A Kirchhoff method for the computation of finite-frequency body wave synthetic seismograms in laterally inhomogeneous media

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Summary. A modified version of the Kirchhoff–Helmholtz integral can be used to synthesize elastic wavefields in media for which velocity is a function of range, $x$, as well as depth, $z$. The essence of the method is that rays are traced from both source and receiver to some intermediate surface, $\Sigma$. The field at the receiver is then given by an integral over $\Sigma$, whose integrand is a particular product of the values of the source and receiver wavefields. The surface $\Sigma$ is not a reflector since the medium is continuous across it. Geometrical ray theory (GRT) is used to calculate the source and receiver wavefields on $\Sigma$. When either the source or receiver wavefield has a caustic in $\Sigma$ then the GRT amplitude is infinite and, in theory, the method breaks down. However, numerical breakdown can be avoided by parameterizing the GRT amplitudes so that their singularities are integrable and choosing $\Sigma$ so that caustics of the source rayfield and caustics of the receiver rayfield do not intersect on $\Sigma$. We refer to this alternative as the extended Kirchhoff–Helmholtz (EKH) method. For reasons of economy EKH may be a practical alternative to the more theoretically correct procedure of using many surfaces: e.g. for two surfaces, tracing rays from the source to the first surface $\Sigma_1$, then from every point on $\Sigma_1$, to every point on the second surface $\Sigma_2$, then from the receiver to $\Sigma_2$, then integrating over the product manifold $\Sigma_1 \times \Sigma_2$.

In this paper we give examples of the errors that arise when caustics on $\Sigma$ are treated as integrable singularities. First the EKH method is compared with the WKBJ method for a stratified medium, then the EKH method is compared with the ordinary Kirchhoff–Helmholtz method where $\Sigma$ intersects no caustics. Errors in the EKH method take the form of small spurious phases which generally arrive later in time than correct arrivals. The arrival times of these error phases can be changed by adjusting $\Sigma$. For some velocity models these phases can be eliminated completely.

The EKH method is not as fast as the Maslov (extended WKBJ) method because of the amount of ray tracing needed. However, one of the attractive features of the EKH procedure is that its underlying theory is very simple.

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Introduction

The Kirchhoff–Helmholtz integral (Helmholtz 1860; Kirchhoff 1883) has long been used in wave propagation problems of all kinds. A review of its applications in the diffraction of elastic waves by cylinders and spheres has been given by Mow & Pao (1971). In reflection seismology it has been used both for the modelling of data (Hilterman 1970, 1975, 1982; Trorey 1970, 1977; Berryhill 1977) and, sometimes implicitly, for the inversion of data (Hagedoorn 1954; French 1974, 1975; Gardner, French & Matzuk 1974; Schneider 1978). Scott & Helmberger (1983) have recently used the integral to model body wave reflections from mountain topography and spall from nuclear blasts. The previous work most closely related to this paper is that of Haddon & Buchen (1981) who used the Kirchhoff–Helmholtz integral to calculate the PKP wavefield in a stratified earth. Here we treat the problem of propagation in an unstratified two-dimensional (2-D) medium; we also propose a practical solution to a problem which occurs when one of the wavefields has a caustic on the surface of integration.

Derivations of the time domain version of the integral can be found in many textbooks (e.g. Officer 1958). For completeness we include here a brief derivation of the frequency domain form of the integral that will be needed in the sequel. As shown in Fig. 1, let \( V \) be an open volume in \( E^N \) and let \( \partial V \) be the boundary of \( V \) with outward pointing unit normal \( \hat{n} \). Then for any two scalar fields \( \psi_1 \) and \( \psi_2 \) it follows from the divergence theorem that

\[
\int_{\partial V} \hat{n} \cdot (\psi_1 \nabla \psi_2 - \nabla \psi_1 \psi_2) \, dA = \int_{V} (\psi_1 \nabla^2 \psi_2 - \nabla^2 \psi_1 \psi_2) \, dV.
\]

This relation is sometimes called Green’s second formula. Now suppose that the scalar fields \( \psi_1 \) and \( \psi_2 \) satisfy the wave equations \( B \psi_1 = f_1 \) and \( B \psi_2 = \delta(x - x_2) \), respectively, where \( f_1 \) is zero except at a point \( x_1 \) outside \( V \), \( x_2 \) is inside \( V \), \( B = \nabla^2 + \omega^2/c^2(x) \) is the Helmholtz operator with inhomogeneous velocity \( c \), and \( \delta \) is the Dirac delta function for \( E^N \). Thus \( \psi_1 \) can be regarded as the field due to a point source at \( x_1 \) and \( \psi_2 \) can be regarded as the field due to a point source at \( x_2 \). Substitution of the wave equations for \( \psi_1 \) and \( \psi_2 \) into Green’s formula then yields the relation

\[
\psi_1(x_2) = \int_{\partial V} \hat{n} \cdot (\psi_1 \nabla \psi_2 - \nabla \psi_1 \psi_2) \, dA
\]

which is our desired form of the Kirchhoff–Helmholtz formula. Equation (1) is exact if \( \psi_1 \) and \( \psi_2 \) are exact solutions of their respective wave equations. When the velocity in the
Helmholtz operator is inhomogeneous then, except for a few special cases such as acoustic waves in a medium of constant density, the physical variable $\psi$ satisfies the wave equation only approximately (e.g. Ansell 1979). Then $\psi$ given by (1) is also a physical approximation. When $\omega$ is large these approximations are usually very good.

In equation (1) we shall regard $x_2$ as the receiver point so that $\psi_1(x_2)$ is the field at $x_2$ due to the source at $x_1$. In many applications of (1) the velocity $c$ is constant or else the volume $V$ containing $x_2$ is taken to be so small that, within $V$, $c$ may be regarded as constant and equal to $c(x_2)$. Then for three dimensions $\psi_2 = \exp [i\omega |x - x_2| / c] / (-4\pi |x - x_2|)$ and for two dimensions $\psi_2 = -i/4H_0^{(1)}(\omega |x - x_2| / c)$ where $H_0^{(1)}$ is the Hankel function of the first kind of order zero. (We have used a Fourier transform

$$\mathcal{F} = \int dt \exp (i\omega t).$$

If instead we had used

$$\mathcal{F} = \int dt \exp (-i\omega t)$$

then these two formulae for $\psi_2$ would be replaced by their complex conjugates.) Here we shall suppose that $c$ is variable within $V$ and we shall let $V$ be arbitrarily large with a surface $\partial V$, shown in Fig. 2, consisting of a more or less planar area, which we shall call $\Sigma$, about half-way between $x_1$ and $x_2$, and a large hemisphere enclosing $x_2$. If the radius of the hemisphere is large enough then the contribution of the hemisphere to the integral is negligible. To see physically why this is so note that for high frequencies the integrand of (1) is proportional to $\exp [i\omega (T_1 + T_2)]$ where $T_1$ is the travel time from $x_1$ to $x$ and $T_2$ is the travel time from $x$ to $x_2$. Thus energy from points on the hemisphere arrives much later than energy from points on the planar area. We will assume that the radius of the hemisphere is infinite so that only the planar area contributes to the integral. Note that the contribution of the hemisphere cannot be neglected by an appeal to geometrical spreading; a large hemisphere will be farther away from $x_2$ than a small hemisphere but both hemispheres subtend nearly the same solid angle at $x_2$.

**Figure 2.** Here the volume $V$ consists of a planar area, $\Sigma$, about half-way between $x_1$ and $x_2$ and a hemisphere containing $x_2$. We assume that as the hemisphere becomes infinite its contribution to the integral (1) becomes negligible.
We first consider the propagation of scalar waves in two dimensions. This problem is a useful one because the theory for two dimensions is simpler than for three, yet the solution can be easily adjusted to give an approximate solution for those 3-D problems where velocity varies only with $x$ and $z$ but is independent of $y$. Later we shall consider the propagation of $P$-waves. Shear waves and a more general theory for three dimensions will be treated elsewhere. The equation to be solved is \( \nabla^2 \psi + \omega^2/c^2 = F_1 \delta(x - x_1) \) in which $F_1$ is the source strength and $\nabla^2 \psi = \partial_x^2 + \partial_z^2$. If $\omega$ is large and $c$ is constant, then the solution is $\psi = F_1 \exp \left[ i\omega r/c - 3\pi/4 \right] / \sqrt{8\pi \omega r/c}$ in which $r = |x - x_1|$. This result is generalized to the case of variable $c$ by making the geometrical optics assumption (Debye 1904) that the energy flux of the wave is conserved along ray tubes like the one shown in Fig. 3. For acoustic and elastic waves the energy density $E = \rho |u|^2$ in which $\rho$ is density and $u$ is velocity. For convenience $\psi$ is chosen to be that combination of physical quantities for which $E = |\psi|^2/c^2$. Then the energy flux is $cE \delta l = |\psi|^2 \delta l/c$ where $\delta l$ is the width of the ray tube. Conservation of energy flux leads to the result

$$\psi(x) = \frac{F_1 \sqrt{c} \exp \left( i\omega T - 3\pi/4 \right)}{\sqrt{8\pi c \delta l/\delta \theta}},$$

in which $T$ is the travel time along the ray from $x_1$ to $x$ and $\delta \theta$ is the angle subtended by the ray tube at the source. If the ray data are collected along a line $\Sigma$ which passes through $x$, then $\delta l/\delta \theta = -\hat{n} \cdot \hat{t}(d\sigma/\delta \theta)$ where $\hat{n}$ is the unit normal to $\Sigma$, $\hat{t}$ is the unit tangent to the ray and $\sigma$ is the distance along $\Sigma$.

When $x$ is located at a focus or a caustic, the ray tube pinches out and $\delta l/\delta \theta = 0$. Equation (2) then predicts an infinite amplitude whereas it can be shown by more exact methods that for any finite $\omega$ the amplitude is also finite. This breakdown of geometrical optics is well known and has been the motivation for the development of alternative techniques such as the rapid WKB methods of Richards (1973) and Chapman (1978). WKB theory has only recently been extended to inhomogeneous media (Maslov 1965; Frazer & Phinney 1980; Sinton & Frazer 1982; Chapman & Drummond 1982) and so geometrical optics is still
commonly used to interpret seismic wavefields for experiments where velocity is known to vary laterally (e.g. Giese, Prodehl & Stein 1976; McMechan & Mooney 1980). The geometrical optics theory has reached a very advanced state of development, both practically and theoretically (e.g. Červený, Molotkov & Pšeničk 1977).

3 The Kirchhoff–GRT solution

In this section we introduce a method which combines the Kirchhoff–Helmholtz integral with geometrical optics to circumvent the failure of the latter at caustics. The essence of the method is that, instead of tracing rays from the source to the receiver as Fig. 3, rays are traced from both source and receiver to a common surface \( \Sigma \) between them as shown in Fig. 4. Geometrical optics are used to evaluate the source field \( \psi_1 \) and the receiver field \( \psi_2 \) along \( \Sigma \) and then \( \psi_1(x_2) \) is calculated by use of equation (1) with \( \partial V \) replaced by \( \Sigma \). Either, or both, of \( \psi_1 \) and \( \psi_2 \) may be singular on \( \Sigma \) but as we know these singularities are non-physical we are justified in assuming that each singularity is integrable. The surface \( \Sigma \) can always be chosen so that no singularity of on \( C \) coincides with any singularity of \( \psi_2 \) on \( \Sigma \). Thus any singularities in the integrand of (1) are integrable.

![Figure 4. The Kirchhoff–GRT method. Instead of tracing rays from \( x_1 \) to \( x_2 \) we trace rays from both points up to \( \Sigma \). Geometrical ray theory is used to evaluate both \( \psi_1 \) and \( \psi_2 \) on \( \Sigma \) then equation (1) is used to calculate \( \psi_1(x_2) \).](https://academic.oup.com/gej/article-abstract/78/2/413/576718/figure-4)
with derivative $\partial_n \psi_1 = \psi_1(\hat{n} \cdot \hat{t}_1)i\omega/c$, and the wavefield on $\Sigma$ due to the receiver is

$$\psi_2(x) = \frac{\sqrt{c} \exp \left( i \omega T_2 - 3\pi i/4 \right)}{8\pi \omega (\hat{n} \cdot \hat{t}_2) d\sigma_2 / d\theta_2}^{1/2}$$  \hspace{1cm} (4)$$

with derivative $\partial_n \psi_2 = \psi_2(\hat{n} \cdot \hat{t}_2)i\omega/c$. These derivatives were obtained by writing $\psi_1$ and $\psi_2$ in the form $\psi = A e^{i\omega T}$ and assuming $|\partial_n A/(A\partial_n T)| \ll \omega$. If neither $\psi_1$ nor $\psi_2$ has a caustic on $\Sigma$ then we may substitute directly into equation (1) to obtain

$$\psi_1(x_2) = \frac{-F}{8\pi} \int_\Sigma \exp \left[ i \omega (T_1 + T_2) \right] \frac{(\hat{n} \cdot (\hat{t}_2 - \hat{t}_1))^{1/2}}{((\hat{n} \cdot \hat{t}_1)(\hat{n} \cdot \hat{t}_2))^{1/2}} \left( \frac{d\theta_1}{d\sigma_1} \frac{d\theta_2}{d\sigma_2} \right)^{1/2} d\sigma_1.$$  \hspace{1cm} (5)$$

The variable of integration in this integral can be changed by noting that the last part of the integrand can be written in the three equivalent ways:

$$\left( \frac{d\theta_1}{d\sigma_1} \frac{d\theta_2}{d\sigma_2} \right)^{1/2} = \left( \frac{d\sigma_1 / d\theta_1}{d\sigma_2 / d\theta_2} \right)^{1/2} d\theta_1 = \left( \frac{d\sigma_2 / d\theta_2}{d\sigma_1 / d\theta_1} \right)^{1/2} d\theta_2. \hspace{1cm} (6)$$

Usually $\theta_1$ or $\theta_2$ is a more convenient variable of integration than $\sigma$. The only assumption made in obtaining the integral (5) is that both $\psi_1$ and $\psi_2$ are regular on $\Sigma$. However, $\psi_1$ is permitted to have a caustic or focus at $x_2$ and $\psi_2$ is permitted to have a caustic or focus at $x_1$. This last feature is what makes Kirchhoff-Helmholtz theory more powerful than geometrical optics.

We now consider how this integral can be adapted for use when $\psi_1$ and $\psi_2$ are allowed to have caustics, cusps and shadows on $\Sigma$. To simplify the discussion we shall assume that each $\psi_1$-ray encounters no more than one caustic on its way from $x_1$ to $\Sigma$ and that each $\psi_2$-ray encounters no more than one caustic on its way from $x_2$ to $\Sigma$. Note that this assumption applies only to the behaviour of individual rays; it does not limit the number of caustics which $\psi_1$ and $\psi_2$ may contain on or off of $\Sigma$. Although it is likely to be satisfied for most of the velocity models of interest in seismology this assumption is only a convenience and can be dispensed with if one is willing to count the number of caustics which each ray encounters on its way from the source or the receiver to $\Sigma$.

Fig. 5(a) illustrates schematically four of the features which the function $\sigma(\theta_1)$ may have when $\psi_1$ is irregular on $\Sigma$. The corresponding behaviour of the derivative $d\sigma_1 / d\theta_1$ is shown in Fig. 5(b). Three of these features – caustics, cusps and shadows – are familiar: at a caustic $d\sigma_1 / d\theta_1$ vanishes; at a cusp $d\sigma_1 / d\theta_1$ has a jump discontinuity but $\sigma_1$ is continuous; at a shadow both $\sigma_1$ and $d\sigma_1 / d\theta_1$ have a jump. The fourth feature, a grazing point, is not a singularity of the $\psi_1$ rayfield but an artefact of our choice for $\Sigma$. Fig. 6 illustrates how at a grazing point the $\psi_1$ rayfield is tangent to $\Sigma$. The integrand of (5) is regular at such a point because the ray tube does not 'pinch out' there and thus the geometrical optics amplitude of $\psi_1(x)$ must be finite. In the integrand of equation (5), as $\sigma$ approaches a grazing point $-\hat{n} \cdot \hat{t}_1$ approaches zero and $d\sigma_1 / d\theta_1$ approaches infinity but the product of these two terms remains finite. The exception to this rule occurs when a caustic is tangent to $\Sigma$. In that case a new $\Sigma$ should be chosen to intersect the caustic at an angle.

It is worth emphasizing that although the behaviour of $\sigma_1(\theta_1)$ and $\sigma_2(\theta_2)$ depends very strongly on the choice made for $\Sigma$ the final result does not. Thus, with two exceptions, the choice made for $\Sigma$ is largely a matter of computational convenience. The first exception is the case of a wavefield containing a focus as shown in Fig. 7. If $\psi_1$ has a focus on $\Sigma$, at $\theta_1(\theta_1^*)$ say, then $d\sigma_1 / d\theta_1$ vanishes at $\theta_1^*$ like $|\theta_1 - \theta_1^*|^{\alpha}$ where $\alpha > 2$ and so the integral (5) cannot be computed. In this case $\Sigma$ should be changed so as not to include the focus.
Figure 5. (a) Possible features of the function $\sigma_1(\theta_1)$. For a ray which leaves the source at angle $\theta_1$ to the vertical, $\sigma_1(\theta_1)$ is the distance along $\Sigma$ from a fixed origin to the point where the ray intersects $\Sigma$. (b) The corresponding behaviour of $d\sigma_1/d\theta_1$.

In practice, as foci are so localized in space, it is highly improbable that $\Sigma$ will ever contain one. The second exception is the case where both the $\psi_1$ rayfield and the $\psi_2$ rayfield have caustics at the same point on $\Sigma$. Then the product of the two geometrical optics amplitudes may have a non-integrable singularity on $\Sigma$. This problem can likewise be avoided by changing $\Sigma$, moving it a little nearer or farther from the source. In fact, like the last problem, this second problem almost never occurs. The one situation in which it could occur is that of a stratified velocity model with the source and receiver at the same depth. Then if $\Sigma$
is chosen to be the perpendicular bisector of the straight line between the source and receiver the caustics of $\psi_1$ and $\psi_2$ on $\Sigma$ will coincide; however, the slightest alteration of $\Sigma$ will destroy this coincidence.

To accommodate the different branches of the rayfields of $\psi_1$ and $\psi_2$ on $\Sigma$ we first decompose the range of the function $\sigma_j(\theta_j)$ into subintervals $\Sigma_{1j}$ on to which $\sigma_j(\theta_j)$ is strictly monotone. Many of these subintervals will overlap as indicated by Fig. 5. Then, letting $\chi_{1j}$ denote the characteristic function of $\Sigma_{1j}$, i.e. $\chi_{1j}(\sigma) = 1$ if $\sigma \in \Sigma_{1j}$ and zero otherwise, we may write

$$\psi_1 = \sum_j \psi_{1j} \chi_{1j}$$

where, for each $j$, $\psi_{1j}$ is given by a formula like (3). Similarly, a decomposition of $\Sigma$ into subintervals $\Sigma_{2k}$ on which $\sigma_2(\theta_2)$ is strictly monotone allows us to write

$$\psi_2 = \sum_k \psi_{2k} \chi_{2k}$$
where $x_{2k}$ is the characteristic function of $\Sigma_{2k}$ and $\psi_{2k}$ is given by a formula like (3). Substitution of these expressions for $\psi_1$ and $\psi_2$ into the Kirchhoff–Helmholtz formula (1) yields a sum of integrals like (5):

$$\psi_1(x_2) = \frac{-F_1}{8\pi} \sum_{jk} \int_{\Sigma_{ij} \cap \Sigma_{2k}} \exp \left[ i \omega (T_{ij} + T_{2k}) \hat{n} \cdot (\hat{t}_{2k} - \hat{t}_{ij}) \right] \left( \frac{d\theta_1}{d\sigma_{ij}} \frac{d\theta_2}{d\sigma_{2k}} \right)^{1/2} d\sigma. \tag{7}$$

In this expression, in order to obtain the correct phase shifts at caustics it is necessary to take $\sqrt{d\sigma_1/d\theta_1} = i |d\sigma_1/d\theta_1|^{1/2}$ wherever $d\sigma_1/d\theta_1 < 0$ and to take $\sqrt{(-\hat{n} \cdot \hat{t}_1)} = -i |\hat{n} \cdot \hat{t}_1|^{1/2}$ whenever $\hat{n} \cdot \hat{t}_1 > 0$. Similarly $\sqrt{d\sigma_2/d\theta_2} = i |d\sigma_2/d\theta_2|^{1/2}$ whenever $d\sigma_2/d\theta_2 < 0$ and $\sqrt{\hat{n} \cdot \hat{t}_2} = -i |\hat{n} \cdot \hat{t}_2|^{1/2}$ whenever $\hat{n} \cdot \hat{t}_2 < 0$. These choices are appropriate for a time dependence $\exp(-i\omega t)$. For a time dependence $\exp(i\omega t)$ the sign of $i$ in these relations should be reversed. As the caustics of $\psi_1$ and $\psi_2$ do not intersect on $\Sigma$, at most one of $d\sigma_1/d\theta_1$ or $d\sigma_2/d\theta_2$ may vanish at the endpoint of an interval. If $d\sigma_1/d\theta_1$ vanishes at an endpoint then we use the second expression in equation (6) and make $\theta_1$ the variable of integration; if $d\sigma_2/d\theta_2$ vanishes we use the third expression in equation (6) and make $\theta_2$ the variable of integration. If $\omega$ is large then on many of these intervals the phase factor $\exp(i\omega (T_{ij} + T_{2k}))$ will be rapidly oscillating and so the quadrature of (7) should be carried out by means of the generalized Filon method (Frazer 1978; Frazer & Gettrusc 1984). With this method each quadrature step may contain many cycles of the phase factor; the step size becomes essentially independent of $\omega$ and is instead limited by the variation of the non-oscillating part of the integrand.

In the sequel integral (7) will be referred to as the EKH (extended Kirchhoff–Helmholtz) integral. This nomenclature is not meant to exclude time domain versions of equation (7). A time domain development of the theory given here would parallel the treatment of Haddon & Buchen (1981) and would probably yield formulae which are faster for computer calculations. However, the same rays must be traced in either case and as the time required for ray tracing is greater than the time required to evaluate (7), even in the frequency domain, this difference in speed is not likely to be significant. On the other hand one of the advantages of a frequency domain development is that it permits the use of frequency-dependent velocities and thus the incorporation of a causal, spatially varying $Q$. The introduction of a spatially varying $Q$ into a time domain asymptotic method can lead to a number of inconsistencies (Frazer 1983).

Like 2-D WKBJ integrals, equation (7) can be adapted to give approximate synthetic seismograms for point sources and receivers provided that the velocity in that medium is independent of some Cartesian coordinate, $y$, say, and the sources and receivers lie within a plane normal to the $y$-axis. The approximation is obtained by multiplying the right side of (7) by a factor $\exp(-in/4)\sqrt{2\pi c_1 r/\omega}$ where $r$ is the distance between source and receiver. This last expression is the ratio of $\exp(-i\omega r/c_1)$ to $(-i/4)H_0^{(1)}(\omega r/c_1)$ when $\omega r/c_1 \gg 1$. It should be pointed out that our procedure for dealing with singularities of $\psi_1$ or $\psi_2$ on $\Sigma$ is theoretically incorrect because we have used geometrical optics expressions at points on $\Sigma$ where they are singular and therefore invalid. A more correct procedure, used by Haddon & Buchen (1981) for a stratified velocity model, is to use more than one surface $\Sigma$. Rays are traced from the source at $x_1$ to each point of $\Sigma_1$, then from each point of $\Sigma_1$ to each point of $\Sigma_2$, ..., then from each point of $\Sigma_{N-1}$ to each point of $\Sigma_N$ then from the receiver at $x_2$ to each point of $\Sigma_N$. The integration is taken over the product manifold

$$\prod_{l=1}^{N} \Sigma_l.$$
If a sufficient number of surfaces (usually two) are used then no rayfield due to a point on $\Sigma_i$ will be singular on $\Sigma_{i+1}$ and so the geometrical optics expressions in the integrand of the final multifold integral will be valid. With this method a great many rays must be traced; in an unstratified earth this ray tracing cannot be carried out analytically and so it is likely to require a considerable amount of computer time.

For application to acoustic waves the factor $F_1$ in equation (7) should be replaced by $[\omega f_0(\omega)\sqrt{\rho_2/\rho_1}]/c_i^2$. Then equation (7) gives the pressure signal at $x_2$ due to an explosion with equivalent body force (Burridge & Knopoff 1964) $f = -f_0(\omega) \nabla \delta(x - x_1)$.

4 A numerical test

Although the derivation of integral (5) is quite rigorous the derivation of the EKH integral (7) was not. The assumption made in obtaining (7) was that an integrand with large finite values can be approximated by one with integrable singularities. Hence it is important to test the performance of the EKH method by comparing it with another method in a problem where the second method is valid. For the EKH method the stratification or lack of stratification of the velocity model is of no particular significance. Therefore a good test of the method can be obtained by comparing it with the WKBJ (phase integral) method (e.g. Richards 1971, 1973) for a model with a smooth stratified velocity function. The accuracy of the WKBJ method for some models has been verified (Choy et al. 1980) by comparison with the reflectivity method (Fuchs & Muller 1971). A comparison of the EKH and WKBJ

![Figure 8. The stratified velocity model used to compute the synthetic seismograms of Fig. 9. For this type of model the WKBJ (phase-integral) method works well. Since the EKH (extended Kirchhoff-Helmholtz) method does not care whether a velocity model is stratified or not, the performance of the method for such a model should be a good indication of its accuracy in general.](https://academic.oup.com/gji/article-abstract/78/2/413/576718)
methods is shown in Fig. 9 for the velocity profile of Fig. 8. The differences between the two record sections in Fig. 9 appear to be as follows:

(1) The first arrivals in the WKBJ synthetics show an apparent phase change from about 18 to 22 km. This phase change is actually due to precursory truncation phase associated with the failure of the non-uniform WKBJ method for the ray which has a turning point at the source. If this truncation phase is removed by use of a smoother window in the WKBJ $p$-integral then the progressive branch in the WKBJ synthetics is completely suppressed up to 30 km.

(2) The EKH synthetics show two spurious phases (third and fourth arrivals past 32 km). These phases are caused by the singularities in the $\psi_1$ and $\psi_2$ rayfields. They are an inherent defect of the EKH method (equation 7) and not a result of any error in our computer implementation of the method.

(3) The EKH second arrivals to the left of the caustic ($t = 1.2\text{ s}$ at 18 and 20 km) are larger than the WKBJ arrivals there. Neither the $\psi_1$- nor the $\psi_2$-rayfields contain caustics on the $\Sigma$ used for this distance so, in theory, both methods are equally valid there. We do not know which one is more correct.

(4) Past the cusp, at about 34 km, the EKH synthetics decay less rapidly in amplitude than the WKBJ synthetics. They are also more broad band and are not Hilbert transformed. The WKBJ theory is correct in such regions (e.g. Choy et al. 1980) and so these differences are a defect in the EKH theory. It may be possible to increase the decay off the cusp by treating the cusp as a caustic in the numerical implementation, however, we have found that both treatments give similar results. In a way this is fortunate as it means that cusps (which are often difficult to distinguish from caustics in numerical work) may be treated as though they were caustics, without ill effect.

On balance, the differences between WKBJ synthetics and the EKH synthetics are slight enough to be encouraging.
5 An example

We now apply the EKH method to a wavefield which cannot be modelled using the WKBJ theory for a stratified medium. The velocity function consists of a linear increase with depth containing a smooth circular low-velocity zone at a depth of 4 km. Ray paths from the source, up to a horizontal line containing the receivers, are shown in Fig. 10. Caustic surfaces, caused by the low-velocity zone, intersect the line of receivers at about 50 and 72 km and these surfaces intersect in a focus at a depth of about 3 km.

For this velocity structure it is always possible to choose the surface of integration $\Sigma$ so that neither $\psi_1$ nor $\psi_2$ has a caustic on $\Sigma$. If the source and receiver are both to the left of the low-velocity zone (LVZ) then a vertical $\Sigma$ half-way between $x_1$ and $x_2$ will intersect no caustics; if the LVZ is between the source and the receiver then a vertical $\Sigma$ through the centre of the LVZ will intersect no caustics. For these choices of $\Sigma$ the integrand of equation (5) is regular and the geometrical formulae used to obtain (5) are valid everywhere on $\Sigma$. Thus the synthetics computed using these choices of $\Sigma$, shown in Fig. 11(b) and enlarged in Fig. 12(a), may be regarded as correct.

In Section 3 we suggested that it is not always necessary to choose $\Sigma$ so that it intersects no caustics and that if it does contain caustics then one can continue to use the geometrical optics formulae for $\psi_1$ and $\psi_2$ and integrate over their singularities. This was the procedure used to compute the synthetics shown in Fig. 11(c) and enlarged in Fig. 12(b). For these synthetics, $\Sigma$ was chosen to be a vertical line approximately half-way between source and receiver. For all receivers at distances greater than about 45 km $\Sigma$ contained two caustics. The $\psi_1$ and $\psi_2$ rayfields for receivers at 50 and 62 km are shown in Fig. 13.

Inspection of Figs 11 and 12 reveals a number of features that must be regarded as errors consequent on our use of a $\Sigma$ containing caustics. The spurious phases which arrive after the correct arrivals in Fig. 11(c) and Fig. 12(b) occur because of the singularities in the integrand. To see why, note that in the normal case of a non-singular integrand a large amplitude portion of the integrand makes a contribution to the integral only if the phase of the

![Figure 10. Ray paths for a velocity model consisting of a linear increase with depth with a smooth, circular, low-velocity zone centred at the point $x = 40$ km, $z = 4$ km. The formula for velocity is $C = 3.0 + 0.02z - 0.4 \exp[-d^2/1.5] \text{ km s}^{-1}$ where $d^2 = [(x - 40)^2 + (z - 4)^2] \text{ km}^2$.](https://academic.oup.com/gji/article-abstract/78/2/413/576718)
Figure 11. Computations for the model of Fig. 10. (a) A travel-time curve. (b) Kirchhoff–Helmholtz synthetics. For these synthetics the surface of integration $\Sigma$, shown schematically in Fig. 4, was adjusted as often as necessary so that it intersected no caustics. (b) Extended Kirchhoff–Helmholtz synthetics. For these synthetics $\Sigma$ was half-way between the source and each receiver and often intersected a caustic.

Figure 12. An enlargement of the portions of Fig. 11(b, c) near the triplication.
integrand does not vary rapidly near the region of large amplitude. However, the large amplitude associated with a singularity of the integrand is so local that its contribution to the integral is relatively unaffected by rapid variations in phase. Most of the other differences between the EKH synthetics (Figs 11c and 12b) and the correct KH synthetics (Figs 11b and 12a) are due to the interference of the spurious ‘singularity phases’ with correct arrivals. The differences in pulse shape of the second arrivals past about 68 km are an instance of this. The KH pulses show a phase shift of about $-\pi/4$ due to the confluence of the progressive branch arrival, with phase shift zero, and the back branch arrival, with phase shift $-\pi/2$. The EKH synthetics in the same region show a phase shift which, although negative, is much closer to zero.

6 P-waves

The theory given in Section 3 was for scalar waves but this theory can be used to compute P-wave motion through the use of displacement or velocity potentials. It will be most convenient to use a velocity potential. To avoid confusion we use $a(x)$ to denote the speed at which a P-wave disturbance passes through the point $x$ and $u(x, \omega)$ to denote the velocity field in the disturbed medium. A rigorous development of the theory of potentials in inhomogeneous media has been given by Ansell (1979). Here it will be sufficient to recall that in a smooth inhomogeneous medium the velocity can be written as

$$u = \rho^{-1/2} \nabla \psi + O(|\psi|/\omega).$$

Then the velocity potential satisfies $(\nabla^2 + \omega^2/\alpha^2) = 0(|\psi|/\omega)$ and the energy density associated with the disturbance is $|\psi|^2/\alpha^2$.

We now convert equations (5) and (7) so that they give the velocity $u(x_2, \omega)$ of the motion at the receiver. Let $\hat{r}_1(\theta_1)$ be the unit tangent at the source to the ray which leaves the source at angle $\theta_1$ to the vertical as shown in Fig. 4 and $\hat{r}_2(\theta_2)$ the unit tangent at the receiver to the ray which leaves the receiver at angle $\theta_2$ to the vertical. Then for a line explosion with equivalent body force density (Burridge & Knopoff 1964) $f = -f_0(\omega)$ $\nabla \delta(x - x_1)$ the factor $-F_1$ in equations (5) and (7) should be replaced by

$$\omega^2 f_0(\omega)/(\sqrt{\rho_1 \rho_2 \alpha_1^2 \alpha_2})$$
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and a factor $\hat{r}_2(\theta_2)$ should be inserted into the integrand. For a source which is a line force in the direction of the unit vector $\hat{m}$ the equivalent body force density is $f = f_0(\omega)\hat{m} \delta(x - x_1)$; the factor $-F_1$ should be replaced by

$$i\omega f_0(\omega)/\sqrt{\rho_1 \rho_2 \alpha_1 \alpha_2}$$

and a factor $\hat{m} \cdot \hat{r}_1(\theta_1) \hat{r}_2(\theta_2)$ should be inserted into the integrand. For a source which is a line dislocation, the equivalent body force density is $f = -M(\omega) \cdot \nabla \delta(\mathbf{r})$ where $M$ is a symmetric second-order tensor. If $M$ is tangent to the vertical plane containing source and receiver then equations (5) and (7) can be used and in this case $-F_1$ should be replaced by $\omega^2/(\sqrt{\rho_1 \rho_2 \alpha_1^2 \alpha_2})$ and a factor $\hat{r}_1(\theta_1) \cdot M(\omega) \cdot \hat{r}_1(\theta_1) \hat{r}_2(\theta_2)$ inserted into the integrand. As noted above, to adapt equations (5) and (7) for approximate use with a point source the right sides of both should be multiplied by $\exp\{-im/4\}$ where $r$ is the distance between source and receiver.

7 Discussion

The EKH (extended Kirchhoff–Helmholtz) method shows some promise as a means of computing wavefields for media in which velocity varies laterally as well as with depth and the wavelengths are very small compared to the distance between source and receiver. The EKH method requires more ray tracing than the EWKBJ method (Maslov 1965; Frazer & Phinney 1980; Chapman & Drummond 1982) but less ray tracing than a theoretically correct Kirchhoff–Helmholtz treatment (Haddon & Buchen 1981). Possible advantages of the EKH method lie in the simplicity of its underlying theory, in which only familiar ray theoretical quantities and relations appear, and the fact that erroneous arrivals (caused by approximations made in the theory) are not precursory. A minor advantage of the EKH method is the likelihood of having to count caustics is reduced. If no ray encounters more than one caustic and no $+2$ ray encounters more than one caustic then the rules given in Section 3 ensure that each arrival will have the correct phase.

We note that the modelling problem treated in this paper can also be addressed by the method of Gaussian beams. For a complete list of references to this method see the paper by Červený (1983). Recent applications of Gaussian beams to seismology by Červený and others have yielded impressive results. We make no attempt to compare our method with Gaussian beams except to note that the latter involves, like the Maslov/EWKBJ method, an integration over the focal sphere of the source whereas our method, like all Kirchhoff–Helmholtz treatment (Haddon & Buchen 1981). Perhaps more fundamental is the difference in ansatz. In order to make a Kirchhoff method efficient one is obliged to use geometrical optics, which have a ray tube ansatz. Gaussian beams, as the name implies, have a beam ansatz. One can imagine Kirchhoff methods with a beam ansatz (integrating over $\Sigma$ the product of each beam from the source with each beam from the receiver) but such methods are probably useful only for the synthesis of reflections from physical discontinuities (e.g. Frazer & Sen 1984).

An important point, which we have not discussed above, is the obvious ease with which the EKH method can be extended to media containing zeroth-order velocity discontinuities. Exploration geophysicists (Hilterman 1970, 1975, 1982; Trorey 1970, 1977) have long used Kirchhoff–Helmholtz theory to model reflection data by choosing the surface of integration to coincide with the physical discontinuity. However, with a few exceptions (Hilterman & Larsen 1975; Berryhill 1979) they have not considered media with variable velocity nor media for which either the source or receiver wavefields were singular on that surface. One of the main points of this paper is that such singularities can be handled without great sacri-
face of either accuracy or economy. This makes feasible the synthesis of wide angle reflection data for media containing many non-planar velocity discontinuities. Such synthetics could be used to model data for unstratified media in much the same way that reflectivity method calculations (Fuchs & Muller 1971) are now used to model data for stratified media.

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References

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