Order \( g \) Gauge Invariance of Witten’s Superstring Field Theory

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Operator expression of the action and the gauge transformation in Witten’s superstring field theory is completed in the fermionic representation of the (super)ghost. The order \( g \) gauge invariance of the action is proved by using this explicit operator expression. This includes the manifestation of how the picture changing operator \( X \) and the inverse picture changing operator \( Y \) satisfy \( XY \sim 1 \) in the fermionic representation.

§1. Introduction

Covariant field theory of open superstring proposed by Witten\(^1\) has recently been investigated from the viewpoint of operator formulation,\(^2\)\(^-\)\(^4\) which was made both in the bosonized and fermionic representations of the (super)ghost. On the one hand in the bosonized representation, Samuel\(^3\) gave a formulation by adopting the technique of Ref. 5). On the other hand, in the fermionic representation, the author\(^3\) gave 3-Neveu-Schwarz (3-NS) and Neveu-Schwarz-Ramond-Ramond (NS-R-R) vertices and proved their BRST invariance. Gross and Jevicki\(^4\) treated the NS sector of the theory in detail and gave the supersymmetric extension of so-called \( K_n \) symmetry.\(^1\)

In this paper, we complete the operator expression in the fermionic representation and prove the order \( g \) gauge invariance of the action. For this purpose, we must first construct ket and bra 2-string vertices, which are used in the action and the gauge transformation law as follows:

\[
S = \langle \Phi | z \Phi | Q_{\beta}^{NS(1)} | (NS)^2 \rangle_{12} + 1 \langle \Phi | z \Phi | Q_{\beta}^{NS(1)} | (R)^2 \rangle_{12} + \frac{2}{3} g \langle \Phi | z \Phi | Q_{\beta}^{NS(1)} | (NS)^3 \rangle_{123} + 2 g \langle \Phi | z \Phi | Q_{\beta}^{NS(1)} | (NS)^3 \rangle_{123} , \tag{1.1}
\]

\[
\delta_1 \langle \Phi | = 1 \langle \lambda | Q_{\beta}^{NS(1)} + g \langle \nu | (NS)^3 \rangle | (NS)^3 \rangle_{123} + g \langle \nu | (NS)^3 \rangle | (NS)^3 \rangle_{123} , \tag{1.2a}
\]

\[
\delta_1 \langle \psi | = 1 \langle \epsilon | Q_{\beta}^{NS(1)} + g \langle \nu | (NS)^3 \rangle | (NS)^3 \rangle_{123} + g \langle \nu | (NS)^3 \rangle | (NS)^3 \rangle_{123} , \tag{1.2b}
\]

where \( \langle \Phi | \) and \( \langle \psi | \) denote NS and R string fields and \( \langle \lambda | \) and \( \langle \epsilon | \) are NS and R gauge parameter fields, respectively. The 3-string vertices \( | (NS)^3 \rangle_{123} \) and \( | (NS)^3 \rangle_{123} \) were given in Refs. 3) and 4). (The vertex \( | (NS)^3 \rangle_{123} \) in Eqs. (1.2) should be cyclically permuted from that in Eq. (1.1) according to the ordering of NS and R fields.) Although construction of 2-string vertex was easily done from the overlapping condition in the case of the bosonic string vertex, the situation is different for the
superstring vertex owing to the presence of picture changing operator. In the picture of superghost vacuum chosen by Witten, \( |R\rangle_{12} \) and \( \nu_1 \langle R| \rangle_{21} \) contain the inverse picture changing operator \( Y \) and the picture changing operator \( X \), respectively. Since they cannot be represented by oscillator modes in the fermionic representation, we cannot separate them from the vertex. Therefore we must directly construct the vertices \( |R\rangle_{12} \) and \( \nu_1 \langle R| \rangle_{21} \) including the operators \( Y \) and \( X \), respectively. This cannot be done from the connection condition alone. Instead we apply the method of Ref. 6), which was useful for the construction of 3-string vertices.\(^3\) Just as we obtained the 3-NS vertex \( |\langle NS\rangle_{123}^a \rangle \) containing the picture changing operator \( X \), we can construct 2-string vertices by using this method.

Invariance of the action (1·1) under the gauge transformations (1·2) should hold in each order of the gauge coupling constant \( g \). In this paper we investigate the order \( g \) gauge invariance. It holds if the vertices are BRST invariant and bra and ket 2-string vertices are inverse of each other in the sense:

\[
12\langle \langle NS\rangle^2 |(NS)^2\rangle_{23} = 1\langle \delta^a_{NS}(1, 3)\rangle_{3}, \quad (1·3a)
\]

\[
12\langle \langle R\rangle^2 |(R)^2\rangle_{23} = 1\langle \delta^a_{R}(1, 3)\rangle_{3}, \quad (1·3b)
\]

where \( 1\langle \delta^a_{NS}(1, 3)\rangle_{3} \) and \( 1\langle \delta^a_{R}(1, 3)\rangle_{3} \) are "\( \delta \)-functions" satisfying

\[
3\langle \Phi |1\langle \delta^a_{NS}(1, 3)\rangle_{3} = 1\langle \Phi |, \quad (1·4a)
\]

\[
3\langle \Psi |1\langle \delta^a_{R}(1, 3)\rangle_{3} = 1\langle \Psi |. \quad (1·4b)
\]

In the case of the bosonic string theory, BRST invariance and relation corresponding to Eq. (1·3) of 2-string vertex were trivial and not taken care of so much. As for the superstring, however, presence of the picture changing operator makes the BRST invariance of 2-R vertices and Eq. (1·3b) nontrivial. Thus, as our second task in this paper, we show them based on the operator expression of the vertices. Note that Eq. (1·3b) implies the realization of the relation

\[
\lim_{z \to w} X(z) Y(w) = 1, \quad (1·5)
\]

in the fermionic representation.

Here a comment should be made on the work by Yamron.\(^7\) He gave R kinetic term in the form that the reality condition was already imposed. We will therefore discuss the reality condition imposed on string fields and the correspondence of our R kinetic term with that of Yamron.

This paper is organized as follows. First in § 2, we review the method developed in Refs. 6), 3) and 8) for constructing BRST invariant vertex on general (super)ghost vacuum. Using this method, explicit operator expression of 2-string vertices is given in § 3. This completes the operator expression of Witten’s superstring field theory. Section 4 is devoted to the proof of the order \( g \) gauge invariance of the action. In § 5 we comment on the reality condition on string fields. At the end of this paper three Appendices are prepared. Concrete formulas for § 2 are given in Appendix A. In Appendix B BRST invariance of the 2-R vertex is shown by using the contour integration method. Finally in Appendix C we show that the R kinetic term in
§ 2. Construction of the BRST invariant vertex

In this section we review the method developed in Refs. 6, 3, and 8 for constructing BRST invariant N-string vertices. In this paper we follow the notation and the convention in Ref. 3. In particular we adopt the convention of the inverse time ordering in the following section.

The method of Ref. 6 is based on the idea of the Neumann function method. We prepare a p-plane (Fig. 1) which represents an N-string scattering process of Fig. 2 and relate it via a conformal mapping with the upper half complex z-plane (Fig. 3) on which we have a simple Neumann (correlation) function. In the case of the Witten-type mid-point interaction, this mapping takes the form

$$\frac{dp}{dz} = N_0 \frac{(z - z_0)(z - z_0^*)}{\prod_{r=1}^{N}(z - Z_r)^{(N-2)/2}}.$$

where the $z_0$ and the real constants $Z_r (r=1, \cdots, N)$ are the locations of the z-plane images of the interaction point and the external states, respectively. Owing to the condition that all strings have the same string length 1 and they interact at the mid-point, only three (real) parameters among the $Z_r$ and $z_0$ are left arbitrary, which coincides with the freedom of $SL(2; R)$ transformations. With this conformal mapping, it is straightforward to obtain the vertex for the physical coordinate $X^a(\sigma)$ and $\phi^\mu(\sigma)$.

The construction of the (super) ghost vertex and the realization of the BRST invariance are of our main concern. As for the construction of the vertex, we concentrate on the ghost coordinate parts and consider generically ghost C and antighost B coordinates of conformal weight $1 - h$ and $h$, respectively, and statistics $\epsilon (\epsilon = +1$ in the fermionic case and $\epsilon = -1$ in the bosonic case). The reparametrization ghost $(RG)(C=c, B=b)$ corresponds to $h=2, \epsilon = +1$ and the superghost (SG) $(C=\gamma, B=\beta)$ has $h=3/2$.
They can be treated as conformal fields on the entire complex \( z \)-plane by doubling the open string in the usual manner. The ghost vertex is uniquely obtained once the correlation function on this plane is determined. This determination is made by taking into account the presence of the ghost number anomaly and the possibility of picture changing operations. For this purpose it is convenient to make use of the bosonization formulas

\[
C = e^\phi \quad \text{and} \quad B = e^{-\phi} \quad \text{for} \quad \epsilon = +1, \tag{2.2}
\]
\[
C = e^{\psi} \eta \quad \text{and} \quad B = \partial \xi e^{-\psi} \quad \text{for} \quad \epsilon = -1, \tag{2.3}
\]

although we finally obtain the vertex written in terms of the oscillators of \( C \) and \( B \). The boson \( \phi \) and the fermions \( \eta \) and \( \xi \) in the above expression satisfy the operator product expansion

\[
\phi(z)\phi(w) \sim \epsilon \ln(z-w),
\]
\[
\eta(z)\xi(w) \sim \frac{1}{z-w}. \tag{2.4}
\]

Since there is a ghost number anomaly

\[
Q = \epsilon(1 - 2\hbar); \quad Q = \begin{cases} -3 & \text{for RG}, \\ 2 & \text{for SG}, \end{cases} \tag{2.5}
\]

we must make ghost insertions at \( Z_r, z_0 \) and \( z_0^* \) in order to cancel it. Thus we insert the ghost source \( e^{p_r\phi(z_r)}e^{q\phi(z_0)} \) and \( e^{q'\phi(z_0^*)} \) with ghost numbers \( p_r, q \) and \( q' \), respectively, satisfying

\[
\sum_{r=1}^N p_r + q + q' = -Q. \tag{2.6}
\]

In the presence of these ghost sources, the correlation function becomes

\[
\langle (\prod_{r=1}^N e^{p_r\phi(z_r)})C(z)B(\bar{z})e^{q\phi(z_0)}e^{q'\phi(z_0^*)} \rangle 
\]
\[
\sim \frac{1}{z - \bar{z}} \prod_{r=1}^N \left( \frac{z - z_r}{\bar{z} - \bar{z}_r} \right)^{p_r} \left( \frac{z - z_0}{\bar{z} - \bar{z}_0} \right)^{q} \left( \frac{z - z_0^*}{\bar{z} - \bar{z}_0^*} \right)^{q'}, \tag{2.7}
\]

as can be easily seen by using Eq. (2.4), where we have normalized the correlation function by \( \langle (\prod_{r=1}^N e^{p_r\phi(z_r)})e^{q\phi(z_0)}e^{q'\phi(z_0^*)} \rangle = 1 \). Here \( p_r \) is determined by the picture on which string fields are constructed, since the ghost insertion at \( Z_r \) has the effect of picture changing of the external vacuum state. On the other hand, \( q \) and \( q' \), which determine the picture at the interaction points, cannot be fixed by Eq. (2.6) only. This freedom is fixed by hand so that we have a vertex corresponding to Witten's formulation.\(^{15}\) Note that, owing to Eqs. (2.5) and (2.6), the correlation function (2.7) automatically satisfies the condition given in Ref. 6) that it should lead to an \( SL(2; \mathbb{R}) \) invariant \( \rho \)-plane correlation function and thus the vertex becomes \( SL(2; \mathbb{R}) \) invariant. We give a concrete expression of the vertex obtained from (2.7) in Appendix A.
Next we must investigate the BRST invariance of the vertex. Although it has the proper overlapping δ-functional structure, they are not generally BRST invariant. This is because the ghost sources $e^{q\phi(z_0)}$ and $e^{q\phi(z_0^*)}$ included in the vertex interfere with the BRST current. From this observation, the following two ways of restoring the BRST invariance can be considered.\(^{(6),(3),(17)}\)

1. To remove the effect of the ghost insertion completely.
2. To make the effect harmless by inserting another operator at the interaction point.

The first way is realized by multiplying a prefactor

$$P_1(z_0) = \lim_{z \to z_0} (z-z_0)^{qz} e^{-q\phi(z)} ,$$

(2.8)

and a similar one at $z_0^*$. Here $(z-z_0)^{qz}$ is required in order to remove the singularity at the coincident point and the limit should be taken after the product with the vertex is evaluated. The second method is realized by multiplying a prefactor $P_2(z_0)$ such that the product $P_2(z_0)e^{q\phi(z_0)}$ becomes an operator commutative with the BRST charge. For example, in Ref. 3) the picture changing operator $X(z_0)$ is decomposed as

$$X(z_0) = \tilde{X}(z_0)e^\phi(z_0) ,$$

(2.9)

and $\tilde{X}(z_0)$ is taken as a prefactor of the 3-NS vertex. We must adopt the first method for the reparametrization ghost sources, since there is no nontrivial operator commutative with the BRST charge like the picture changing operator $X$ and the inverse picture changing operator $Y$. In this case the prefactor (2.8) can be written in the fermionic representation as

$$P_1(z_0) = \begin{cases} 
\tilde{b}(z_0)\frac{d}{dz} \tilde{b}(z_0) \ldots \frac{d^{qz}}{dz^{qz}} \tilde{b}(z_0) , & \tilde{b}(z) = (z-z_0)^{qz} b(z) \text{ for } q > 0 , \\
\tilde{c}(z_0)\frac{d}{dz} \tilde{c}(z_0) \ldots \frac{d^{qz}}{dz^{qz}} \tilde{c}(z_0) , & \tilde{c}(z) = (z-z_0)^{qz} c(z) \text{ for } q < 0 .
\end{cases}$$

(2.10)

On the other hand, superghost sources cannot be rewritten into the fermionic representation. Therefore in order to write the vertex fully in the fermionic representation, we must use the second way for the superghost in which we can always find an appropriate prefactor by using $X$ or $Y$.

Although the above observation help the understanding, we must prove the BRST invariance of the vertex explicitly. This can be most easily and systematically done by using the contour integration method developed in Refs. 11) and 18). Further the concrete form of the prefactor relating with the picture changing operator $X$ can be easily obtained in the process of the proof (see Refs. 18), 3) and Appendix B).

§ 3. 2-string vertices

In this section we give a concrete operator expression for the 2-string vertices in Eqs. (1.1) and (1.2). Although 2-NS vertices are almost trivial, we also treat them...
The simplest choice of the conformal mapping which takes the $\rho$-plane (Fig. 4) into the complex $z$-plane is $\rho = \ln z$ ($Z_1 = 0, Z_2 = \infty$). However we use the $SL(2; \mathbb{R})$ transformed one
\[
\rho = \alpha_1 \ln (z - Z_1) + \alpha_2 \ln (z - Z_2) + \ln M; \quad \frac{d\rho}{dz} = \frac{Z_1 - Z_2}{(z - Z_1)(z - Z_2)} ,
\]
\[\alpha_1 = - \alpha_2 = 1 , \quad M = \frac{Z_0 - Z_2}{Z_0 - Z_1} i ,
\]
\[\rho (Z_0) = i \sigma_0 ; \quad \sigma_0 = \frac{\pi}{2} , \tag{3.1}
\]
so that the correspondence with the previous section and the $Z_r$ dependence of the prefactor become clearer. In Eq. (3.1), $Z_0$ should be taken so that $M$ becomes real. In the following the phases of the square roots of $\alpha_1$ and $\alpha_2$ are fixed by
\[\alpha_1^{1/2} = 1 , \quad \alpha_2^{1/2} = i . \tag{3.3}
\]
As can be seen from Eq. (A.6), the vacuum state for the reparametrization ghost (RG):
\[\langle 0 | c_{-n} = 0 \text{ for } n \geq 1 , \quad \langle 0 | b_{-n} = 0 \text{ for } n \geq 0 , \tag{3.4}
\]
that for the NS superghost (SG):
\[\langle 0 | \gamma_{-(n + 1/2)} = \langle 0 | \beta_{-(n + 1/2)} = 0 \text{ for } n \geq 0 , \tag{3.5}
\]
and that for the R-SG:
\[\langle 0 | \gamma_{-n} = 0 \text{ for } n \geq 1 , \quad \langle 0 | \beta_{-n} = 0 \text{ for } n \geq 0 , \tag{3.6}
\]
have ghost number $+1$, $-1$ and $-1/2$, respectively.* Therefore we denote them by $\langle +1 |, \langle -1 |$ and $\langle -1/2 |$ and their adjoint vacuum (A.7) by $| +1 \rangle = (\langle +2 |)^*, | -1 \rangle = (\langle -1 |)^*$ and $| -1/2 \rangle = (\langle -3/2 |)^*$. Note that the adjoint vacuum is identical with the hermitian conjugate one only for $\langle -1 |$.

Now we begin with the construction of the ket vertices $| (NS)^2 \rangle_{12}$ and $| (R)^2 \rangle_{12}$ in (1.1). First we must fix the value of $q$ and $q'$ for each case. We should take $q' = 0$ for the SG so that the picture changing operator is written by the right-moving component in accordance with Witten's choice. Further we take $q' = 0$ also for the RG although other choices lead to the same result in the final expression. From Eqs. (2.6) and (2.5) we have

\begin{figure}[h]
\centering
\begin{tabular}{c|c}
\hline
$P$ & $1$ \\
\hline
$10\sigma_0$ & $2$ \\
\hline
$-1\sigma_0$ & $-$
\end{tabular}
\caption{The extended $\rho$-plane representing the 2-string scattering.}
\end{figure}

* We use ghost number assignments determined by conformal field theory. Since bra and ket exchange their roles in our convention, the ghost number of the $SL(2; C)$ invariant bra vacuum is assigned to be zero.
\[ p_1=p_2=+1, \quad q=+1 \quad \text{and} \quad q'=0 \quad \text{for RG}, \quad (3\cdot7a) \]
\[ p_1=p_2=-1 \quad \text{and} \quad q=q'=0 \quad \text{for NS·SG}, \quad (3\cdot7b) \]
\[ p_1=p_2=-\frac{1}{2}, \quad q=-1 \quad \text{and} \quad q'=0 \quad \text{for R·SG}. \quad (3\cdot7c) \]

Then we can easily obtain the concrete expression of the vertex from Eqs. (A·8) and (A·11) and the result is

\[
\begin{align*}
|\langle RG \rangle_{12}^2| &= \exp\{E_c(1, 2)\}|\sim 1|\sim 1_{2}, \quad (3\cdot8) \\
E_c(1, 2) &= \sum_{n \geq 1} (-)^{n+1} (b_{(-1)}^{(1)} c_{-n}^{(2)} + b_{(-1)}^{(2)} c_{-n}^{(1)}) \\
&\quad - (b_0^{(1)} - b_0^{(2)}) \sum_{n \geq 1} (c_{-n}^{(1)} e^{-in\sigma_0} - c_{-n}^{(2)} (-)^n e^{in\sigma_0}), \\

|\langle NS·SG \rangle_{12}^2| &= \exp\{E_y^{NS}(1, 2)\}|\sim 1|\sim 1_{2}, \quad (3\cdot9) \\
E_y^{NS}(1, 2) &= \sum_{n > 0} (-)^{n} (\beta_{(n+1/2)}^{(1)} \gamma_{(n+1/2)}^{(2)} - \beta_{(n+1/2)}^{(2)} \gamma_{(n+1/2)}^{(1)}), \\

|\langle R·SG \rangle_{12}^2| &= \exp\{E_y^{R}(1, 2)\}|\sim \frac{1}{2}_{1}, \sim \frac{1}{2}_{2}, \quad (3\cdot10) \\
E_y^{R}(1, 2) &= \sum_{n \geq 1} (-)^{n} i (\gamma_{(n)}^{(1,1)} \gamma_{-n}^{(2,2)} - \gamma_{(n+1/2)}^{(2,2)} \gamma_{-n}^{(1,1)}), \\
&\quad - (\beta_{0}^{(1)} - i\beta_{0}^{(2)}) \sum_{n \geq 1} (\gamma_{-n}^{(1)} e^{-in\sigma_0} + i\gamma_{-n}^{(2)} (-)^n e^{in\sigma_0}).
\end{align*}
\]

In the case of the 2-string vertices, connection conditions are easily seen without using the property of the correlation function. We have

\[
\begin{align*}
\{c^{(1)}(\sigma) + c^{(2)}(\pi \text{sgn}\sigma - \sigma)\}|\langle RG \rangle_{12}^2| &= 0, \quad (3\cdot11a) \\
\{b^{(1)}(\sigma) - b^{(2)}(\pi \text{sgn}\sigma - \sigma)\}|\langle RG \rangle_{12}^2| &= (b_0^{(1)} - b_0^{(2)})|\langle RG \rangle_{12}^2| 2\pi \delta(\sigma - \sigma_0), \quad (3\cdot11b) \\
\{\gamma^{(1)}(\sigma) - i\text{sgn}\sigma \gamma^{(2)}(\pi \text{sgn}\sigma - \sigma)\}|\langle NS·SG \rangle_{12}^2| &= 0, \quad (3\cdot12a) \\
\{\beta^{(1)}(\sigma) - i\text{sgn}\sigma \beta^{(2)}(\pi \text{sgn}\sigma - \sigma)\}|\langle NS·SG \rangle_{12}^2| &= 0, \quad (3\cdot12b) \\
\{\gamma^{(1)}(\sigma) - i\gamma^{(2)}(\pi \text{sgn}\sigma - \sigma)\}|\langle R·SG \rangle_{12}^2| &= 0, \quad (3\cdot13a) \\
\{\beta^{(1)}(\sigma) - i\beta^{(2)}(\pi \text{sgn}\sigma - \sigma)\}|\langle R·SG \rangle_{12}^2| &= (\beta_0^{(1)} - i\beta_0^{(2)})|\langle R·SG \rangle_{12}^2| 2\pi \delta(\sigma - \sigma_0), \quad (3\cdot13b)
\end{align*}
\]

where the \(\delta\)-functions in Eqs. (3·11b) and (3·13b) reflect the effect of the ghost source inserted at the interaction point. It should be noted that the RG vertex (3·8) can be rewritten as

\[
\begin{align*}
|\langle RG \rangle_{12}^2| &= c^{(1)}(\sigma_0)|\langle RG \rangle_{0}^2| - c^{(2)}(\sigma_0)|\langle RG \rangle_{0}^2| 2\pi \delta(\sigma - \sigma_0), \quad (3\cdot14) \\
|\langle RG \rangle_{0}^2| &= (b_0^{(1)} - b_0^{(2)})|\langle \exp E_c(1, 2) \rangle_{12}^2 = |\sim 1|\sim 1_{2}, \quad (3\cdot15)
\end{align*}
\]
\[ E^{0}(1, 2) = \sum_{n \geq 1} (-)^{n+1}(b^{(1)}_{n}c^{(2)}_{-n} + b^{(2)}_{n}c^{(1)}_{-n}), \]

where \( |(RG_{0})^{2}\rangle_{12} \) is just the 2-RG vertex usually made use of and we have replaced \( \pi - \sigma_{0} \) by \( \sigma_{0} \) since \( \sigma_{0} \) is just \( \pi/2 \). Equation (3·14) shows explicitly the ghost source included in the vertex. Similarly we can extract the ghost source in the R-SG vertex using the coordinate representation of R-SG zero mode (see Appendix C).

The naive vertex without prefactor is obtained by combining the (super)ghost part Eqs. (3·8)~(3·10) with the physical part. In addition to the well-known 2-string vertex for the physical coordinate \( x_{\mu}(\sigma) \):

\[ |(\text{Phys})^{2}\rangle_{12} = \{\text{exp}E_{\text{x}}(1, 2)\} |0\rangle_{1} |0\rangle_{2} (2\pi)^{d} \delta^{d}(p_{1} + p_{2}), \quad (3·16) \]

\[ E_{\text{x}}(1, 2) = \sum_{n \geq 1} (-)^{n+1} \frac{1}{n} a^{(1)}_{-n} a^{(2)}_{-n}, \]

we have those for the fermionic coordinate \( \psi^{\nu}(\sigma) :*) ***)

\[ |(\text{NS})^{2}\rangle_{12} = \{\text{exp}E_{\psi}^{\text{NS}}(1, 2)\} |0\rangle_{1} |0\rangle_{2}, \quad (3·17) \]

\[ E_{\psi}^{\text{NS}}(1, 2) = \sum_{n \geq 0} (-)^{n+1} d^{(1)}_{(n+(1/2))} \cdot d^{(2)}_{-(n+(1/2))}, \]

\[ |(R_{0})^{2}\rangle_{12} = \{\text{exp}E_{\psi}^{R}(1, 2)\} |0\rangle_{2}, \quad (3·18) \]

\[ E_{\psi}^{R}(1, 2) = \sum_{n \geq 1} (-)^{n+1} id^{(1)}_{-n} \cdot d^{(2)}_{n}, \]

\[ |0\rangle_{2} = \sum_{a_{\alpha}, \beta} |\bar{a}_{\alpha}| |\beta\rangle_{\alpha}(C^{-1})_{\alpha\beta} + i \sum_{a_{\alpha}, \beta} |a_{\alpha}| |\bar{\beta}\rangle_{\beta}(\overline{C^{-1}})_{\alpha\beta}. \]

These vertices are obtained from the correlation function

\[ \langle \phi^{\mu}(z) \phi^{\nu}(\bar{z}) \rangle = \frac{1}{z - \bar{z}} \eta^{\mu\nu}, \quad \text{for (NS)}^{2}, \quad (3·19a) \]

\[ \langle \phi^{\mu}(z) \phi^{\nu}(\bar{z}) \rangle = \frac{1}{z - \bar{z}} \frac{1}{2} \left( \frac{z - z_{1}}{\bar{z} - z_{1}} \frac{\bar{z} - z_{2}}{z - z_{2}} \right)^{1/2} \left( \frac{\bar{z} - z_{1}}{z - z_{1}} \frac{z - z_{2}}{\bar{z} - z_{2}} \right)^{1/2} \eta^{\mu\nu}, \quad \text{for (R)}^{2}, \quad (3·19b) \]

as in Refs. 18 and 3). Thus the naive vertex can be written as

\[ |(\text{NS}_{0})^{2}\rangle_{12} = |(\text{Phys})^{2}\rangle_{12} \otimes |(\text{RG})^{2}\rangle_{12} \otimes |(\text{NS}_{\psi})^{2}\rangle_{12} \otimes |(\text{NS} \cdot \text{SG})^{2}\rangle_{12}, \quad (3·20a) \]

\[ |(R_{0})^{2}\rangle_{12} = |(\text{Phys})^{2}\rangle_{12} \otimes |(\text{RG})^{2}\rangle_{12} \otimes |(R_{\psi})^{2}\rangle_{12} \otimes |(R \cdot \text{SG})^{2}\rangle_{12}. \quad (3·20b) \]

Next we must determine the prefactor and examine the BRST invariance of the vertex. According to the procedure in § 2, the prefactor for the 2-NS vertex (3·20a) should be taken as

\[ (*) \text{The explanation for the } R \text{ vacuum state } |0\rangle_{2} \text{ was given in Refs. 18 and 3). Here, we have changed its overall normalization and phase for later convenience in § 5.} \]

\[ (***) \text{The same result as (3·17) and (3·18) was already given in Ref. 4).} \]
\[ P_{\text{NS}} = \lim_{z \to z_0} \{(z - z_0) b(z)\} , \quad (3\cdot21) \]

since the ghost source included in the vertex is \( c(z_0) \) as can be seen from Eqs. (3\cdot7a) and (3\cdot7b). This can be evaluated on the vertex by using Eqs. (A\cdot2), (A\cdot8) and (A\cdot9). We take the limit from the sth string region and obtain

\[ P_{\text{NS}} = \lim_{z \to z_0} \left[ (z - z_0) \left( \frac{d}{dz} \right)^2 \left\{ \sum_{n \geq 0} b^{(n)} \left( e^{-\eta_1} - \sum_{r \geq 0} N_{n,n}^r \left( e^{m_1} \right) b^{(r)}_n \right) \right\} \right] \]

\[ = \lim_{z \to z_0} \left[ (z - z_0) \left( \frac{d}{dz} \right)^2 \left\{ \sum_{n \geq 0} \frac{1}{a_r} \int \frac{d\sigma_r}{2\pi} e^{-\eta_1} N(\rho, \bar{\rho}) b^{(r)}_n \right\} \right] \]

\[ = - a \sum_{n \geq 0} \frac{1}{a_r} \int \frac{d\sigma_r}{2\pi} e^{-\eta_1} b^{(r)}_n \]

\[ = - a \times (b^{(1)}_0 - b^{(n)}_0) , \quad (3\cdot22) \]

\[ a = \frac{d\rho}{dz} \bigg|_{z = z_0} , \quad (3\cdot23) \]

where we have substituted (3\cdot7a) and \( k = 2, \epsilon = +1 \). After removing the SL(2; \( R \)) dependent normalization factor \(-a\), we have the final form of the ket 2-NS vertex:

\[ |(\text{NS})^{2}_{12} = \mathcal{P}_{\text{NS}} |(\text{Phys})^{2}_{12} \otimes |(\text{RG})_{12}^{2} \otimes |(\text{NS}_{0})^{2}_{12} \otimes |(\text{NS} \cdot \text{SG})^{2}_{12} , \quad (3\cdot24) \]

where \(|(\text{RG})^{2}_{12}\) in (3\cdot20a) is replaced by \(|(\text{RG})^{2}_{12}|\) owing to the presence of the prefactor. (Note that the second term of \( E_c(1, 2) \) in Eq. (3\cdot8) vanishes because of the prefactor.) We have multiplied (3\cdot24) by the covariantized GSO projection operator:\(19), 20\)

\[ \mathcal{P}_{\text{NS}} = \frac{1 + G}{2} , \quad (3\cdot25) \]

\[ G = (-)^{N_{\text{NS}}} , \quad (3\cdot26) \]

\[ N_{\text{NS}} = \sum_{k \geq 0} (d_{-k} d_k + \beta_{-k} \gamma_k - \gamma_{-k} \beta_k) - 1 , \quad k \in \mathbb{Z} + \frac{1}{2} . \]

The concrete operator expression of (3\cdot24) tells us its BRST invariance trivially and it is not necessary to resort to the contour integration method.

As for the prefactor for the 2-R vertex (3\cdot20b), the following decomposition of the picture changing operator is made use of:

\[ Y(z_0) = c(z_0) \partial \xi(z_0) e^{-\eta(z_0)} = \lim_{z \to z_0} \{(z - z_0) \beta(z)\} \times c(z_0) e^{-\eta(z_0)} . \quad (3\cdot27) \]

Since \( c(z_0) e^{-\eta(z_0)} \) is included in the vertex (3\cdot20b), we should take

\[ P_{\text{NS}} = \lim_{z \to z_0} \{(z - z_0) \beta(z)\} \quad (3\cdot28) \]

as the prefactor, following the procedure in § 2. We can evaluate it on the vertex similarly to Eq. (3\cdot22) and obtain
\[ P_R = -a^{1/2} \times (\beta_0^{(i)} - i\beta_0^{(j)}) . \]  

(3·29)

Therefore we have

\[ |(R)^{2}\rangle_{12} = \mathcal{D}_R^{(i)} \mathcal{D}_R^{(j)} (\beta_0^{(i)} + i\beta_0^{(j)}) c^{(i)}(\alpha) |(\text{Phys})^{2}\rangle_{12} \otimes |(\text{RG})^{2}\rangle_{12} \]

(3·30)

where the \( Z_r \) dependent normalization factor was removed and the phase of the prefactor was chosen so that the \( R \) kinetic term becomes hermitian when reality condition is imposed in § 5. We have again multiplied (3·30) by the covariantized GSO projection operator:

\[ \mathcal{D}_R = \frac{1 + z}{2} , \]

(3·31)

\[ z = \Gamma \eta (-)^{\eta} , \]

(3·32)

\[ \eta_n = \sum_{\eta \geq 1} (d_n \cdot d_n + \beta_n \gamma_n - \gamma_n \beta_n) + \beta_0 \gamma_0 . \]

(3·33)

It is an easy practice to show the BRST invariance of (3·28) by using the contour integration method. (In this case it is BRST invariant at any space-time dimension.)

The remaining task in this section is to construct the bra vertices in Eq. (1·2). Instead of constructing them directly from correlation functions, we investigate their hermitian conjugate vertices \(|(\bar{NS})^{2}\rangle_{12} \) and \(|(\bar{R})^{2}\rangle_{12} \) so that the formulas in Appendix A become available. Hereafter we attach \( \sim \) in order to distinguish them from vertices in Eq. (1·1). Their physical parts are, of course, identical with those of \(|(NS)^{2}\rangle_{12} \) and \(|(R)^{2}\rangle_{12} \), respectively. On the other hand, as for the (super)ghost parts, the external vacuum states become different in the case of the RG and R-SG. Since RG, NS-SG and R-SG parts of the vertices \(|(NS)^{2}\rangle_{12} \) and \(|(R)^{2}\rangle_{12} \) are constructed on \(|+1\rangle = \langle +1\rangle \rangle = |+1\rangle , |\bar{1}\rangle = \langle \bar{1}\rangle \rangle = |\bar{1}\rangle \) and \(|-1/2\rangle = \langle -1/2\rangle \rangle = |-1/2\rangle \), respectively, the external vacuum states must be taken as \( <+|2\rangle , \langle -1| \) and \( \langle -3/2| \) for each case. Therefore by setting \( q'=0 \), we have

\[ p_1 = p_2 = +2 , \quad q = -1 \quad \text{and} \quad q' = 0 \quad \text{for RG} , \]

(3·33a)

\[ p_1 = p_2 = -1 \quad \text{and} \quad q = q' = 0 \quad \text{for NS-SG} , \]

(3·33b)

\[ p_1 = p_2 = -\frac{3}{2} , \quad q = +1 \quad \text{and} \quad q' = 0 \quad \text{for R-SG} . \]

(3·33c)

Since (3·33b) is just identical with (3·7b), we have only to treat RG and R-SG vertices. We can again obtain them from Eqs. (A·8) and (A·11). After taking the hermitian conjugation, the result becomes

\[ 12 \langle (RG)^{2}\rangle = 
\]

\[ \mathcal{E}_c(1, 2) = \sum_{n \geq 1} (-)^n (b_n^{(1)} c_n^{(2)} + b_n^{(2)} c_n^{(1)}) \]

\[ - \sum_{n \geq 1} (b_n^{(1)} e^{i\sigma} + b_n^{(2)} (-)^{n} e^{-i\sigma})(c_0^{(1)} + c_0^{(2)}) , \]

(3·34)
where extra signs have appeared in $\tilde{E}_c(1, 2)$ and $\tilde{E}_r(1, 2)$ because of the fermionic property of the RG and the hermiticity $(\beta_n)^* = -\beta_n$, respectively. These vertices satisfy the connection conditions

$$l_2\langle \overline{R\cdot SG} \rangle^2 = l_2\langle \overline{R\cdot GO} \rangle^2 = l_2\langle \overline{R\cdot NS} \rangle^2 = l_2\langle \overline{R\cdot Phys} \rangle^2 = 12\langle (\beta_n^{(1)}\beta_n^{(2)}) \rangle = 0 ,$$

and the RG vertex (3.34) can be rewritten as

$$12\langle \overline{R\cdot SG} \rangle^2 = 12\langle \overline{R\cdot GO} \rangle^2 = 12\langle \overline{R\cdot NS} \rangle^2 = 12\langle \overline{R\cdot Phys} \rangle^2 = 12\langle \overline{R\cdotGO} \rangle^2 = 12\langle \overline{R\cdotNS} \rangle^2 = 12\langle \overline{R\cdot Phys} \rangle^2 = 12\langle \overline{R\cdotGO} \rangle^2 = 12\langle \overline{R\cdotNS} \rangle^2 = 12\langle \overline{R\cdotPhys} \rangle^2 ,$$

corresponding to Eqs. (3.11)~(3.15).

The prefactor of the bra 2-NS vertex is now obtained quite similarly to that of the ket 2-NS vertex. In this case it should be taken as

$$P_{NS} = \lim_{z \to z_0} (z - z_0) c(z) ,$$

and this becomes

$$P_{NS} = -\frac{1}{a^3} \times (c^{(1)} + c^{(2)}),$$

on $|(NS)^2\rangle_{12}$. Therefore the final expression of the bra 2-NS vertex becomes

$$12\langle (\overline{NS})^2 | = 12\langle \overline{(Phys)^2} \rangle \otimes 12\langle \overline{(R\cdot GO)^2} \rangle \otimes 12\langle \overline{(NS\cdot SG)^2} \rangle \otimes 12\langle \overline{(NS\cdot Phys)^2} \rangle ,$$

where $12\langle (Phys)^2 |$, $12\langle (NS\cdot Phys)^2 |$ and $12\langle (NS\cdot SG)^2 |$ are the hermitic conjugates of Eqs. (3.16), (3.17) and (3.9), respectively.

Finally we determine the prefactor of the bra 2-R vertex. It should be an operator $\tilde{P}_r$ which becomes $X(z_0)$ when it is multiplied by the ghost source $b(z_0) e^{\phi(z_0)}$.

* Our normalization of the superghost is different from that of Ref. 16) by factor 2, which makes the coefficients in $X(z_0)$ different from the usual one. 10, 28
$\tilde{P}_k \times b(z_0) e^{\theta(z_0)} = X(z_0)$

$$= e^{\theta} \mathcal{A} + \partial \eta e^{2 \theta} b + \partial (\eta e^{2 \theta} b) + c \partial \xi |_{z = z_0}.$$ (3.43)

However it is difficult to solve this equation directly, since we must finally have $\tilde{P}_k$ written in the fermionic representation and well-defined on the vertex. Therefore, instead, we use the contour integration method not only for proving the BRST invariance but also for seeking the expression of the prefactor. This is performed in Appendix B. The result is

$$\tilde{P}_k = \tilde{c}(z_0) \left( -\psi(z_0) A(z_0) + \tilde{b}(z_0) \tilde{\varphi}(z_0) + \frac{d\tilde{c}}{dz}(z_0) \tilde{\beta}(z_0) + 2 \frac{d\tilde{\varphi}}{dz}(z_0) + I_0(z_0) \tilde{\gamma}(z_0) \right),$$ (3.44)

where operators with $\sim$ are defined in Eqs. (B-3) and are regular on the vertex, including at the interaction point. They are written only by creation operators in the sense of Eq. (B-5). $I_0(z_0)$ in the last term is defined in Eq. (B-4b):

$$I_0(z) = \frac{\partial}{\partial \tilde{z}} \left( \frac{(z - Z_1)(z - Z_2)}{z - Z_1} \right) |_{z = z} = 2 \left( \frac{d \varrho}{dz} \right)^{-1} \left( \frac{d^2 \varrho}{dz^2} \right),$$ (3.45)

$$I_0(z_0) = 2 e_1 = -2 \left( \frac{1}{z_0 - Z_1} + \frac{1}{z_0 - Z_2} \right),$$ (3.46)

where $e_1$ is the coefficient of the expansion of $\varrho(z)$ around $z_0$ defined in Eq. (B-13). This last term can be understood to compensate the change of the correlation function between $\tilde{c}$ and $\tilde{b}$ owing to the external ghost sources. At first sight, prefactor (3.44) seems to have nontrivial $SL(2; \mathbb{R})$ dependence owing to this term. However the truth is that the dependence exists only in the normalization factor again. In order to check this and obtain the expression of the prefactor independent of the $z$-plane, we rewrite $\tilde{P}_k$ in a way similar to Eq. (3.22). From Eqs. (3.33), (A.2), (A.8) and (A.9), we obtain

$$\frac{d\tilde{c}}{dz}(z_0) = \frac{1}{a} \left\{ \frac{1}{2} (c_0^{(1)} - c_0^{(2)}) + \sum_{n \geq 1} \left( c_n^{(1)} e^{-in\sigma_0} - c_n^{(2)} e^{-in\sigma_0} \right) \right\} + \frac{3 e_1}{2a_2} (c_0^{(1)} + c_0^{(2)}),$$ (3.47a)

$$\tilde{\gamma}(z_0) = -\frac{1}{a^{3/2}} (\gamma_0^{(1)} - i \gamma_0^{(2)}),$$ (3.47b)

$$\frac{d\tilde{\varphi}}{dz}(z_0) = \frac{1}{a^{1/2}} \left\{ \frac{1}{2} (\gamma_0^{(1)} + i \gamma_0^{(2)}) + \sum_{n \geq 1} \left( \gamma_n^{(1)} e^{-in\sigma_0} + i \gamma_n^{(2)} e^{-in\sigma_0} \right) \right\} + \frac{e_1}{a^{3/2}} (\gamma_0^{(1)} - i \gamma_0^{(2)}),$$ (3.47c)

$$\tilde{b}(z_0) = -a^3 \sum_{n \geq 1} n (b_n^{(1)} e^{-in\sigma_0} - b_n^{(2)} e^{-in\sigma_0}),$$ (3.47d)
\[ \tilde{\beta}(z_0) = -a^{5/2} \sum_{n \geq 1} n(\beta^{(1)}_n e^{-in\delta_0} - i\beta^{(2)}_n e^{-in\delta_0}), \quad (3.47e) \]
in addition to Eq. (3.41), where we have rewritten the result using \( \delta_0 = \pi/2 \). (We have omitted the expression for \( \phi^a(z_0) \) and \( A^a(z_0) \), since they trivially reproduce \( \phi^a(z_0) = -a^{1/2} \phi^a(z_0) \) and \( A^a(z_0) = aA^a(z_0) = -aA^a(z_0) \) on the vertex). Note that the combination \( 2d\tilde{\gamma}/dz(z_0) + \lambda_0(z_0) \tilde{\gamma}(z_0) \) becomes independent of \( \epsilon_1 \) and the last term in Eq. (3.47a) vanishes when it is multiplied by \( \tilde{c}(z_0) \propto (c_0^{(1)} + c_0^{(2)}) \). Thus the \( Z_r \) dependence indeed factorizes as \( a^{-1/2} \). Now we give the final expression of the bra 2-R vertex. Utilizing Eq. (3.38) and the explicit operator expression of the vertex, we have

\[ \begin{align*}
\tilde{X}(z_0) &= -\phi^{(2)}(z_0) \cdot A^{(2)}(z_0) - i \frac{db^{(2)}}{d\sigma}(z_0)(\gamma^{(2)}_0 - i\gamma^{(1)}_0) \\
&\quad - i \frac{d\bar{b}^{(2)}}{d\sigma}(z_0) c^{(2)}(z_0) + 2\gamma^{\text{reg}}(z_0) b^{(2)}(z_0), \\
\gamma^{\text{reg}}(z_0) &= \frac{1}{2}[\gamma^{(2)}_0 + i\gamma^{(1)}_0] + \sum_{n \geq 1} (\gamma^{(2)}_n e^{in\delta_0} + i\gamma^{(1)}_n e^{in\delta_0}),
\end{align*} \quad (3.49) \]
after taking the hermitian conjugation, where the overall normalization and phase are taken so that Eq. (1.3b) holds in § 4. This completes the construction of the 2-string vertices.

§ 4. Order \( g \) gauge invariance

In this section we prove the order \( g \) invariance of the action (1.1) under the gauge transformations (1.2).

First we comment on the statistics of string fields. The relative minus signs in the braces in Eq. (1.2) are correct only when string fields \( \langle \Phi \rangle \) and \( \langle \Psi \rangle \) are Grassmann odd. (Thus \( \langle \lambda \rangle \) and \( \langle \epsilon \rangle \) become Grassmann even.) In order to realize these statistics, it is most suitable to take all the (super)ghost vacua \( \langle \lambda^+1, \lambda^{-1}, \lambda^{-1/2} \rangle \) and their adjoints) as Grassmann even. Then NS field \( \langle \Phi \rangle \) becomes odd because physical states contain odd number of fermionic oscillators \( d^{(2)}_{\text{H}(1/2)} \). Further, since the GSO permitted right handed ground state \( \langle \alpha^+| \times \langle 1/2 \rangle = (\alpha^+| \times \langle -1/2 \rangle) \in \text{grassmann even in our convention,}^*) \) R field \( \langle \Psi \rangle \) becomes odd owing to the statistics of the component spinor field. The following massless states typically represent the fermionic statistics of NS and R string fields:**)

\[ \begin{align*}
\langle \Phi \rangle &= A_{\alpha^+} |d^{(2)}_{\text{H}(1/2)} + \cdots , \\
\langle \Psi \rangle &= \phi_\epsilon C_{\alpha^+} \langle \alpha^+ | + \cdots .
\end{align*} \quad (4.1) \]

*) The statistics of left and right-handed spinor ground states \( \langle \alpha \rangle \) and \( \langle \bar{\alpha} \rangle \) must be taken to be opposite.** In Ref. 3) the right-handed state \( \langle \alpha \rangle \) was fixed to be even.

**) The presence of the charge conjugation matrix \( C_{\alpha^+} \) is necessary where string field is represented by bra states.
Note that, with this convention, the ket 2-string vertices become symmetric thanks to
the GSO projection operator and the antisymmetry of the charge conjugation matrix:
\[ |(\text{NS})^2\rangle_{12} = |(\text{NS})^2\rangle_{21}, \quad |(\text{R})^2\rangle_{12} = |(\text{R})^2\rangle_{21}, \] (4.2)
which is necessary for the consistency of the kinetic terms.

Now since all the vertices appearing in Eqs. (1.1) and (1.2) have turned out to be
BRST invariant, we have only to show Eqs. (1.3a) and (1.3b). Then the order \( g \)
invariance follows from the cyclic symmetry of the 3-string vertices and Eq. (4.2).

In the NS sector, Eq. (1.3a) holds rather trivially. Using the concrete expression
of the vertices, we obtain
\[ \langle 12 \rangle (\text{NS})^2 \langle (\text{NS})^2 \rangle_{23} = \mathcal{P}_{\text{NS}}(12) \langle 0 | \exp \{ D_\text{NS}(1, 3) + D_{\text{NS}}(1, 3) \} | 0 \rangle, \] (4.3)
\[ D_\text{NS}(1, 3) = \sum_{n \geq 1} \left( \frac{1}{n} a_n^{(3)} \cdot a_n \right) + \frac{1}{n} b_n^{(3)} \gamma_n, \] (4.4)
\[ D_{\text{NS}}(1, 3) = \sum_{k \geq 0} \left( \frac{1}{k+1} a_k^{(3)} d_k^{(1)} + \frac{1}{k+1} b_k^{(3)} \gamma_k \right), \quad k \in \mathbb{Z} + \frac{1}{2}, \] (4.5)
where we have used the properties
\[ \mathcal{P}_{\text{NS}}(12)_{12} = \mathcal{P}_{\text{NS}}(12)_{12}, \]
\[ \langle 12 \rangle (\text{NS})^2 \mathcal{P}_{\text{NS}}(12)_{12} = \langle 12 \rangle (\text{NS})^2 \mathcal{P}_{\text{NS}}(12)_{12}. \] (4.6)
Equation (4.3) clearly satisfies Eq. (1.4a).

On the other hand, Eq. (1.3b) in the R sector is not trivial owing to the complicated
prefactor in Eqs. (3.30) and (3.48) and the presence of the bilinear term of the
same string oscillators in Eqs. (3.10) and (3.35). The product of the prefactor contains
divergent products \((db/d\omega)(\alpha_0) \cdot (c_0)\) and \((b_0) \cdot (\alpha_0)\) at the coincident point,
while the product of the R-SG part vertices \( \langle 12 \rangle (\text{R-SG})^2 \langle (\text{R-SG})^2 \rangle_{23} \) is expected to
vanish reflecting the operator product of the ghost sources included in the vertex:
\[ e^{\phi(z)} e^{\rho(w)} \sim \frac{z}{w} \rightarrow 0 \quad \text{as } z \rightarrow w. \] (4.7)
Therefore, although we can expect that Eq. (1.3b) follows from \( \lim_{z \rightarrow w} X(z) Y(w) = 1 \),
we must show it by regularizing the divergence properly. For this purpose we split
the interaction points along the "time" direction by \( r \) (Fig. 5), i.e., insert \( \exp(-\tau L^{(2)}) \)
between the vertices and evaluate the product
\[ \langle 12 \rangle (\text{R-SG})^2 e^{-\tau L^{(2)}} (\text{R-SG})^2 \langle (\text{R-SG})^2 \rangle_{23}, \] (4.8)
where
\[ L = \frac{1}{2} p^2 + \sum_{n \geq 1} \left( a_n \cdot a_n + n(b_n c_n + c_n b_n + d_n \cdot d_n + b_n \gamma_n - \gamma_n b_n) \right) \] (4.9)
is the zeroth component of the Virasoro operator. This insertion has the effect of
transforming the oscillators \( \mathcal{O}^{(2)}_n \) in the ket vertices into \( \mathcal{O}^{(2)}_n e^{-\tau r} \). After calculating
the product we finally take the limit \( \tau \rightarrow +0 \).

First we evaluate the product of the R-SG part vertices. This can be done by
using the following formula shown by the coherent state technique. In matrix notation it is

\[ \langle 0 | \exp\{ (\gamma | U_2 | \beta) + (\beta | V_2) + (\gamma | W_2) \} \exp \{ (\beta^* | U_1 | \gamma^*) + (\gamma^* | V_1) + (\beta^* | W_1) \} | 0 \rangle = (\det M)^{-1} \exp\{ (V_1 | M^{-1} | W_2) \\
+ (V_2 | U_1 M^{-1} | W_2) \\
+ (V_1 | M^{-1} U_2 | W_1) \\
+ (V_2 | (1 + U_1 M^{-1} U_2) | W_1) \} , \]

Fig. 5. The coincident interaction points are split by \( \tau \).

where components of the vectors \( |\gamma\rangle, |\gamma^*\rangle, |\beta\rangle \) and \( |\beta^*\rangle \) satisfy

\[ [\gamma_n, \gamma_{m}^*] = [\beta_n, \beta_m^*] = \delta_{n,m} . \]

In this case these vectors are taken as

\[ |\gamma\rangle = \begin{pmatrix} \gamma_0^{(2)} \\ \gamma_1^{(2)} \\ \gamma_2^{(2)} \\ \vdots \end{pmatrix} , \quad |\gamma^*\rangle = \begin{pmatrix} \beta_0^{(2)} \\ \beta_1^{(2)} \\ \beta_2^{(2)} \\ \vdots \end{pmatrix} , \quad |\beta\rangle = \begin{pmatrix} -\beta_1^{(2)} \\ \vdots \end{pmatrix} , \quad |\beta^*\rangle = \begin{pmatrix} \gamma_0^{(2)} \\ \gamma_1^{(2)} \\ \gamma_2^{(2)} \\ \vdots \end{pmatrix} , \]

respectively. Then since the matrix \( U_2 U_1 \) becomes diagonal, the calculation is straightforward and we obtain

\[ \langle \langle (\hat{R} \cdot \hat{S} G)^2 | e^{-\tau \cdot (\hat{S} G)} | (\hat{R} \cdot \hat{S} G)^2 \rangle \gamma_{23} \]

\[ = (1 - e^{-\tau}) \left\{ -\frac{1}{2} \exp\{ \sum_{n \geq 0} \beta_n^{(2)} \gamma_n^{(1)} - \sum_{n \geq 1} \gamma_n^{(2)} \beta_n^{(1)} \} e^{-\tau} \right. \]

\[ - (1 - e^{-\tau})^2 \gamma^* \beta^* (\sigma_0, \tau) \left| \frac{1}{2} \right|^3 , \]

where

\[ \gamma^* \beta^* (\sigma_0, \tau) = (\sum_{n \geq 0} \gamma_n^{(1)} e^{n(\tau + i\sigma_0)}) (\sum_{n \geq 0} \beta_n^{(2)} e^{-n(\tau + i\sigma_0)}) + (\sum_{n \geq 1} \gamma_n^{(2)} e^{-n\tau}) (\sum_{n \geq 1} \beta_n^{(1)} e^{n\tau}) \]

\[ + (\sum_{n \geq 0} \gamma_n^{(1)} e^{n\tau}) (\sum_{n \geq 0} \beta_n^{(2)} e^{-n\tau}) + (\sum_{n \geq 1} \gamma_n^{(2)} e^{-n\tau}) (\sum_{n \geq 1} \beta_n^{(1)} e^{n\tau}) . \]

Therefore we have

\[ \lim_{\tau \to 0} \{ (1 - e^{-\tau})^{-1} \langle \langle (\hat{R} \cdot \hat{S} G)^2 | e^{-\tau \cdot (\hat{S} G)} | (\hat{R} \cdot \hat{S} G)^2 \rangle \gamma_{23} \}
\]

\[ = \left\{ -\frac{1}{2} \exp(D_{\tau}(1, 3)) \left| \frac{1}{2} \right|^3 \right\} \]

\[ (4.15) \]
with
\[
D_\tau^R(1, 3) = \sum_{n \geq 0} \beta_-(^{(3)} \gamma_n^{(1)}) - \sum_{n \geq 1} \gamma_-(^{(3)} \beta_n^{(1)}).
\] (4·16)

Equation (4·13) just takes the form expected from the operator product expansion (4·7). Further by shifting \(|W_2\rangle\) and \(|V_1\rangle\) into
\[
|W_2\rangle = |W_2\rangle + \begin{pmatrix} x \\ 0 \\ 0 \\ \vdots \end{pmatrix}, \quad |V_1\rangle = |V_1\rangle + \begin{pmatrix} y \\ 0 \\ 0 \\ \vdots \end{pmatrix},
\] (4·17)

and differentiate with respect to \(x\) and/or \(y\), we can calculate the product of the vertices with \(\gamma_0^{(2)}\) and/or \(\beta_0^{(2)}\) inserted between. Thus we obtain
\[
\lim_{r \to 0} [(1 - e^{-r})^{-1} 1_{12} \langle \hat{R} \cdot \hat{S} \rangle^2 |e^{-i \tau \hat{L}}(\beta_0^{(3)} + i \beta_0^{(2)})| (\hat{R} \cdot \hat{S})^2 \rangle_{23}]
\]
\[
= \langle \left( - \frac{1}{2} \right) \exp[D_\tau^R(1, 3)] \left( \frac{1}{2} \right) \rangle \gamma^{(1)}(\sigma_0),
\]
\[
\lim_{r \to 0} [(1 - e^{-r})^{-1} 1_{12} \langle \hat{R} \cdot \hat{S} \rangle^2 |(\gamma_0^{(1)} + i \gamma_0^{(2)}) e^{-i \tau \hat{L}}(\beta_0^{(3)} + i \beta_0^{(2)})| (\hat{R} \cdot \hat{S})^2 \rangle_{23}]
\]
\[
= \langle \left( - \frac{1}{2} \right) \exp[D_\tau^R(1, 3)] \left( \frac{1}{2} \right) \rangle \gamma_0^{(1)}(\sigma_0),
\]
\[
\lim_{r \to 0} [(1 - e^{-r})^{-1} 1_{12} \langle \hat{R} \cdot \hat{S} \rangle^2 |( - i ) \frac{d \beta^{(1)}}{d \sigma}(\sigma_0) e^{-i \tau \hat{L}}(\beta_0^{(3)} + i \beta_0^{(2)})| (\hat{R} \cdot \hat{S})^2 \rangle_{23}]
\]
\[
= \langle \left( - \frac{1}{2} \right) \exp[D_\tau^R(1, 3)] \left( \frac{1}{2} \right) \rangle \left( \frac{d \beta^{(1)}}{d \sigma}(\sigma_0) \right)^2,
\] (4·18)

which can also be understood to reflect the operator product expansions properly as seen from Eqs. (B·3b), (3·47), (3·28), (3·29), (2·3) and (2·4). From Eqs. (4·15) and (4·18) we can see that only the second term of the prefactor (3·49c) survive the limit \(r \to 0\) since the product of \(- i \frac{d \beta^{(2)}}{d \sigma}(\sigma_0)\) and \(e^{(2)}(\sigma_0 - i r)\) has the singularity \((1 - e^{-r})^{-2}\). Thus by taking into account the extra phase appearing from
\[
\langle 0 | \langle \bar{a} | (\bar{a} a)^2 | 3\rangle_{\bar{a}},
\]
we finally obtain
\[
1_{12} \langle \hat{R} \rangle \langle \hat{R} \rangle_{23} = \lim_{r \to 0} 1_{12} \langle \hat{R} \rangle \langle e^{-i \tau \hat{L}} | (\hat{R})^2 \rangle_{23}
\]
\[
= \mathcal{P} \left[ \sum_{u=\bar{a},a} (-)^{|w|} \langle u | \exp \{ D_B(1, 3) + D^R_B(1, 3) + D^R_R(1, 3) \} | u \rangle_3 ,
\]

\[
D^B_R(1, 3) = \sum_{n \geq 1} d^{(3)}_n \cdot d^{(1)}_n , \quad (-)^{|w|} = \begin{cases} 
1 & \text{for } u = \bar{a} , \\
-1 & \text{for } u = a .
\end{cases}
\] (4.20)

This satisfies the property (1.3b) and completes the order \( g \) gauge invariance proof.

§ 5. Reality condition on string fields

In this section we comment on the reality condition imposed on string fields. Although our interest lies in the part related with the R-SG, we treat the other parts for completeness.

The reality condition should be set by using 2-string vertex in the form: \(^{(1,1,2)}\)

\[
(2\langle \Phi |)^{1} = i \langle \Phi | V_{(3)} \rangle_{12} .
\] (5.1)

As for the physical part, vertices (3.16)~(3.18) are made use of without any problem. On the other hand, as for the (super)ghost parts, vacuum states on which vertices are constructed are different from the previous ones in the case of the RG and R-SG. Since the ghost vacua \( \langle +1 |, \langle -1 | \) and \( \langle -1/2 | \) are taken into \( | +1 \rangle = | \tilde{+2} \rangle, | -1 \rangle = | \tilde{-1} \rangle \) and \( | -1/2 \rangle = | \tilde{-3/2} \rangle \), respectively, by the hermitian conjugation, 2-string vertices should be constructed on \( | +1 \rangle | \tilde{+2} \rangle_2, | \tilde{-1} \rangle | \tilde{-1} \rangle_2 \) and \( | -1/2 \rangle | \tilde{-3/2} \rangle_2 \) for the RG, NS-SG and R-SG, respectively. Thus we are led to construct vertices by taking

\( p_1 = +1, \quad p_2 = +2 \) and \( q = q' = 0 \) \quad for RG , \hspace{1cm} (5.2a)

\( p_1 = p_2 = -1 \) \quad and \( q = q' = 0 \) \quad for NS·SG , \hspace{1cm} (5.2b)

\( p_1 = -\frac{1}{2}, \quad p_2 = -\frac{3}{2} \) and \( q = q' = 0 \) \quad for R·SG . \hspace{1cm} (5.2c)

In these cases we do not need ghost sources at the interaction points \( (q = q' = 0) \).

Since (5.2b) is again the same as (3.7b), only RG and R-SG vertices should be given here. From Eqs. (A.8) and (A.11), we obtain

\[
| (\text{RG})^x \rangle_{12} = \exp \{ R_c(1, 2) \} | +1 \rangle | +2 \rangle_2 ,
\] (5.3)

\[
R_c(1, 2) = \sum_{n \geq 1} (-)^{n+1} (b^{(1)}_n c^{(2)}_n + b^{(2)}_n c^{(1)}_n) - b^{(1)}_0 c^{(2)}_0 ,
\]

\[
| (\text{R·SG})^x \rangle_{12} = \exp \{ R^R(1, 2) \} \left| \tilde{\frac{1}{2}} \right|_1 \left| \tilde{\frac{3}{2}} \right|_2 ,
\] (5.4)

\[
R^R(1, 2) = \sum_{n \geq 1} (-)^n i (\beta^{(1)}_{-n} \gamma^{(2)}_n - \beta^{(2)}_{-n} \gamma^{(1)}_n) + i \beta^{(1)}_0 \gamma^{(2)}_0 .
\]

These vertices satisfy the connection conditions (3.11) and (3.13) with the \( \delta \)-function singularity replaced by 0. Note that the RG vertex (5.3) can be rewritten as

\[
| (\text{RG})^x \rangle_{12} = -| (\text{RG}_0)^x \rangle_{12} = | (\text{RG}_0)^x \rangle_{12}
\] (5.5)
by using the relation
\[ |\bar{1}\rangle = c_0 |\bar{2}\rangle, \quad |\bar{2}\rangle = b_0 |\bar{1}\rangle. \quad (5\cdot6) \]

Using the above vertices, reality condition for NS-field $\langle \phi \rangle$ and R-field $\langle \psi \rangle$ is imposed by
\[
(\langle \phi \rangle)^* = 1 \langle \phi | (NS)_R^{2}\rangle_\odot_{12}, \quad (5\cdot7a)
\]
\[
(\langle \psi | d_0^{(2)} \rangle)^* = 1 \langle \psi | (R)_R^{2}\rangle_\odot_{12}, \quad (5\cdot7b)
\]
\[
|(NS)_R^{2}\rangle_\odot_{12} = \mathcal{P}_R^{(1)} \mathcal{P}_R^{(2)} |(\text{Phys})^2\rangle_\odot_{12} \otimes |(RG)_R^{2}\rangle_\odot_{12} \otimes |(NS)_R^{2}\rangle_\odot_{12} \otimes |(NS\cdot SG)^2\rangle_\odot_{12}, \quad (5\cdot8a)
\]
\[
|(R)_R^{2}\rangle_\odot_{12} = \mathcal{P}_R^{(1)} \mathcal{P}_R^{(2)} |(\text{Phys})^2\rangle_\odot_{12} \otimes |(RG)_R^{2}\rangle_\odot_{12} \otimes |(R_s)^2\rangle_\odot_{12} \otimes |(R\cdot SG)_R^{2}\rangle_\odot_{12}, \quad (5\cdot8b)
\]

where zero mode $d_0^0$ of the fermionic coordinate $\varphi^0(\sigma)$ is inserted in $(5\cdot7b)$, which is necessary for imposing hermiticity on spinor component fields. From Eq. $(5\cdot7)$, we can see, for example, massless vector field $A_\mu$ in Eq. $(4\cdot1)$ becomes purely imaginary and massless spinor field $\psi_\alpha$ becomes real in the usual representation $C = \Gamma^0$.

§ 6. Conclusions and Discussion

In this paper operator expression of the action and the gauge transformation in Witten's superstring field theory has been completed in the fermionic representation of the (super)ghost. Vertices in the formulation have been constructed by making use of the technique in the conformal field theory. Then the order $g$ gauge invariance has been proved based on this operator expression. Since the order $g^0$ invariance follows from the nilpotency of the BRST charge at the critical dimension\(^{23}\), the gauge invariance was shown up to order $g^2$.

Strictly speaking, however, the order $g$ invariance proof was done by fixing the order of multiplication of the vertices. In the R sector in which SG part vertices take a bit nontrivial form, the "associativity" of the multiplication should be investigated. This is also a complication owing to the picture changing operators $X$ and $Y$.

Further in order to complete the gauge invariance proof, it is necessary to demonstrate the associativity of the 3-string $\ast$-product rigorously, which has not been done even for the bosonic string theory. For this purpose we will have to exploit a powerful method for evaluating determinants and products of infinite dimensional matrices.\(^*)\)

On the other hand, this associativity is expected to break down also in the superstring field theory when string states are taken as bad states outside the open string Fock space\(^{27}\). Relating with this phenomena, closed superstrings appearing from open superstrings are an interesting subject.\(^{28}\)

Another interesting topic is the supersymmetry in the context of string field theory. This seems easier to be investigated in the bosonized formulation.\(^2\) There-

\(^*)\) In this respect, the author was informed that LeClair, Peskin and Preitschopf\(^{26}\) have been exploiting a technique based on the conformal field theory (see also Refs. 25 and 26)).
fore it will be important to connect directly the fermionic formulation with the bosonized formulation.

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Appendix A

Operator Expression of the Vertex

In this appendix we give a concrete operator expression of the vertex by using the correlation function (2.7).

The $z$-plane fields $C(z)$ and $B(z)$ are represented by the oscillator modes of the $r$-th string via intermediate $\rho$-plane fields as

$$C(\rho) = \left(\frac{d\rho}{dz}\right)^{-h} C(z), \quad B(\rho) = \left(\frac{d\rho}{dz}\right)^{-h} B(z), \quad (A.1)$$

$$C(\rho) = \left(\frac{d\rho}{dz}\right)^{-h} \mathcal{C}(r) \left(\sigma_r - i\xi_r\right) = \epsilon_r(\sigma_r) \alpha_r^{-h} \sum_{n=-\infty}^{\infty} C_{hr}^{(r)} e^{n\tau r}, \quad (A.2)$$

$$B(\rho) = \left(\frac{d\rho}{dz}\right)^{-h} \mathcal{B}(r) \left(\sigma_r - i\xi_r\right) = \epsilon_r(\sigma_r) \alpha_r^{-h} \sum_{n=-\infty}^{\infty} B_{hr}^{(r)} e^{n\tau r}, \quad (A.2)$$

where $\rho(z)$ is assumed to lie in the $r$th string region and represented as

$$\rho(z) = \alpha_r \xi_r + \beta_r,$$

$$\xi_r = \xi_r + i\sigma_r; \quad \xi_r \leq 0, \quad \left\{ \begin{array}{ll} 0 \leq \sigma_r \leq \pi & (\text{Im } \rho \geq 0), \\ -\pi \leq \sigma_r \leq 0 & (\text{Im } \rho \leq 0), \end{array} \right. \quad (A.3)$$

$$\alpha_r = \pm 1, \quad \beta_r = \pi \theta(-\alpha_r) \text{sgn} \sigma_r$$

with the integration constant for Eq. (2.1) fixed by

$$\rho(\omega_0) = i\omega_0; \quad \omega_0 = \frac{\pi}{2}. \quad (A.4)$$

In Eq. (A.2), $n_r$ takes integral values for the reparametrization ghost and the R superghost and half-integral values for the NS superghost. In the case of the superghost ($h=3/2$), the cut structure of $(d\rho/dz)^{1/2}$ requires the sign factor $\epsilon_r(\sigma_r)$, which can be taken as

$$\epsilon_r(\sigma_r) = \begin{cases} (-)^{\theta(-\sigma_r) \theta(\sigma_r)} & r \text{th string } = \text{NS}, \\ 1 & r \text{th string } = \text{R} \end{cases} \quad (A.5)$$

for $N \leq 3$ by a suitable choice of the cut structure. We must further fix the phases of $\alpha_r^{1/2}$ in order to obtain a definite expression. We can freely fix them, however, since any choice leads to the same result in the presence of the covariantized GSO projection operator.\(^{19,20}\)
Since the external vacuum states are taken as\(^{(16, *)}\)

\[ p_r |0\> = r |0\> e^{br(z_r)}, \]

\[ p_r |C_{-r}\> = 0 \quad n_r \geq h - \epsilon p_r \equiv m_0^{(r)}, \quad (A.6) \]

\[ p_r |B_{-r}\> = 0 \quad n_r \geq (1 - h) + \epsilon p_r \equiv n_0^{(r)}, \]

the vertex is constructed on the adjoint vacuum \( |0\>^{p_r} \):

\[ |0\>^{p_r} \propto (-q - p_r |0\>)^t; \quad p_r |0\>^{p_r} = 1, \quad (A.7) \]

where \( \dagger \) denotes to take the hermitian conjugation. Note that \( p_r \) takes integral values for the RG and the NS-SG and half-integral values for the R-SG. Then the vertex is given by using the Fourier component of the \( \rho \)-plane correlation function as\(^{(6, 3)}\)

\[ |V(1, \cdots, N)\> = \exp \{E(1, \cdots, N)|0\>^{p_1} \cdots |0\>^{p_N} , \]

\[ E(1, \cdots, N) = \sum_r \sum_{s=1}^{N} \sum_{m_0^{(r)} m_0^{(s)}} N^{rs}_{n_r m_s} B_{-r}^{(s)} C_{-m_s}^{(s)}, \quad (A.8) \]

where the Fourier components \( N^{rs}_{n_r m_s} \) are defined by\(^{(**)}\)

\[ N(\rho(z), \bar{\rho}(\bar{z})) = \left( \frac{d\rho(z)}{dz} \right)^{-1} \left( \frac{d\bar{\rho}(\bar{z})}{d\bar{z}} \right)^{-1} N(z, \bar{z}) \]

\[ = \left[ -\frac{1}{a_r} \delta_{rs} \left( \theta(\xi_s - \xi_r) \sum_{n_r} e^{-n_r (\xi_s - \bar{s})} - \theta(\xi_r - \xi_s) \sum_{m_s} e^{-m_s (\xi_r - \xi_r)} \right) \right. \]

\[ - a_r^{-1} \delta_{rs} \sum_{n_r} e^{-n_r (\xi_r - \bar{s})} \sum_{m_s} N^{rs}_{n_r m_s} e^{n_r \xi_r + m_s \xi_s} \] \( \beta_r(\sigma_r) \beta_s(\bar{\sigma}_s) \), \quad (A.9) \]

\[ N(z, \bar{z}) = -\frac{1}{z - \bar{z}} \left( \prod_{t=1}^{N} \left( \frac{z - z_t}{\bar{z} - z_t} \right)^{\epsilon p_t} \right) \left( \frac{z - z_0}{\bar{z} - z_0} \right)^{\epsilon q} \left( \frac{z - z_0^*}{\bar{z} - z_0^*} \right)^{\epsilon q} \exp \left( \frac{z - z_0}{\bar{z} - z_0} \right)^{\epsilon p_t} \exp \left( \frac{z - z_0^*}{\bar{z} - z_0^*} \right)^{\epsilon q}, \quad (A.10) \]

In Eq. (A.10), \( \rho(z) \) and \( \bar{\rho}(\bar{z}) \) are assumed to belong to the \( r \)th and \( s \)th string regions, respectively. By making the Fourier transformation, \( N^{rs}_{n_r m_s} \) are represented by the following integration formula:

\[ N^{rs}_{n_r m_s} = -\alpha_r^{-1} a_r^{-1} \int_{\mathbb{R}} \int_{\mathbb{R}} e^{-n_r \xi_r m_s \xi_s} N(z, \bar{z}) \left( \frac{d\rho(z)}{dz} \right)^{-1} \left( \frac{d\bar{\rho}(\bar{z})}{d\bar{z}} \right)^{-1} \quad (A.11) \]

Here \( \xi_r \) (and similarly \( \xi_s \)) is given by Eq. (A.3) with the \( \beta_r \) replaced by \( \pi \text{sgn} \sigma_r \) as a result of absorbing the sign factor.

It is easy to show that the vertex (A.8) satisfies

\[ p_1 |0\> \cdots p_N |0\>^{TC(x) B(z)} |V(1, \cdots, N)\> = N(z, \bar{z}), \quad (A.12) \]

where \( T \) denotes taking the inverse time ordering with respect to \( \xi_r \) and \( \bar{\xi}_r \) when \( z \) and \( \bar{z} \) are in the same \( r \)th string region. We can also easily show that the vertex satisfies a proper connection condition from the single-valuedness of the correlation.

\(^{*}\) The hermitian conjugate expressions are made use of in the usual convention.

\(^{(**)}\) The minus sign in \( N(z, \bar{z}) \) is due to the convention of the inverse time ordering.\(^{6, 3}\)
function at the boundary of the string regions.\(^{(11,6,3)}\)

**Appendix B**

**Prefactor and BRST Invariance of \(|(\bar{R})^2\rangle_{12}\)**

In this appendix, we find an appropriate prefactor of \(|(\bar{R})^2\rangle_{12}\) (the naive part of \(|(\bar{R})^2\rangle_{12}\)) and prove the BRST invariance of \(|(\bar{R})^2\rangle_{12}\) by using the contour integration method.\(^{(11,16,3)}\) Since the procedure is quite similar to that in Refs. 18) and 3), we omit the details of the calculation.

As usual the sum of the BRST charge \(\sum_{r=1}^{2} Q_{\theta}^{(r)}\) can be rewritten as

\[
\sum_{r=1}^{2} Q_{\theta}^{(r)} \langle (\bar{R})^2 \rangle_{12}
\]

\[
= \int_{C_{12}} \frac{d\rho}{2\pi i} \left[ c(\rho) \left\{ -\frac{1}{2} A(\rho)^2 + \frac{dc}{d\rho} b(\rho) - \frac{1}{2} \phi(\rho) \cdot \frac{d\phi}{d\rho} - \frac{d\eta}{d\rho} \beta(\rho) \right\} 
+ \gamma(\rho) \left\{ -\phi(\rho) \cdot A(\rho) + b(\rho) \gamma(\rho) + \frac{1}{2} \frac{dc}{d\rho} \beta(\rho) \right\} \right] \langle (\bar{R})^2 \rangle_{12}
\]

\[
= \int_{C_{12}} \frac{dz}{2\pi i} \left[ \frac{1}{z-\bar{z}_0} \tilde{c}(z) \left\{ -\frac{1}{2} A(z)^2 + \frac{d\tilde{c}}{dz} \tilde{b}(z) - \frac{1}{2} \phi(z) \cdot \frac{d\phi}{dz} - \frac{d\eta}{dz} \beta(z) \right\} 
+ \frac{1}{z-\bar{z}_0} \tilde{\gamma}(z) \left\{ -\phi(z) \cdot A(z) + \tilde{b}(z) \tilde{\gamma}(z) + \frac{1}{2} \frac{d\tilde{c}}{dz} \tilde{\beta}(z) + \frac{1}{2} \frac{1}{z-\bar{z}_0} \tilde{c}(z) \tilde{\beta}(z) \right\} 
+ \left( \frac{d\rho}{dz} \right)^{-1} \left( \frac{d^2\rho}{dz^2} \right) \frac{1}{4(z-\bar{z}_0)} \tilde{c}(z) \phi(z) \cdot \phi(z) \right] \langle (\bar{R})^2 \rangle_{12},
\]

where we have introduced new operators

\[
\tilde{c}(z)=(z-\bar{z}_0) c(z), \quad \tilde{b}(z)=\frac{1}{z-\bar{z}_0} b(z), \quad \tilde{\gamma}(z)=(z-\bar{z}_0) \gamma(z), \quad \tilde{\beta}(z)=\frac{1}{z-\bar{z}_0} \beta(z),
\]

which are regular at \(z=\bar{z}_0\) on the vertex. The integration contours in Eqs. (B·1) and (B·2) are depicted in Figs. 6 and 7, respectively. Here the \(z\)-plane contour could be deformed into a small circle surrounding \(\bar{z}_0\), since the singularity exists only at \(z=\bar{z}_0\).

The integrand in Eq. (B·2) is then rewritten in terms of the creation operators by eliminating the annihilation operator parts by passing them through to the vacuum in the vertex. This yields contractions between the operators which give the \(z\)-plane correlation functions. In this case we have

\[
A^\mu(z) A^\nu(z) = \frac{1}{(z-\bar{z})^2} \eta^{\mu\nu}, \quad \eta^{\mu\nu}.
\]

\[
\bar{c}(z) \bar{b}(z) = \frac{1}{z-\bar{z}} \frac{(z-\bar{z}_1)(z-\bar{z}_2)}{(\bar{z}-\bar{z}_1)(\bar{z}-\bar{z}_2)} \]

\[
= \frac{1}{z-\bar{z}} + \sum_{n=0}^{\infty} I_n(z)(\bar{z}-z)^n,
\]

\[
\tilde{c}(z) \phi(z) \cdot \phi(z) = \frac{(z-Z_1)(z-Z_2)}{(\bar{z}-Z_1)(\bar{z}-Z_2)} \]

\[
\tilde{\gamma}(z) \tilde{\beta}(z) = \frac{1}{z-\bar{z}} + \sum_{n=0}^{\infty} \bar{I}_n(z)(\bar{z}-z)^n.
\]
where we have defined \( I_n, K_n \) and \( J_n \) by Taylor expansion. The operator \( \mathcal{O}(z) \) which survives the contraction is replaced by
\[
\mathcal{O}^{(-)}(z) + [\mathcal{O}^{(+)}(z), E(1, 2)],
\]
where \( \mathcal{O}^{(+)} \) and \( \mathcal{O}^{(-)} \) denote the annihilation and creation operator part of \( \mathcal{O} \), respectively, and \( E(1, 2) \) is the exponent operator in the vertex defined in § 3. We represent the operator (B·5) by the same symbol \( \mathcal{O}(z) \) for notational simplicity.

First of all we have no-contraction terms which take the same form as (B·2) but are written only by the creation operator (B·5). They contribute to the contour integration by \( Q^{(0)}(\bar{R}_0, 2\gamma_{12}) \):

\[
Q^{(0)} = - \left[ \bar{c}(z_0) \left\{ - \frac{1}{2} A(z_0) + \frac{d\bar{c}}{dz}(z_0) \bar{b}(z_0) - \frac{1}{2} \psi(z_0) \cdot \frac{d\psi}{dz}(z_0) - \frac{d\bar{\psi}}{dz}(z_0) \bar{\beta}(z_0) \right\} \right.
\]

\[
+ \bar{\gamma}(z_0) \left\{ - \psi(z_0) \cdot A(z_0) + \bar{b}(z_0) \bar{\gamma}(z_0) + \frac{d\bar{c}}{dz}(z_0) \bar{\beta}(z_0) \right\} \]

\[
\left. + \frac{1}{2} \bar{c}(z_0) \left\{ \frac{d\bar{\gamma}}{dz}(z_0) \bar{\beta}(z_0) + \bar{\gamma}(z_0) \frac{d\bar{\beta}}{dz}(z_0) \right\} \right].
\]

From this we are led to the prefactor
\[
\bar{X}_1(z_0) = \bar{c}(z_0) \left\{ - \psi(z_0) \cdot A(z_0) + \bar{b}(z_0) \bar{\gamma}(z_0) + \frac{d\bar{c}}{dz}(z_0) \bar{\beta}(z_0) \right\}.
\]

The no-contraction terms (B·6) vanish on the vertex \( \bar{X}_1(z_0) \) owing to the Grassmann property \( \{ c(z_0) \}^2 = - \{ \psi(z_0) \cdot A(z_0) + \cdots \}^2 = 0 \). Note that \( \bar{X}_1(z_0) \) roughly takes the form expected from the argument in § 2. The product of \( \bar{X}_1(z_0) \) with the ghost source \( b(z_0) e^{\bar{x}(z_0)} \) included in the vertex partly reproduces the picture changing operator \( X(z_0) \).

Next we evaluate the one-contraction terms in (B·2) with prefactor (B·7) put in
front of \(|\langle \tilde{R}_0 \rangle^2 \rangle_{12}\). For the terms for which the contraction is made inside the integrand, we need a regularization to treat the contraction of operators at a coincident point properly. This is made as usual by going back to the expression (B·1) in \(\rho\)-variables and shifting the coordinate \(\rho(z)\) of one of the contracted operators to \(\rho(z') = \rho(z) + \delta\). This \(\delta\) is put to zero after the integral is evaluated. From the lengthy but straightforward calculation, we obtain the contribution \(Q^{(1)}(\tilde{R}_0)^2 \rangle_{12}\) of the one-contraction terms:

\[
Q^{(1)} = -Q^{(0)} \times \left( \frac{2}{d} \frac{d^2}{dz^2}(z_0) + I_0(z_0) \tilde{\gamma}(z_0) \right) ,
\]

where \(I_0\) is the function defined in Eq. (B·4b). This shows that the vertex \(\tilde{X}_1(z_0) \times \langle |\tilde{R}_0 \rangle^2 \rangle_{12}\) is still not BRST invariant. It should be noted, however, that \(Q^{(1)}\) is proportional to \(Q^{(0)}\) encountered in the calculation of the no-contraction terms. Therefore if we modify our prefactor from \(\tilde{X}_1(z_0)\) to

\[
\tilde{P}_b(z_0) = \tilde{X}_1(z_0) + \tilde{X}_2(z_0) ,
\]

\[
\tilde{X}_2(z_0) = 2 \frac{d^2}{dz^2}(z_0) + I_0(z_0) \tilde{\gamma}(z_0) ,
\]

then the one-contraction contribution \(Q^{(1)}(\tilde{R}_0)^2 \rangle_{12}\) is canceled by the no-contraction terms on \(\tilde{X}_2(z_0)|\langle \tilde{R}_0 \rangle^2 \rangle_{12}\).

Finally we must show that the vertex

\[
|\langle \tilde{R} \rangle^2 \rangle_{12} = \tilde{P}_b(z_0)|\langle \tilde{R}_0 \rangle^2 \rangle_{12}
\]

is BRST invariant. For this it is sufficient to evaluate the two-contraction terms on \(\tilde{X}_1(z_0)|\langle \tilde{R}_0 \rangle^2 \rangle_{12}\) and the one-contraction terms on \(\tilde{X}_2(z_0)|\langle \tilde{R}_0 \rangle^2 \rangle_{12}\). We have from the straightforward calculation

\[
\sum_{\tau = 1}^2 Q^{(\tau)}(\tilde{R}_0)^2 \rangle_{12} = (d - 10) \left[ \left\{ e_1 e_2 - \frac{3}{4} e_3 - \frac{3}{8} (e_1)^3 + \left( \frac{1}{4} e_2 - \frac{3}{16} (e_1)^2 \right) I_0(z_0) \right\} \tilde{c}(z_0) \tilde{\gamma}(z_0) - \frac{1}{6} \tilde{c}(z_0) \frac{d^3}{dz^3}(z_0) \right] |\langle \tilde{R}_0 \rangle^2 \rangle_{12} ,
\]

where \(b, c\) and \(e\) are defined by the expansion of \(\rho(z)\) around \(z = z_0\):

\[
\rho(z) - \rho(z_0) = a(z-z_0) \left\{ 1 + \frac{1}{2} e_1 (z-z_0) + \frac{1}{3} e_2 (z-z_0)^2 + \frac{1}{4} e_3 (z-z_0)^3 + \cdots \right\} .
\]

Equation (B·12) shows that the vertex \(|\langle \tilde{R} \rangle^2 \rangle_{12}\) is indeed BRST invariant at the space-time dimension \(d = 10\).

**Appendix C**

--- **Correspondence with Yamron's Ramond Kinetic Term** ---

In this appendix we show that our R kinetic term defined by using the 2-R vertex (3·30) just corresponds with Yamron's one\(^7\) when the reality condition (5·7b) is imposed. This is done by relating the R-SG vertex (3·10) with (5·4). For this purpose we change the representation of the R-SG zero mode into the coordinate
Using the relation
\[
\left| -\frac{1}{2} \right> = \delta(\gamma_0) \left| -\frac{3}{2} \right>, \tag{C.1}
\]
we can rewrite the vertex (3.10) as
\[
| (R \cdot SG) \rangle_{12} = \exp\left\{ - (\beta_0^{(2)} + i\beta_0^{(1)})(\gamma^{(2)}(\sigma_0) - \gamma^{(2)}) \right\}
\times \exp\left\{ \sum_{n \geq 1} (-)^n i(\beta_n^{(1)} \gamma^{(2)}_n - \beta_n^{(2)} \gamma^{(1)}_n) \right\} \delta(\gamma^{(2)}) \left| -\frac{1}{2} \right> \left| -\frac{3}{2} \right>, \tag{C.2}
\]
where we have used \( \pi \sigma_0 = \sigma_0 \) and \( \beta_0^{(2)} = -\partial/\partial \gamma^{(2)} \). The term \(- i\beta_0^{(1)} \gamma^{(2)}(\sigma_0)\) has vanished in the second equality owing to the \( \delta \)-function. Note that we have just extracted the ghost source \( e^{-\gamma(\sigma_0)} \sim \delta(\gamma(\sigma_0)) \) in the vertex, which corresponds to Eqs. (3.14) and (5.5). Now 2-R vertex (3.30) is written as
\[
| (R)^2 \rangle_{12} = - c^{(2)}(\sigma_0) \delta'(\gamma^{(2)}(\sigma_0)) | (R) \rangle_{12}. \tag{C.3}
\]
Therefore the R kinetic term in Eq. (1.1) can be represented as
\[
\langle \Psi | Q \Psi | \Psi \rangle,
\]
\[
Y = c(\sigma_0) \delta'(\gamma(\sigma_0)), \tag{C.4}
\]
when the reality condition (5.7b) is imposed. Equation (C.3) is just the one given by Yamron.

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