snaps at the two tangent limits, as shown by the solid lines in Fig. 3. This is the characteristic model for snap action or "oil canning." It is of interest to note that the value $\tau_1 = -8$, although crudely derived, represents a reasonably good estimate of the first critical buckling load.

Qualitatively, therefore, flat thin plates may be used in any one of the following five regimes, which are determined both by their geometric and elastic properties, and the method of clamping: (*a*) membranes, (*b*) thin plates, (*c*) drum heads, (*d*) edge compressed plates, (*e*) slack or snap action diaphragms.

It may be noted, in Fig. 2, that the only large displacement regimes (i.e., w_o/h appreciably exceeding unity) of considerable deflection sensitivity are the membrane and slack diaphragm regimes. The remainder of the discussion will be concerned, primarily, with these two large deflection regimes.

The approximate equation applicable to problems neglecting bending is (24) or, the "exact" equation

$$y = \frac{[\rho z\tau]_{\rho_o} - \varphi_0 x/2}{[\rho^2\tau]_{\rho_o} + \frac{x}{2}\left[\frac{d}{d\rho}(\rho^2\tau)\right]_1 + \frac{1}{8}\int_o^x \int_x^{x_1} y^2 dx dx} \tag{34}$$

where, in either case, it is no longer necessary to satisfy the bending boundary conditions ($z = 0$, at $\rho = \rho_o$ and $\rho = 1$).

More Accurate Solutions. A better approximation to the large displacement regimes, neglecting bending, can be obtained from equation (24). A solution of the simple form $y = b_o$ is in fact an exact solution. It leads to the conditional equation

$$\frac{[3 - \nu]}{16} b_o^3 + [1 - \nu]\tau_1 b_o + \frac{[1 - \nu]}{2} \varphi_o = 0 \tag{35}$$

Integration of equation (35) ultimately leads to a relationship between central deflection and pressure

$$\left[\frac{w_o}{h}\right]^3 + \frac{4}{3 - \nu} \frac{u}{a} \left[\frac{a}{h}\right]^2 \left[\frac{w_o}{h}\right] = \frac{1 - \nu}{3 - \nu} \left[\frac{a}{h}\right]^4 \frac{p}{E} \tag{36}$$

For membranes

$$\frac{w_o}{a} = \left[\frac{1 - \nu}{3 - \nu}\right]^{1/3} \left[\frac{a}{h} \frac{p}{E}\right]^{1/3} \tag{37}$$

For slack diaphragms, using the slackness approximation equation (11)

$$\left[\frac{w_o}{h}\right]^3 - \frac{8}{9 - 3\nu} \left[\frac{w_s}{h}\right] \left[\frac{w_o}{h}\right] = \frac{1 - \nu}{3 - \nu} \left[\frac{a}{h}\right]^4 \frac{p}{E} \tag{38}$$

One final higher order approximation will be attempted. Let

$$y = b_o + b_1 x \tag{39}$$

Equation (34) ultimately leads to a central deflection of

$$\left[\frac{w_o}{h}\right]^3 + \frac{4}{3-\nu}\,\frac{\left[1+\frac{f}{2}\right]^2 \frac{u}{a}\left[\frac{a}{h}\right]^2}{\left[1+\frac{4}{3}\frac{5-\nu}{3-\nu}f+\frac{1}{2}\frac{7-\nu}{3-\nu}f^2\right]}\,\frac{w_o}{h} = f\left[1+\frac{f}{2}\right]^3\left[\frac{a}{h}\right]^4\frac{p}{E} \tag{40}$$

where f is the root of

$$1 - \frac{3-\nu}{1-\nu}f - \frac{2}{3}\frac{5-\nu}{1-\nu}f^2 - \frac{1}{6}\frac{7-\nu}{1-\nu}f^3 = 0 \tag{41}$$

For membranes

$$\frac{w_o}{a} = f^{1/3}\left[1+\frac{f}{2}\right]\left[\frac{a}{h}\frac{p}{E}\right]^{1/3} \tag{42}$$

For slack diaphragms

$$\left[\frac{w_o}{h}\right]^3 - \frac{8}{9-3\nu}\,\frac{\left[1+\frac{f}{2}\right]^2\left[\frac{w_s}{h}\right]^2\frac{w_o}{h}}{\left[1+\frac{4}{3}\frac{5-\nu}{3-\nu}f+\frac{1}{2}\frac{7-\nu}{3-\nu}f^2\right]} = f\left[1+\frac{f}{2}\right]^3\left[\frac{a}{h}\right]^4\frac{p}{E} \tag{43}$$

As a measure of the degree of success of these approximations, consider the various membrane solutions of equations (33), (37), and (42).

They may be compared with a more precise value of

$$\frac{w_o}{a} = 0.662\left[\frac{a}{h}\frac{p}{E}\right]^{1/3} \tag{44}$$

given by Hencky [6] for $\nu = 1/4$. For $\nu = 1/4$, equations (33), (37), and (42) have coefficients of 0.510, 0.648, and 0.664, respectively. The degree of improvement of the various approximations may be noted. A one or two-term approximation for the solution neglecting bending, therefore, may be expected to give reasonably good results for applicable problems.

Loaded Diaphragms

There are many configurations of transmitting discs and diaphragms, with corresponding boundary conditions of more or less exactness that may exist in the large displacement regime. Two were discussed.

In the first configuration (see Fig. 4) a rigid disc of equal or greater diameter than a clamped diaphragm with zero or built-in edge compression is pressed against the diaphragm. The disc balances at some displacement determined both by the concentrated load applied to the rigid disc and the opposing pressure load applied to the diaphragm. Since the rigid disc is tangent to the diaphragm at the last circle of contact, this configuration may be referred to as "tangent disc" loading.

In the second configuration (see Fig. 5), a diaphragm is stretched (either positively or negatively), a rigid center support disc is "cemented" on, the diaphragm is restretched, and the edge is clamped. This configuration may be referred to as "cemented disc" loading.

These cases are treated in the original paper, and the results are shown in Figs. 4 and 5.

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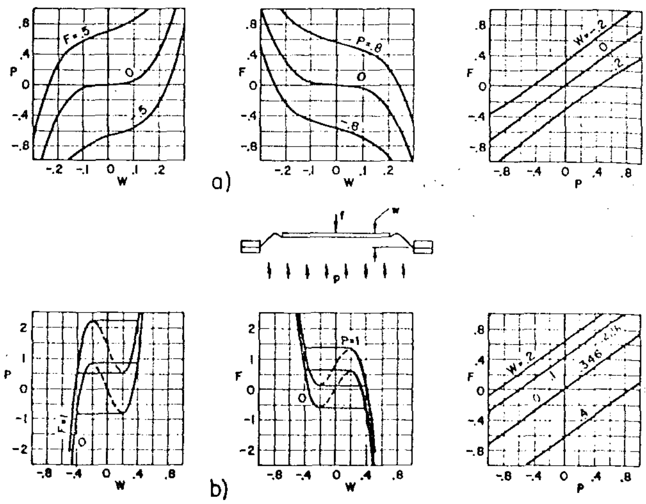

Fig. 5 Relations among dimensionless force *F*, pressure *P*, deflection *W* for diaphragm ($\nu = 1/3$) loaded through cemented disc (dimensionless radius $\rho_o^2 = 1/2$): (*a*) with dimensionless edge displacement $u/a = 0$; (*b*) with dimensionless edge displacement $u/a = -1/5$.

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Authors' Closure

We would like to thank the discusser for his contributions. They furnish a useful supplement to our paper. However, we feel that we ought to make some clarifying comments.

1 It seems to us that we not only differ from the discusser in our solution methods but also somewhat in our basic problem. The Föppl-von Karmán equations that the discusser uses are only valid for the large deflections of plates (as admitted by the discusser) and constitute a subset of the more general Marguerre equations for shallow shells used by us. The way the discusser introduces what he called the "slack" diaphragm is by assuming that the initial transverse deflection was introduced by a radial compression (buckling). It is true that this is an occurring and technically important case, and it is obviously also treatable by our approach; but we do go a step beyond that by allowing, for instance, for the possibility that the buckled diaphragm undergoes an annealing process that produces a stress free initial shallow shell surface, or that the diaphragm was intentionally designed as a shallow shell. Obviously, the Föppl-von Karman equations do not apply any longer in such a case and the Marguerre equations for the large deflections of shallow shells are to be employed. Summing up this point: The discusser's treatment is limited to a subset of the problems that are treatable by a solution of Marguerre's equations.

2 A point not entered into by the discusser is the treatment of the pressure dependent boundary condition that occurs because of the varying radius of contact between diaphragm and piston. We would like to think that this is also a contribution to the state of the art in diaphragm design. It will be interesting to see, for the subset of problems treated by the discusser, how his solution approach can be adapted to this type of problem.

In closing, we would like to reiterate that we are pleased about the discusser's contribution, since it helps to clarify some of the difficult but practically important problems of diaphragm design.