Chiral Anomalies of Anti-Symmetric Tensor Gauge Fields in Higher Dimensions*

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The chiral $U(1)$ anomalies and the covariant gravitational anomalies of the anti-symmetric tensor gauge fields in arbitrary even dimensions are derived on the basis of Fujikawa’s path integral method. Schwinger’s proper time method is applied to evaluate the anomalous path integral Jacobians.

§ 1. Introduction

It is known that the gravitational anomalies\(^1\) in $4N-2$ dimensions are closely related to the gravitational chiral $U(1)$ anomalies\(^2\) in $4N$ dimensions.\(^1,3\)–\(^8\) Covariant form of the anomaly (covariant anomaly)\(^4\) for general coordinate transformations in $4N-2$ dimensions can be written as

$$D_v T^\mu\nu = \pm \frac{1}{2} 4\pi D_v \left\{ \frac{\partial}{\partial R_{\mu\nu}} \mathcal{A}(R) \right\},$$

where $T^\mu\nu$ is a symmetric energy-momentum tensor,\(^9\) the sign $\pm$ corresponds to the chirality of the matter fields, and $\mathcal{A}(R)$, a local polynomial of curvature 2-form $R_{\mu\nu} = (1/2)R_{\mu\nu\rho\sigma} dx^\rho / dx^\sigma$, is the chiral $U(1)$ anomaly in $4N$ dimensions. A similar relation holds for the consistent anomalies.\(^3\)–\(^6\)

The relation (1·1) was originally derived by Alvarez-Gaumé and Witten\(^9\) by two methods, Feynman diagram technique and Fujikawa’s path integral method.\(^10\) Path integral approaches to (1·1) were also given by Fujikawa\(^7\) and Matsuki.\(^8\) More mathematical approach was given by Bardeen and Zumino.\(^4\)

In the path integral approach, we may take two steps to verify (1·1). First, we derive a general expression for the chiral $U(1)$ anomalies in arbitrary dimensions. For this purpose, we should develop a method to calculate the path integral Jacobians for the chiral transformation; as is known, the anomalies are ascribable to the non-invariance of the path integral measure.\(^10\) Next, we generalize the method to the evaluation of the Jacobians for general coordinate transformation. Then we compare the results with the $U(1)$ anomalies.\(^3\)

In the first step, there are two simple methods to evaluate the anomalous Jacobians for the chiral $U(1)$ transformation. One is based on the 1+0 dimensional supersymmetric non-linear sigma model, and was applied to spin 1/2, 3/2 and antisymmetric tensor fields in Ref. 1). The other is Schwinger’s proper time method, and

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*** Fujikawa derived (1·1) in a slightly different way.\(^7\)
was used for spin 1/2 and 3/2 by the present authors.\[11\]

The above two methods have been generalized to the gravitational anomalies. Using the method of supersymmetric $\sigma$-model, Alvarez-Gaumé and Witten evaluated the Jacobians of gravitational anomalies and derived (1·1) (up to an overall factor 1/2) for spin 1/2, 3/2 and anti-symmetric tensor fields.\[1\] Matsuki showed that the proper time method is also applicable to the gravitational anomalies, and derived (1·1) for spin 1/2 and 3/2 fields by this method.\[8\]

The purpose of this paper is to apply the proper time method to both the chiral $U(1)$ and the gravitational anomalies of the anti-symmetric tensor gauge fields. First, we calculate the chiral $U(1)$ anomaly in § 2. The calculation is analogous to that used for spin 3/2 fields.\[11\] Next, imitating Matsuki's arguments,\[8\] we evaluate the gravitational anomalies of self-dual anti-symmetric tensor gauge fields in § 3. This presents a proof of (1·1) for these fields.

§ 2. Chiral $U(1)$ anomaly

We shall first investigate the meaning of the chiral $U(1)$ anomaly of the anti-symmetric tensor gauge fields. These anomalies can exist in $4N$ dimensional curved space-time for the fields of rank $2N-1$ (for example, the electromagnetic gauge field in four dimensions). Since the chiral transformation cannot be defined for the gauge potential (see (2·6)), we should deal with the field strength as an independent variable in the path integral. Thus, we consider the path integral in the first order formalism, where both gauge potential $A_{\mu_1\cdots\mu_{2N-1}}$ and field strength $F_{\mu_1\cdots\mu_{2N}}$ are treated as independent variables.

The action in the first order formalism is given by

$$S = \int d^4x \sqrt{g} \left( \frac{1}{2} F \cdot F - F \cdot dA \right), \quad (2·1)$$

where $g = |\det g_{\mu\nu}|$ with $g_{\mu\nu}$ being the background metric and $dA$ stands for the curl (exterior derivative) of $A$:

$$(dA)_{\mu_1\cdots\mu_{2N}} = 2N \partial_{[\mu_1} A_{\mu_2\cdots\mu_{2N}]} \quad (2·2)$$

The dot $\cdot$ denotes the contraction over tensor indices, e.g.,

$$F \cdot dA = \frac{1}{(2N)!} g^{\mu_1\nu_1} \cdots g^{\mu_{2N}\nu_{2N}} F_{\mu_1\cdots\mu_{2N}} (dA)_{\nu_1\cdots\nu_{2N}}. \quad (2·3)$$

In the classical theory, the equivalence of the first order formalism to the second order one is guaranteed by the field equation $\delta S/\delta F = 0$,

$$F - dA = 0. \quad (2·4)$$

But in the quantum theory, the equivalence would be expressed as

$$\langle (F - dA) O \rangle = \int [\mathcal{D} F] [\mathcal{D} A] (F - dA) O e^{iS} = 0, \quad (2·5)$$

where $O$ is an arbitrary function of $A$ and $F$. In the following, however, we shall see
that (2·5) does not hold in general.

Let us consider the chiral $U(1)$ transformation

$$F \rightarrow F' = F \cos \theta + *F \sin \theta,$$

$$A \rightarrow A' = A,$$  \hspace{1cm} (2·6)

where $*F$ is a dual tensor of $F$:

$$(*F)_{\mu_1 \cdots \mu_{2N}} = \frac{\sqrt{g}}{(2N)!} \varepsilon_{\nu_1 \cdots \nu_{2N}} F_{\mu_1 \cdots \mu_{2N}} F^{\nu_1 \cdots \nu_{2N}}$$  \hspace{1cm} (2·7)

with $\varepsilon_{\mu_1 \cdots \mu_{2N}}$ being the totally anti-symmetric tensor density ($\varepsilon_{01 \cdots 4N-1} = 1$). Under the transformation (2·6), the action (2·1) is transformed as

$$S \rightarrow S + \int d^{4N} x \sqrt{g} \theta(x) *F \cdot (F - dA),$$  \hspace{1cm} (2·8)

where we took $\theta$ to be infinitesimal and $x$-dependent. The chiral anomaly arises when the path integral measure $[\mathcal{D} F]$ is not invariant under (2·6). We shall write the non-trivial Jacobian as

$$[\mathcal{D} F] \rightarrow [\mathcal{D} F'] = [\mathcal{D} F] \exp \left\{ \int d^{4N} x \sqrt{g} \theta(x) \mathcal{A}(x) \right\}.$$  \hspace{1cm} (2·9)

From (2·8) and (2·9), we have an anomalous "Ward-Takahashi identity"

$$\langle *F(x) \cdot (F(x) - dA(x)) \rangle = i \mathcal{A}(x).$$  \hspace{1cm} (2·10)

This shows that (2·5) does not hold in general.$^*$ Thus we may think that the chiral $U(1)$ anomaly indicates the breakdown of the equivalence between the first and the second order formalism.

We can interpret the above inequivalence from a viewpoint of the zero-modes.$^{12}$ In Euclidean version of the path integral, the measure $[\mathcal{D} F]$ is defined through a mode expansion of $F$. Usually the mode functions are assumed to be eigenfunctions of the Laplacian for the $2N$-forms. Thus, the (off-shell) field strength $F$ in the first order formalism includes the modes of the harmonic $2N$-forms (zero-modes). On the contrary, in the second order formalism the field strength $dA$ cannot have the modes of the harmonic $2N$-forms; a harmonic $2N$-form cannot be expressed as an exterior derivative of any $(2N-1)$-form by definition. From these facts, the inequivalence $F - dA \neq 0$ may be attributable to the zero-modes mismatch between $F$ and $dA$. In fact, we can show that the anomaly (2·10) is nothing but the density of the topological signature$^{12}$ (the number of self-dual harmonic $2N$-forms minus the number of anti-self-dual ones).

However, as expected by a formal path integration over $F$, this inequivalence (and that in the footnote on this page) cannot affect the physical quantities. In fact, $^*$ We can get a more general equation which indicates the inequivalence. For example, in four dimensions, the $\zeta$ function method leads to

$$\langle F(x)_{\mu \nu} (F_{\alpha \beta}(x) - dA_{\alpha \beta}(x)) \rangle \propto R_{\mu \nu \alpha \beta}(x) R_{\alpha \beta}^{\gamma \delta}(x) + \cdots,$$

where the first term becomes $U(1)$ anomaly when $\varepsilon$-tensor is contracted and the dots denote the terms that do not contribute to the chiral $U(1)$ anomaly.
an explicit evaluation leads to

\[ \langle T_{\mu \nu}(1\text{st order formalism}) - T_{\mu \nu}(2\text{nd order formalism}) \rangle = 0, \] (2.11)

which proves the equivalence of the gravitational couplings. As for other interactions, the equivalence is trivial since the gauge potential alone (not the field strength of the first order formalism) appears in the interactions.

To evaluate the Jacobian in (2.9), we shall treat a bi-spinor field \( \psi_{ab} \), the so-called Kähler-Dirac (KD) field. This makes the calculation simple as discussed in Ref. 1. The KD field \( \psi_{ab} \) (in \( 2n \) dimensional space-time) can be considered as a set of anti-symmetric tensor fields \( F_{\mu_1 \cdots \mu_p}^{(p)}(p = 0, 1, \ldots, 2n) \) through the correspondence

\[ \psi_{ab} = \sum_{p=0}^{2n} \frac{i^p}{p!} (\gamma^{a_1} \cdots \gamma^{a_p})_{ab} F_{\mu_1 \cdots \mu_p}^{(p)}, \] (2.12)

where the \( \gamma^{a} \)'s are the Dirac matrices and \( F_{\mu_1 \cdots \mu_p}^{(p)} \) is the component of the world tensor \( F_{\mu_1 \cdots \mu_p}^{(p)} \) in the local Lorentz frame \( \epsilon_{\mu}^{a} \) (Latin indices are for the local Lorentz frame). Among these tensors of various ranks, we identify especially the middle rank tensor \( F_{\mu_1 \cdots \mu_n}^{(n)} \) with the field strength \( F_{\mu_1 \cdots \mu_2n}^{\mu} \) of the tensor gauge field (when \( 2n = 4N \)). As seen below, the other rank tensors \( F_{\mu_1 \cdots \mu_{2n-p}}^{(n)}(p \neq n) \) do not contribute to the chiral \( U(1) \) anomaly. Thus we can get the anomaly of the anti-symmetric tensor gauge field by treating the real KD field.

The chiral \( U(1) \) transformation for the KD field is given by

\[ \psi \rightarrow \psi' = e^{i \gamma_{2n+1}} \psi, \] (2.13)

where \( \gamma_{2n+1} = i^{n-1} \gamma^0 \gamma^1 \cdots \gamma^{2n-1} \). For infinitesimal parameter \( \theta \), this corresponds to the following chiral rotation of the tensor fields:

\[ \delta F^{(p)} = i^n (-1)^{(p+1)(1/2)p} \theta^* F^{(2n-p)} \] (2.14)

with

\[ *F_{\mu_1 \cdots \mu_p}^{(2n-p)} = \frac{\sqrt{g}}{(2n-p)!} \epsilon_{\nu_1 \cdots \nu_{2n-p}} e_{\mu_1 \cdots \mu_p} F_{\nu_1 \cdots \nu_{2n-p}}^{(2n-p)}. \]

The transformation property of the middle rank tensor \( F^{(n)} \) agrees with (2.6) when \( n = 2N \). As for the other rank tensor fields, the \( p \)- and \( (2n-p) \)-rank \((p \neq n)\) tensor fields are transformed into each other. This can be written as

\[ \delta \begin{bmatrix} F^{(p)} \\ F^{(2n-p)} \end{bmatrix} = i^n \begin{bmatrix} 0 \\ (-1)^{(p+1)} \end{bmatrix} \theta^* \begin{bmatrix} F^{(p)} \\ F^{(2n-p)} \end{bmatrix}. \] (2.15)

This shows that the path integral Jacobian of these fields is trivial for the chiral rotation. Thus the chiral anomaly of the KD field comes only from the Jacobian of \( F^{(n)} \) fields.

Now, we shall evaluate the anomaly of KD fields associated to the chiral rotation (2.13) by using the algorithm developed in a previous paper. The path integral

\[ *F_{\mu_1 \cdots \mu_p}^{(2n-p)} = \frac{\sqrt{g}}{(2n-p)!} \epsilon_{\nu_1 \cdots \nu_{2n-p}} e_{\mu_1 \cdots \mu_p} F_{\nu_1 \cdots \nu_{2n-p}}^{(2n-p)}. \]
measure in the Euclidean version is defined as \([D \phi] = \prod k d a_k\) through a mode expansion \(\phi = \sum k d a_k \phi_k\). Here, the \(\phi_k's\) are the eigenfunctions of the field operator \(D = \gamma^\mu D_\mu\) for the KD fields, where the covariant derivative \(D_\mu\) is given by

\[
D_\mu \phi = \partial_\mu \phi + \frac{1}{2} \omega_{\mu ab} [\sigma^{ab}, \phi],
\]

in which \(\omega_{\mu ab}\) is the spinor connection and \(\sigma^{ab} = (1/4) [\gamma^a, \gamma^b]\). By an argument similar to that of Fujikawa,\(^{10}\) we have an expression for the anomaly as

\[
\mathcal{A}(x) = \frac{i}{\sqrt{g(x)}} \lim_{t \to 0} \text{Tr} \left[ (\gamma_{2n+1} 1) K(x, x'; t) \right].
\]  

Here \(K_{a'b'; a'b}(x, x'; t)\) is the heat kernel for the KD fields, and characterized by the following heat equation and boundary condition:

\[
\frac{\partial}{\partial t} K_{a'b'; a'b}(x, x'; t) = - D^a K_{a'b'; a'b}(x, x'; t),
\]

\[
K_{a'b'; a'b}(x, x'; 0) = \delta_{a'a} \delta_{b'b} \delta(x, x').
\]

If we define a product operation \(\otimes\), in a slightly unusual manner, by

\[
(X \otimes Y)\phi = X\phi Y,
\]

then the covariant derivative becomes

\[
D_\mu \phi = \partial_\mu \phi + \frac{1}{2} \omega_{\mu ab} (\sigma^{ab} \otimes 1 - 1 \otimes \sigma^{ab}) \phi,
\]

so that

\[
[D_\mu, D_\nu] = R_{\mu \nu} \otimes 1 - 1 \otimes R_{\mu \nu},
\]

\[
D^a D_\mu - R_{\mu \nu} \sigma^{\mu \nu} + \frac{1}{4} R
\]

with

\[
R_{\mu \nu} = \frac{1}{2} R_{\mu \nu ab} \sigma^{ab}.
\]

The anomaly (2·17) can be written as

\[
\mathcal{A}(x) = \frac{i}{\sqrt{g(x)}} \lim_{t \to 0} \text{Tr} \left[ (\gamma_{2n+1} \otimes 1) K(x, x'; t) \right].
\]  

In order to evaluate the anomaly (2·24), we need not have a full expression of the solution to (2·18) and (2·19). Various terms are omitted through the limit \(t \to 0\) and the trace operation. The terms which survive after the trace operation over the spinor indices have at least \(n\) factors of the matrices \(\sigma^{\mu \nu} \otimes 1\), because of the factor \(\gamma_{2n+1} \otimes 1\). From this fact combined with a dimensional analysis of the surviving terms after the limit \(t \to 0\), we can see that the surviving sector of the heat kernel must be a polynomial of \(R_{\mu \nu} \otimes 1\) and \(R_{\mu \nu} \otimes \sigma^{\mu \nu}\), and that it cannot have other terms such as the
derivatives of the curvature tensor or the scalar curvature $R$. Thus to get the anomaly, we can assume the following effective equations:

$$
\begin{align*}
[D_\mu, D_\nu] &= R_{\mu\nu} \otimes 1, \\
[D_\mu, R_{\mu\nu} \otimes 1] &= 0, \\
[R_{\mu\nu} \otimes 1, R_{\rho\sigma} \otimes 1] &= 0, \\
R &= 0.
\end{align*}
$$

(2.25)

Under the assumption (2.25), the heat equation (2.18) with the boundary condition (2.19) can be easily solved in a fashion similar to that used for the Rarita-Schwinger fields. From (2.22) and (2.25), we can write the formal solution to (2.18) and (2.19) in terms of the Schwinger-DeWitt notation as

$$
K_{\alpha\beta; \epsilon \gamma}(x, x'; t) = \langle x; \alpha, \beta \mid e^{-i(D^\alpha D^\beta - R_{\mu\nu} \otimes \sigma^{\mu\nu})} \mid x' ; \epsilon \gamma \rangle \\
= \langle x; \alpha, \beta \mid e^{-iD^\alpha D^\beta} \mid x' ; \alpha'' \beta'' \rangle [e^{iR_{\mu\nu} \otimes \sigma^{\mu\nu}}]_{\alpha'' \beta''},
$$

(2.26)

The second equality is due to the commutativity of $D_\mu$ with $R_{\mu\nu} \otimes \sigma^{\mu\nu} = (R_{\mu\nu} \otimes 1)(1 \otimes \sigma^{\mu\nu})$. Equation (2.25) also implies that we can treat the derivative $D_\mu$ appearing on the last line in (2.26) as

$$
D_\mu = \hat{D}_\mu \otimes 1 = \left( \partial_\mu + \frac{1}{2} \omega_{\mu \alpha \beta} \sigma^{\alpha \beta} \right) \otimes 1,
$$

(2.27)

where $\hat{D}_\mu$ denotes the covariant derivative for spin 1/2 fields. Consequently, we can write

$$
\langle x; \alpha, \beta \mid e^{-iD^\alpha D^\beta} \mid x' ; \alpha'' \beta'' \rangle = \delta_{\alpha\beta} \langle x; \alpha \mid e^{-iD^\alpha D^\beta} \mid x' ; \alpha'' \rangle
$$

$$
= [K_{1/2}(x, x'; t) \otimes 1]_{\alpha\beta; \alpha'' \beta''},
$$

(2.28)

where $K_{1/2}(x, x'; t) = \langle x; \alpha \mid \exp(-i\hat{D}^2) \mid x' ; \alpha' \rangle$ is the heat kernel for the spin 1/2 fields. As shown in Ref. 11), under the effective equation (2.25) it can be solved as

$$
K_{1/2}(x, x'; t) = \frac{\sqrt{g(x)}}{(4\pi t)^{3/2}} \operatorname{det} \left[ \frac{R(t)}{\sinh R(t)} \right]^{1/2}.
$$

(2.29)

Here, we have used the matrix notation $(R)^{\mu\nu} = R_{\mu\nu}$ and det represents the determinant over these matrix indices. From (2.24), (2.26) and (2.28), we have

$$
\mathcal{A}(x) = \frac{i}{\sqrt{g(x)}} \lim_{t \to 0} \operatorname{Tr}_1 \left[ \gamma_2 \gamma_{1/2}(x, x; t) \operatorname{Tr}_2 (e^{iR_{\mu\nu} \otimes \sigma^{\mu\nu}}) \right],
$$

(2.30)

where $\operatorname{Tr}_1$ and $\operatorname{Tr}_2$ denote trace operations over the first and second spinor indices, respectively. Assuming the representation of the gamma matrices in which

$$
\sigma_{2j-1,2j} = \frac{1}{2} \sigma_1 \otimes \cdots \otimes \sigma_{2j-1} \otimes \sigma_{2j} \otimes \cdots \otimes \sigma_1
$$

(2.31)

with $\sigma_3$ being the Pauli matrix, and block-diagonalization*) of $R_{\mu\nu}$ as

*) We regard $R_{\mu\nu} = (R)^{\mu\nu}$ as a c-number matrix, owing to (2.25).
we can verify\(^3\)

\[
\text{Tr} e^{i R_{\mu \nu} \otimes \sigma_{\mu \nu}} = 2^n \det [\cosh R_t]^{1/2}.
\] (2.32)

Substituting (2.29) and (2.32) into (2.30), we have a general expression for the anomaly

\[
\mathcal{A}(x) = \lim_{t \to 0} \frac{2^n i}{(4 \pi t)^n} \text{Tr} \gamma_{2n+1} \text{det} \left[ \begin{array}{c}
R_t \\
\tanh R_t
\end{array} \right]^{1/2} = \lim_{t \to 0} \frac{2^n i}{(4 \pi t)^n} \text{Tr} \gamma_{2n+1} \exp \left\{ - \frac{1}{2} \text{tr} \ln \left[ \frac{\tanh R_t}{R_t} \right] \right\},
\] (2.33)

where tr denotes the trace over the tensor indices, e.g., \(\text{tr} R^2 = R_{\mu \nu} R_{\mu \nu}^\prime\). Using the formula\(^5\)

\[
\ln \frac{\tanh z}{z} = \sum_{k=1}^\infty (-1)^k \frac{2^{2k} (2^{2k-1} - 1) B_k}{(2k)!} z^{2k}
\] (2.34)

with \(B_k\) being the \(k\)-th Bernoulli number \((B_1 = 1/6, B_2 = 1/30, B_3 = 1/42, \cdots)\), we obtain an explicit form of the anomaly \(\mathcal{A}(x) = \mathcal{A}^{(4N)}(x)\) in \(4N\)-dimensions,

\[
\mathcal{A}^{(4N)}(x) = \sum_{j_1 + 2j_2 + \cdots + N j_N = N} c(j_1, \cdots, j_N) T(j_1, \cdots, j_N),
\] (2.35)

where

\[
c(j_1, \cdots, j_N) = \frac{i}{(2 \pi)^{2N}} \prod_{m=1}^N \frac{1}{j_m!} \left[ \frac{(-1)^{m+1} (2^{2m-1} - 1) B_m}{(2m)! 2m} \right]^{j_m},
\]

\[
T(j_1, \cdots, j_N) = 2^{2N} \text{Tr} \gamma_{2N+1} (\text{tr} R^2)^{j_1} (\text{tr} R^4)^{j_2} \cdots (\text{tr} R^{2N})^{j_N}
\]

\[
= \varepsilon \langle R^2 \rangle^{j_1} \langle R^4 \rangle^{j_2} \cdots \langle R^{2N} \rangle^{j_N}
\] (2.36)

with the notation of Ref. 11), for example,

\[
\varepsilon \langle R^2 \rangle^2 = \varepsilon_{\mu_1 \cdots \mu_8} R^a_{\mu_1 \rho_1} R^b_{\rho_1 \rho_2} R^c_{\rho_2 \rho_3} R^d_{\rho_3 \rho_4} R^e_{\rho_4 \rho_5} R^f_{\rho_5 \rho_6} R^g_{\rho_6 \rho_7} R^h_{\rho_7 \rho_8}.
\]

Note that the anomaly exists only in \(4N\) dimensions as expected. Our result (2.35), expressed explicitly in terms of the Riemann tensor, agrees with that of Delbourgo
and Matsuki\textsuperscript{18,41,49}) who derived it starting from the expression of the index theorem. The explicit forms in some lower dimensions are

\[ \mathcal{A}^{(4)} = \frac{i}{(2\pi)^2} \left[ \frac{1}{24} \varepsilon \langle R^2 \rangle \right] , \]

\[ \mathcal{A}^{(8)} = \frac{i}{(2\pi)^4} \left[ \frac{1}{1152} \varepsilon \langle R^8 \rangle^2 - \frac{7}{2880} \varepsilon \langle R^4 \rangle \right] , \]

\[ \mathcal{A}^{(12)} = \frac{i}{(2\pi)^6} \left[ \frac{1}{82944} \varepsilon \langle R^{24} \rangle^2 - \frac{7}{69120} \varepsilon \langle R^8 \rangle \langle R^4 \rangle + \frac{31}{181440} \varepsilon \langle R^6 \rangle \right] . \quad (2.37) \]

These agree with the previous results\textsuperscript{1,41,49)}.

\section*{§ 3. Gravitational anomaly}

Before calculating the gravitational anomaly, we shall start with a remark on the symmetric energy-momentum tensor \( T_{\mu\nu}^{sym}(x) \) appearing in (1.1). \( T_{\mu\nu}^{sym}(x) \) is the symmetric part of the usual energy-momentum tensor \( T_{\mu\nu}(x) \):

\[ T_{\mu\nu}^{sym}(x) = \frac{1}{2} \left[ T_{\mu\nu}(x) + T_{\nu\mu}(x) \right] , \]

\[ \sqrt{g} T_{\mu\nu}(x) = - \varepsilon^{\mu\nu}(x) \frac{\delta S}{\delta e_{\mu\nu}(x)} . \quad (3.1) \]

As stressed by Fujikawa et al.,\textsuperscript{9)} \( D_\mu T_{\mu\nu}^{sym}(x) \) (not \( D_\mu T_{\mu\nu}(x) \)) is the general coordinate anomaly calculated perturbatively in Ref. 1). This is because in Ref. 1) the local Lorentz frame is taken to be

\[ e_{\mu}^a = \delta_{\mu}^a + \frac{1}{2} h_{\mu}^a + \cdots , \quad (3.2) \]

where \( h_{\mu\nu}(= h_{\nu\mu}) \) is the weak gravitational field defined by \( g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}(\eta_{\mu\nu} = \text{diag}(+1, -1, \cdots -1)) \). Thus \( T_{\mu\nu}^{sym}(x) = -2 \delta S/\delta h_{\mu\nu} \) couples to \( h_{\mu\nu} \).

General coordinate transformations, however, break the frame choice (3.2). Corresponding to a coordinate transformation \( x^\mu \rightarrow x'^\mu = x^\mu - \xi^\mu(x) \), we usually transform \( e_{\mu}^a \) as

\[ \delta_{\text{cov}}(\xi) e_{\mu}^a = D_\mu \xi^a + D_\nu \xi^b \eta_{ab} , \quad (3.3) \]

where \( \delta_{\text{cov}}(\xi) \) denotes the general coordinate transformation up to local Lorentz one: \( \delta_{\text{cov}}(\xi) = \delta_{\text{G}}(\xi) + \delta_{\text{L}} (\xi^b \omega_{ab}) \).\textsuperscript{9)} Clearly this spoils (3.2); \( e_{\mu a} = \eta_{ab} e_{\mu b} \) in (3.2) is symmetric with respect to \( \mu \) and \( a \) whereas \( \delta_{\text{cov}}(\xi) e_{\mu a} \) is not. To keep the symmetry of the vielbein, we must further rotate the local Lorentz frame as

\textsuperscript{4)} A bit of arguments are needed to compare our result to theirs. Their result is expressed in terms of \( B_{2m}(1/2) \) rather than \( B_m \), where \( B_m(x) \) is the Bernoulli polynomial defined by

\[ \frac{z e^{xz}}{e^z - 1} = \sum_{m=0}^{\infty} B_m(x) \frac{z^n}{m!} . \]

By using this generating function, we can easily get

\[ B_m \left( \frac{1}{2} \right) = (-1)^m \frac{2 z^{2m} - 2}{2^{2m}} B_m . \quad (m \geq 1) \]

This formula makes the comparison possible.
\[ \delta_{\text{sym}}(\xi) = \delta_{\text{cov}}(\xi) + \delta_{\text{LL}}(D_\mu \xi_a), \]

so that
\[ \delta_{\text{sym}}(\xi) e_\mu^a = \frac{1}{2} (D_\mu \xi^a + D^a \xi_\mu). \]

Thus \( T^\mu_\nu(x) \) appears in the variation of the effective action \( \Gamma'[e_\mu^a] \) under the transformation (3.4):
\[ \delta_{\text{sym}}(\xi) \Gamma'[e_\mu^a] = \int dx \sqrt{g} \xi^\nu(x) D_\mu \langle T^\mu_\nu(x) \rangle. \]

Consequently, to compare our calculation with the perturbative results of Ref. 1, we should evaluate the Jacobian for the transformation (3.4).

Among various bosonic fields, only (anti-)self-dual anti-symmetric tensor fields contribute to the gravitational anomalies in \( 2n = 4N - 2 \) dimensional space-time. This can be understood from the fact that there are no invariant inner products on the functional space of the (anti-)self-dual tensor fields. If it were, we could easily present an invariant path integral measure. For example, for a scalar field \( \phi \) the functional measure \( \prod_x \mathcal{D}[g^{1/2}(x) \phi(x)] \) is invariant because of the invariance of the line element \( ||\delta \phi||^2 = (\delta \phi, \delta \phi) = \int dx g^{1/2}(x) \delta \phi(x) \delta \phi(x). \)

On the contrary, for the self-dual tensor field \( F^{\mu_1 \cdots \mu_{2N-1}} \), usual inner product vanishes in \( 4N - 2 \) dimensional space-time:
\[ \int dx \sqrt{g} F \cdot F = \frac{1}{(2N-1)!} \int dx \sqrt{g} F_{\mu_1 \cdots \mu_{2N-1}} F^{\mu_1 \cdots \mu_{2N-1}} = 0. \]

Thus the path integral measure \( \mathcal{D} F \) has a non-trivial Jacobian for the general coordinate transformation.

Instead of evaluating the Jacobian of \( \mathcal{D} F \), we may calculate the Jacobian of a chiral KD field \( \psi \). The chirality condition on the KD field,
\[ \gamma_{2n+1} \psi = - \psi, \]

corresponds to
\[ F^{(p)}_{\mu_1 \cdots \mu_{2N-1}} = i^{n-1}(-1)^{4(1/2)(p+1)} \bar{F}^{(2n-p)}_{\mu_1 \cdots \mu_{2N-1}} \]

in terms of the tensor fields \( F^{(p)}(p=0, 1, \cdots, 2n = 4N - 2) \) defined by (2.12). Especially, the middle rank tensor \( F^{(2n-1)} \) obeys the self-duality condition \( *F^{(2n-1)} = F^{(2n-1)} \).

Thus the chiral KD field and the self-dual tensor field yield the same anomalous Jacobian for the general coordinate transformation; we know that the other rank tensor fields \( F^{(p)}(p \neq 2N-1) \) have invariant path integral measures.

To evaluate the anomaly of the chiral KD field in Fujikawa's method, we shall treat the Euclidean version of the path integral. The Lagrangian is given by
\[ \text{Lagrangian} \]

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\*\*\* In the path integral derivation of the anomalies in Ref. 1, \( \delta_{\text{sym}}(\xi) \) was not used. They took \( \delta_{\text{cov}}(\xi) \) for spin 1/2 fields and \( \delta_{\text{cov}}(\xi) + 2 \delta_{\text{LL}}(D_\mu \xi_a) \) for spin 3/2 fields. Here, \( \delta_{\text{LL}} \) is the local Lorentz transformation which applies only to the vector indices (and not to the spinor indices) of the spin 3/2 fields — even the classical action is not invariant under \( \delta_{\text{LL}} \). Anomalies associated to these transformations differ by a factor 2 from the perturbative one. Matsuki\( ^8 \) used \( \delta_{\text{cov}}(\xi) \) for spin 1/2 and 3/2 fields. Especially for spin 3/2 fields, \( \delta_{\text{cov}}(\xi) \) leads to a different form of the anomaly, though he eventually got (1.1) by ignoring some relevant terms.
\[ \mathcal{L} = \sqrt{g} \text{Tr} \ \bar{\psi} i D^\alpha \gamma_\alpha \frac{1}{2} \gamma_{2n+1} \psi , \]  

(3·10)

where \( \bar{\psi} \) and \( \psi \) are independent of each other in the Euclidean version. Since the field \( \bar{\psi} = \frac{1}{2} (1 + \gamma_{2n+1}) \psi \) also includes a self-dual tensor field, the measure \( [D \bar{\psi}] \) yields the same anomaly of \( [D \psi] \).

Under the coordinate transformation (3·4), the KD field \( \psi \) transforms as

\[ \delta_{\text{sym}} (\xi') \psi = \xi' D \psi + \frac{1}{2} (D_a \xi^a) [\sigma^{ab}, \psi] . \]  

(3·11)

By an argument similar to that in Ref. 9), we obtain the Jacobian factor \( J[\xi] \) of \([D \bar{\psi} D \psi]\) for the transformation (3·11)

\[ \ln J[\xi] = - \frac{1}{2} \sum_m \int dx / \sqrt{g} \ \text{Tr} \ \phi_m ^* \gamma_{2n+1} \{ \xi'^a \bar{D}_a - \bar{D}_a \xi'^a \} + (D_a \xi^a)(\sigma^{ab} \otimes 1 - 1 \otimes \sigma^{ab}) \} \phi_m \]  

\[ = - \frac{1}{4} \sum_m \int dx / \sqrt{g} \ \text{Tr} \ \phi_m ^* \gamma_{2n+1} \{ \xi'^a \bar{D}_a - \bar{D}_a \xi'^a \} - 2 (D_a \xi^a) \otimes \sigma^{ab} \} \phi_m , \]  

(3·12)

where the \( \phi_m \)'s are eigenfunctions of the operator \( D \),

\[ D \phi_m = \lambda_m \phi_m . \]  

(3·13)

The second equality of (3·12) can be derived by the help of (3·13). Using the Gaussian cutoff to regulate the infinite sum in (3·12), we have

\[ \ln J[\xi] = - \frac{1}{4} \lim_{t \to 0} \text{Tr}_{a,x} (\gamma_{2n+1} \otimes 1) \{ [\xi'^a D_a + D_a \xi'^a - 2 (D_a \xi^a) \otimes \sigma^{ab}] e^{-tH} \} , \]  

(3·14)

in which \( \text{Tr}_{a,x} \) denotes the trace over both the spinor indices and space-time points.

By imitating the arguments of Matsuki, 8) we shall now evaluate (3·14) by the proper time method. Owing to the trace property of \( \text{Tr}_{a,x} \), we can write (3·14) as

\[ \ln J[\xi] = - \frac{1}{4} \lim_{t \to 0} \frac{i}{t} \text{Tr}_{a,x} (\gamma_{2n+1} \otimes 1) \]  

\[ \times \exp \{ - t [D^2 - i \xi'^a D_a - i D_a \xi'^a + 2 i (D_a \xi^a) \otimes \sigma^{ab}] \} , \]  

(3·15)

where the only terms linear in \( \xi'^a \) are assumed to be picked up on the right-hand side. We regard the square bracket in the exponent as a proper time Hamiltonian and the exponential function as a heat kernel. We can express the Hamiltonian as

\[ \bar{D}^a D_a - \bar{R}_{\mu \nu} \otimes \sigma^{\mu \nu} + \frac{R}{4} + O(\xi^2) = H \]  

(3·16)

with

\[ \bar{D}_\mu = D_\mu - i \xi_\mu , \]  

(3·17)

\[ \bar{R}_{\mu \nu} = R_{\mu \nu} - i (D_\mu \xi_\nu - D_\nu \xi_\mu) . \]  

(3·18)

As seen below, the heat equation for the heat kernel \( \exp (-tH) = \tilde{K}(t) \) can be treated analogously to the case of \( U(1) \) anomaly.
To evaluate (3.15), we can use effective equations similar to (2·25). In the heat kernel $\tilde{K}$, the terms which will survive after the trace operation include at least $2N - 1$ factors of $\sigma_{\mu\nu}\otimes 1$, because of the factor $\gamma_{2N+1}\otimes 1$. Among these terms we need the only terms linear in $\tilde{\xi}^\mu$. Both these factors $\sigma_{\mu\nu}\otimes 1$ and $\tilde{\xi}^\mu$ are included only in the factor $\tilde{R}_{\mu\nu}\otimes 1$, which appears in $[\tilde{D}_\mu, \tilde{D}_\nu]$ and the second term in (3·16). Moreover, a dimensional analysis shows that the terms surviving after the limit $t \to 0$ have at most $2N$ factors of $\tilde{R}_{\mu\nu}\otimes 1$. From these facts, all the terms we need are those including just $2N$ factors of $\tilde{R}_{\mu\nu}\otimes 1$; no other terms such as derivatives of $\tilde{R}_{\mu\nu}$ or scalar curvature are needed. Thus, in evaluating (3·15), we can use the following effective equations:

\[
\begin{aligned}
[\tilde{D}_\mu, \tilde{D}_\nu] &= \tilde{R}_{\mu\nu}\otimes 1, \\
[\tilde{D}_\mu, \tilde{R}_{\mu\nu}\otimes 1] &= 0, \\
[\tilde{R}_{\mu\nu}\otimes 1, \tilde{R}_{\rho\sigma}\otimes 1] &= 0, \\
R &= 0.
\end{aligned}
\] (3·19)

Under the assumption (3·19) the solution of $\tilde{K}$ is given by replacing $D_\mu$ and $R_{\mu\nu}$ with $\tilde{D}_\mu$ and $\tilde{R}_{\mu\nu}$ in the heat kernel derived in the previous section:

\[
\text{Tr}(\gamma_{2N+1}\otimes 1)\langle x|e^{-t\tilde{K}}|x\rangle = \frac{\sqrt{g}}{(2\pi t)^{2N-1}}\text{Tr}\gamma_{2N+1}\text{det}\frac{\tilde{R}_t}{\tanh \frac{\tilde{R}_t}{t}}^{1/2}.
\] (3·20)

Substituting (3·20) into (3·15) and picking up the terms linear in $\tilde{\xi}^\mu$, we have

\[
\ln J[\tilde{\xi}] = \frac{1}{4}\lim_{t \to 0} \int dx \sqrt{g} \frac{2^{2N-1}}{t(4\pi t)^{2N-1}} \text{Tr}\gamma_{2N+1}\text{det}\frac{\tilde{R}_t}{\tanh \frac{\tilde{R}_t}{t}}^{1/2} = \int dx \sqrt{g} \tilde{\xi}^\mu D_\mu \left\{ \lim_{t \to 0} \frac{1}{2} \frac{2^{2N-1}}{(4\pi t)^{2N-1}} \text{Tr}\gamma_{2N+1} \left( \frac{\delta}{\delta (R_{\mu\nu})} \right) \text{det}\frac{R_t}{\tanh \frac{R_t}{t}} \right\}^{1/2}.
\] (3·21)

This gives the gravitational anomaly for the chiral KD field. Since both $\phi$ and $\bar{\phi}$ contribute to the anomaly in the Euclidean formulation, we should divide (3·21) by 2 to get the anomaly of a real KD field of the Minkowskian version. (This is quite parallel to the factor 1/2 for the anomaly of the Majorana-Weyl spinor.)

To get the anomaly of the self-dual tensor gauge field, we need another factor 1/2. We can explain this additional factor in the perturbative scheme.17) A real chiral KD field (in the Minkowskian version) has a real $(2N-2)$-rank tensor $F^{(2N-2)}$ as well as the real self-dual tensor $F^{(2N-1)}$. As a result of the field equation of the KD field, the curl of the $(2N-2)$-rank tensor $dF^{(2N-2)}$ becomes self-dual. Consequently, roughly speaking, a real KD field includes two self-dual tensor gauge fields $F^{(2N-1)}$ and $dF^{(2N-2)}$, both of which contribute to the anomalous diagram. This is the origin of the factor 1/2.\footnote{In the previous literature,13,20,23,25 the origin of the additional factor was explained as a result of the Wick rotation: A real self-dual tensor becomes complex in the Euclidean version, thus getting twice degree of freedom.} More detailed analysis will be reported elsewhere.17)

By the above arguments, we eventually have a general expression for the gravitational anomaly of the self-dual anti-symmetric tensor gauge field in $4N-2$ dimensions,
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\[ D_\mu \langle T^{\mu \nu}_s(x) \rangle = \frac{1}{2} 4\pi D_\mu \left\{ \lim_{\tau \to 0} \frac{1}{8} \left( \frac{2^N}{(4\pi)^{2N}} \text{Tr} \gamma_{2n+1} \right) \delta \frac{\delta }{\delta R_{\mu \nu}} \det \left[ \frac{R^I}{\tanh R^I} \right] \right\}^{1/2} \text{.} \] (3.22)

Thus (1.1) holds if we take \( \mathcal{A}(R) \) to be 1/8 times\(^3\) the 4N dimensional chiral U(1) anomaly (2.33).

§ 4. Summary

In this paper we have applied the proper time method to the evaluation of both the chiral U(1) and the gravitational anomalies of the anti-symmetric tensor gauge fields. The results confirm (1.1) for the anti-symmetric tensor gauge fields.

It should be noted that (1.1) holds only for the symmetric energy-momentum tensors \( T^{\mu \nu}_s \). These tensors correspond to the transformation \( \delta_{\text{sym}} \), a general coordinate transformation with a compensating local Lorentz transformation to preserve the symmetry of the vielbein.\(^3\) The other transformation does not lead to (1.1). For example, \( \delta_{\text{cov}} \) leads to the anomalies (=\( D_\mu T^{\mu \nu} \)) different from (1.1) in the normalization factor\(^3,4,7,9\) for spin 1/2 fields and in the form itself for spin 3/2 fields. \( (T^{\mu \nu}_s) \) is suitable for the study of the anomaly cancellation. Unlike \( T^{\mu \nu}_s \), \( T^{\mu \nu}_s \) has no local Lorentz anomaly;\(^9\) it may be considered as the \( T^{\mu \nu} \) whose anti-symmetric part (=local Lorentz anomaly) is subtracted.\(^9\)

As seen in § 2, the chiral U(1) anomalies of the anti-symmetric tensor gauge fields appear only in the first order formalism. The anomalous Jacobian indicates the violation of the (classical) equivalence between the first and the second order formalism. However, as far as usual interactions are concerned, this inequivalence induces no difference in any physical quantities.

As assumed in the previous literature, a chiral KD field has the gravitational anomaly twice as large as a self-dual tensor gauge field. This is due to the fact that a chiral KD field includes two self-dual tensor gauge fields.\(^7\)

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\(^3\) The factor 1/8 consists of three factors of 1/2. Two of them are explained above: The reality condition for the KD field and the appearance of the two propagating self-dual tensor modes in a KD field. The last factor comes from the dimension-depending factor 2\(^n\) in (2.32), which requires a factor 1/2 when we change the space-time dimension from 2n=4N to 4N−2. (Similarly, the factor \( 1/(4\pi)^n \) in (2.33) yields the second factor 4\pi on the right-hand side in (3.22) or (1.1). But this factor is common to the anomalies of spin 1/2, 3/2 and anti-symmetric tensor fields.) Similar care should be paid for the anomalies of spin 3/2 fields.
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