Collective Pair Excitations in Dynamical Nuclear Field Theory. II

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(Received July 23, 1987)

For the study of collective excitations in a superconducting nuclear system with the pairing force, the dynamical nuclear field theory (DNFT) is extended so as to include higher-order effects. Effective number and pair-creation operators are evaluated up to second-order in the amplitudes of the pair rotation or vibration. The number equation and the self-consistency condition are extended to the same order accordingly, while the effective Hamiltonian is calculated up to third-order. Physical interpretation of the results becomes easy if renormalized collective parameters are used. Decoupling of the pair rotation and the pair vibration is shown to persist in the higher-order approximation.

§ 1. Introduction

In previous papers\(^1,2\) (from now on, referred to as I and II, respectively), we have studied collective excitations of a nuclear system with the pairing force by the dynamical nuclear field theory (DNFT) in the lowest-order approximation. Collective modes are composed of the pair rotation and the pair vibration. They have the same properties as those of the random phase approximation,\(^3\) but now are evaluated in the basis system taken as the starting point for the perturbative expansion. In this paper we discuss corrections for these results by applying DNFT extended to higher-order.

As will be shown in § 2, two collective parameters, the energy gap and the chemical potential, are expanded up to second-order in the small amplitudes of the pair rotation and vibration, and thus include terms up to quadratic phonon numbers. Consequently, the perturbing interaction between a particle and collective modes in the driving Hamiltonian consists of the first- and second-order terms.

By using this interaction we calculate effective operators for the pair-creation and the nucleon number up to second-order in § 3. In § 4 we postulate the self-consistency between the former and the energy gap. We demand rather drastically all the second-order terms of the effective number operator to vanish. The coefficients of these conditions include complicated linear combinations of functions defined in Appendix A. In order to get some physical insight, we rewrite these equations more transparently in § 5. We find that renormalizations of the collective parameters are very much helpful.

In § 6 we calculate the effective Hamiltonian up to third-order by using the formula for the perturbative expansion given in Appendix C. We obtain two kinds of the interactions between the pair rotation and the pair vibration, linear and quadratic in phonon numbers. The former can be shown to vanish, while the latter can be brought to a renormalization correction for the second-order Hamiltonian. Finally conclusions are contained in § 7.
§ 2. Hamiltonians and collective parameters

As in I and II, we start with the Hamiltonian

$$H = H_{sp} - G\tilde{P}^\dagger\tilde{P},$$  \hspace{1cm} (2.1)$$

in which the single-particle Hamiltonian and the pair-creation operator are given by

$$H_{sp} = \sum_a e_a c_a^\dagger c_a, \quad \tilde{P}^\dagger = \sum_{a\neq 0} c_a^\dagger c_a^\dagger,$$  \hspace{1cm} (2.2)$$

respectively. The time-dependence of the Hartree-field Hamiltonian can be eliminated by transforming to the intrinsic frame. The resultant driving-Hamiltonian is decomposed as

$$\tilde{H} = H_0 + V.$$  \hspace{1cm} (2.3)$$

Here

$$H_0 = H_{sp} - \Delta_0 (\tilde{P}^\dagger + \tilde{P}) - \lambda_0 \tilde{N} + \omega b^\dagger b,$$  \hspace{1cm} (2.4)$$

$$V = - (\Delta - \Delta_0)^\dagger \tilde{P}^\dagger - (\Delta - \Delta_0) \tilde{P} - (\lambda - \lambda_0) \tilde{N},$$  \hspace{1cm} (2.5)$$

in which $\tilde{N}$ is the number operator and $\omega$ is the phonon energy ($\hbar = 1$).

In the present paper, we expand the differences $\Delta - \Delta_0$ and $\lambda - \lambda_0$ of Eq. (2.5) up to second-order in the small amplitudes of the pair vibration and the pair rotation. The difference of the energy gap consists of the rotational and vibrational parts, and their coupling

$$\Delta - \Delta_0 = (\text{rot}) + (\text{vib}) + (\text{coupl}).$$  \hspace{1cm} (2.6)$$

The rotational part is expressed as

$$(\text{rot}) = \delta \Delta + c_\delta (\delta \Delta)^2,$$  \hspace{1cm} (2.7)$$

in which $c_\delta (\delta \Delta)^2$ means a correction for $\delta \Delta$ determined in I in the lowest-order approximation. When the vibrational part is divided as

$$(\text{vib}) = \text{vib}[1] + \text{vib}[2],$$  \hspace{1cm} (2.8)$$

the first term is linear in $b^\dagger$ or $b$, as given in II

$$\text{vib}[1] = \delta \Delta_{\text{vib}}[1] = \xi (b^\dagger + b) - \eta (b^\dagger - b),$$  \hspace{1cm} (2.9)$$

while the second term, being quadratic in phonons, takes the general form

$$\text{vib}[2] = \delta \Delta_{\text{vib}}[2]$$
$$= \xi_{20} (b^\dagger b^\dagger + bb) - \eta_{20} (b^\dagger b^\dagger - bb) + \xi_{11} (b^\dagger b + bb^\dagger) - \eta_{11} (b^\dagger b - bb^\dagger).$$  \hspace{1cm} (2.10)$$

Because of the boson property of the phonon, the last term on the right-hand side of Eq. (2.10) expresses a constant shift of $\Delta$ arising from the pair vibration. The coupling of the pair rotation with the pair vibration may be expressed as
in which $\delta \xi$ and $\delta \eta$ are expected to be proportional to $\delta \mathcal{A}$.

Similarly we write $\lambda - \lambda_0$ as

$$\lambda - \lambda_0 = (\text{rot}) + (\text{vib}).$$

(2.12)

The rotational part is expanded as

$$(\text{rot}) = \delta \lambda + c_4 (\delta \lambda)^2,$$

in which $\delta \lambda$ is the first-order difference determined in I. To meet the restriction of DNFT that the velocity $\lambda = \phi$ does not vary in time, the vibrational part cannot contain a first-order contribution, but begins with a linear combination of $b^\dagger b$ and $bb^\dagger$

(we have associated the time dependence $e^{i\omega t}$ and $e^{-i\omega t}$ with $b^\dagger$ and $b$, respectively):

$$(\text{vib}) = \delta \lambda_{\text{vib}} [2] = \zeta_{11} (b^\dagger b + bb^\dagger) - \delta_{11} (b^\dagger b - bb^\dagger).$$

(2.14)

Corresponding to these expansions, we can write $V$ as


(2.15)

As given in II, the first-order interaction $V[1]$ consists of the rotational and vibrational parts


(2.16)

while the second-order interaction $V[2]$ includes also the coupling of the pair rotation with the pair vibration


(2.17)

Written out explicitly, they are

$$V_r[2] = - c_4 (\delta \mathcal{A})^2 (\hat{P}^\dagger + \hat{P}) - c_4 (\delta \lambda)^2 \hat{N},$$

$$V_v[2] = - \xi_{20} (\hat{P}^\dagger + \hat{P}) (b^\dagger b^\dagger + bb) - \eta_{20} (\hat{P}^\dagger - \hat{P}) (b^\dagger b^\dagger - bb)$$

$$- [\xi_{11} (\hat{P}^\dagger + \hat{P}) + \zeta_{11} \hat{N}] (b^\dagger b + bb^\dagger)$$

$$+ [\eta_{11} (\hat{P}^\dagger + \hat{P}) + \delta_{11} \hat{N}] (b^\dagger b - bb^\dagger),$$

$$V_c[2] = - \delta \xi (\hat{P}^\dagger + \hat{P}) (b^\dagger + b) - \delta \eta (\hat{P}^\dagger - \hat{P}) (b^\dagger - b).$$

(2.18)

It should be noted that the phonon energy $\omega$ of Eq. (2.4), which will be determined in the higher-order approximation, is different from the lowest-order value. The difference will be denoted as $\delta \omega$.

§ 3. Effective one-body operators

We here construct the effective operators for the creation of a pair and the number of quasiparticles in a state $\alpha$, denoted as $\hat{p}_\alpha^\dagger = a_\alpha^\dagger a_\alpha^\dagger$ and $\hat{n}_\alpha = a_\alpha^\dagger a_\alpha$, respectively. We aim at deriving a simple relation for the second-order $\hat{n}_{\alpha,\text{eff}}$.

First-order terms of $\hat{p}_{\alpha,\text{eff}}$ were given in II as
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\[ \tilde{\rho}_{a,\text{eff}}[1] = \text{rot}[1] + \text{vib}[1] \]  \quad (3.1)

with

\[ \text{rot}[1] = -\frac{V_a^{(20)}}{2E_a}, \quad \text{vib}[1] = \frac{W_a^{(11)}}{2E_a} b^* + \frac{W_a^{(20)}}{2E_a + \omega} b, \]  \quad (3.2)

in which \( E_a \) is the quasiparticle energy and the definitions of \( V_a^{(20)}, W_a^{(20)} \), etc. can be found in II. The interaction \( V[1] \) contributes to the second-order effective operator by acting twice, while the \( V[2] \) interaction gives terms from the first-order perturbation. Each contribution is given in the first and second lines on the right-hand side of the following equation, respectively:

\[ \tilde{\rho}_{a,\text{eff}}[2] = \text{rot}[2] + \text{vib}[2] + \text{coup}[2] + \text{rot}[2'] + \text{vib}[2'] + \text{coup}[2'], \]  \quad (3.3)

in which

\[ \text{rot}[2] = -\frac{V_a^{(20)} V_a^{(11)}}{2E_a^2}, \]  \quad (3.4a)

\[ \text{vib}[2] = b^* b^* \left\{ \frac{2W_a^{(02)} W_a^{(11)}}{(2E_a - \omega)(2E_a - \omega)} + bb \left( \frac{2W_a^{(20)} W_a^{(11)}}{2E_a + \omega(2E_a + \omega)} \right) \right. \]
\[ + b^* b \left\{ \frac{W_a^{(02)}(W_a^{(11)} - 2W_a^{(11)})}{(2E_a - \omega)2E_a} - \frac{W_a^{(20)} W_a^{(11)}}{2E_a(2E_a + \omega)} \right\} \]
\[ + bb^* \left\{ \frac{W_a^{(02)}(W_a^{(11)} - 2W_a^{(11)})}{2E_a(2E_a + \omega)} - \frac{W_a^{(20)} W_a^{(11)}}{2E_a(2E_a - \omega)} \right\}, \]  \quad (3.4b)

\[ \text{coup}[2] = -b^* \left\{ \frac{2V_a^{(02)} W_a^{(11)}}{(2E_a - \omega)2E_a} + \frac{2V_a^{(11)} W_a^{(02)}}{(2E_a - \omega)^2} \right\} \]
\[ - b \left\{ \frac{2V_a^{(02)} W_a^{(11)}}{2E_a(2E_a + \omega)} + \frac{2V_a^{(11)} W_a^{(20)}}{(2E_a + \omega)^2} \right\}, \]  \quad (3.4c)

\[ \text{rot}[2'] = c_5(\delta \lambda)^2 \frac{n_a}{2E_a} + c_6(\delta \lambda)^2 \frac{m_a}{2E_a}, \]  \quad (3.5a)

\[ \text{vib}[2'] = b^* b^* \frac{\xi_{20}^{(+)}}{2E_a - \omega} + bb \frac{\xi_{20}^{(-)}}{2E_a + \omega} + (b^* b + bb^*) \left( \xi_{11} \frac{n_a}{2E_a} + \xi_{11} \frac{m_a}{2E_a} \right) \]
\[ + (b^* b - bb^*) \left( \delta_{11} \frac{n_a}{2E_a} + \gamma_{11} \frac{m_a}{2E_a} \right), \]  \quad (3.5b)

\[ \text{coup}[2'] = b^* \frac{\xi_{112}^{(-)}}{2E_a - \omega} + b - \frac{\xi_{112}^{(+)}}{2E_a + \omega}. \]  \quad (3.5c)

Here we have defined

\[ n_a = 2u_a v_a, \quad m_a = u_a^2 - v_a^2, \]  \quad (3.6a)

\[ \xi_{20}^{(+)} = \xi_{20} m_a \pm \eta_{20}, \quad \xi_{11}^{(+)} = \delta \xi m_a \pm \delta \eta. \]  \quad (3.6b)

Next, we write down \( \tilde{n}_{a,\text{eff}} \). It vanishes in first-order

\[ \text{rot}[1] = \text{vib}[1] = 0. \]  \quad (3.7)
Second-order terms are very simple

\[
\text{rot}[2] = \left[ \frac{V_a^{(20)}}{2E_a} \right]^2,
\]

\[
\text{vib}[2] = (b^\dagger b^\dagger + bb) \frac{W_a^{(02)} W_a^{(20)}}{(2E_a - \omega)(2E_a + \omega)}
\]

\[
+ b^\dagger b \left[ \frac{W_a^{(02)}}{2E_a - \omega} \right]^2 + bb^\dagger \left[ \frac{W_a^{(20)}}{2E_a + \omega} \right]^2
\]

\[
= \{ \tilde{p}_{a, \text{eff}}(\text{vib}[1]) \} \{ \tilde{p}_{a, \text{eff}}(\text{vib}[1]) \},
\]

\[
\text{coup}[2] = -b^\dagger \left\{ \frac{V_a^{(02)} W_a^{(20)}}{2E_a (2E_a + \omega)} + \frac{V_a^{(20)} W_a^{(02)}}{(2E_a - \omega) 2E_a} \right\}
\]

\[
- b \left\{ \frac{V_a^{(02)} W_a^{(02)}}{(2E_a - \omega) 2E_a} + \frac{V_a^{(20)} W_a^{(20)}}{2E_a (2E_a + \omega)} \right\}
\]

\[
= -\frac{V_a^{(02)}}{2E_a} \tilde{p}_{a, \text{eff}}(\text{vib}[1]) - \frac{V_a^{(20)}}{2E_a} \tilde{p}_{a, \text{eff}}(\text{vib}[1]),
\]

\[
\text{rot}[2'] = \text{vib}[2'] = \text{coup}[2'] = 0.
\]

Therefore the second-order \( \tilde{n}_{a, \text{eff}} \) can be arranged in the form

\[
\tilde{n}_{a, \text{eff}}[2] = \tilde{p}_{a, \text{eff}}[1] \tilde{p}_{a, \text{eff}}[1].
\]

In the above discussion we have neglected the effect due to the change in the phonon frequency.

Using the above expansions, we can derive the effective operators for \( \tilde{P}^\dagger \) and \( \tilde{N} \) up to second-order. We add the effect due to the change in \( \omega \). Its contribution, denoted as \( \text{coup}[2'] \), is given by \( \delta_\omega F = (\partial F / \partial \omega) \delta \omega \), where \( F \) is the first-order vibrational result. Details of the expressions are omitted here, because we can identify them from the self-consistency condition and the number equation given in § 4.

§ 4. Self-consistency condition and number equation

4.1. Self-consistency condition

The requirement of the consistency between \( \tilde{P}_{\text{eff}}^\dagger \) and the gap parameter

\[
\tilde{P}_{\text{eff}}^\dagger = \frac{A}{G}
\]

gives the following set of equations.

First, the part independent of the collective parameters is written as

\[
P^{(0)}(\lambda_0, \Delta_0) = \sum_{a \geq 0} u_a(\lambda_0, \Delta_0) v_a(\lambda_0, \Delta_0) = \frac{A_0}{G}.
\]

The rotational part of Eq. (4·1) gives
rot[1]: \[ \frac{1}{2} (R_2 \delta \lambda + R_3 \delta \lambda) = \frac{1}{G} \delta \Delta, \tag{4.3} \]

\[ \begin{align*} 
\text{rot[2]:} & \quad \frac{1}{2} R_2 c_1(\delta \lambda)^2 + \frac{1}{2} R_3 c_4(\delta \lambda)^2 + \left( -\frac{1}{4} S_1 + \frac{1}{2} S_3 \right)(\delta \lambda)^2 \\
& \quad + \left( -S_2 + \frac{1}{2} S_4 \right) \delta \lambda \delta \Delta - \frac{3}{4} S_3 (\delta \Delta)^2 = \frac{C_4}{G} (\delta \Delta)^2. \tag{4.4} \end{align*} \]

The functions \( R_i \) and \( S_i \) are defined in Appendix A. Equation (4.4) is composed of rot[2'] and rot[2] parts of \( \tilde{P}^{* \text{eff}} \). According to Eq. (4.3), \( \delta \Delta \) is proportional to \( \delta \lambda \):

\[ \delta \Delta = r \delta \lambda, \quad r = R_2/R_1. \tag{4.5} \]

The first-order vibrational part gives two equations for \( \xi \) and \( \eta \)

\[ \begin{align*} 
\text{vib[1]:} & \quad \xi R_{23}(\omega) - \eta S_{12}(\omega) = \frac{2}{G} \xi, \tag{4.6a} \\
\xi S_{12}(\omega) - \eta R_0(\omega) = -\frac{2}{G} \eta. \tag{4.6b} \end{align*} \]

The second-order vibrational part (vib[2'] + vib[2]) gives the following four equations for the coefficients of \( (1/2)(b^+ b + bb), (1/2)(b^+ b^+ - bb), (1/2)(b^+ b + bb'), \) and \( (1/2)(b^+ b^+ - bb^+), \) in this order:

\[ \begin{align*} 
\xi_{20} R_{23}(2\omega) - \eta_{20} S_{12}(2\omega) & \quad - \frac{\xi^2}{\omega} (2S_{33}(2\omega) - S_{33}(\omega)) \\
& \quad + \frac{2 \xi \eta}{\omega} (R_{23}(2\omega) - R_{23}(\omega)) + \frac{\eta^2}{\omega} S_{11}(\omega) = \frac{2}{G} \xi_{20}, \tag{4.7a} \\
\xi_{20} S_{12}(2\omega) - \eta_{20} R_0(2\omega) & \quad - \frac{2 \xi^2}{\omega} (R_{23}(2\omega) - R_{23}(\omega)) \\
& \quad + \frac{2 \xi \eta}{\omega} (S_{11}(2\omega) - S_{11}(\omega)) = -\frac{2}{G} \eta_{20}, \tag{4.7b} \\
\xi_{11} R_3 + \xi_{11} R_2 & \quad - \frac{\xi^2}{\omega} (2S_{33}(\omega) + \omega S_{55}(\omega)) \\
& \quad + \frac{2 \xi \eta}{\omega} (R_{23}(\omega) - R_3 + \omega R_{23}(\omega)) - \eta^2 S_{11}(\omega) = \frac{2}{G} \xi_{11}, \tag{4.7c} \\
- \eta_{11} R_3 - \delta_{11} R_2 + \frac{\xi^2}{\omega} [\omega P^{(0)}(R_{23}(\omega) - R_3) - 2(R_{33}(\omega) - R_{33}(0)) - \omega R_{33}(\omega)] \\
& \quad + \frac{2 \xi \eta}{\omega} [-2\omega P^{(0)} S_{12}(\omega) + S_{22}(\omega) + \omega S_{22}(\omega)] - \eta^2 R_{11}(\omega) = -\frac{2}{G} \eta_{11}. \tag{4.7d} \end{align*} \]

Finally, we obtain two equations from the coefficients of \( (1/2)(b^+ + b) \) and \( (1/2)(b^+ - b) \) arising from the pair rotation-vibration coupling and the change in \( \omega(\text{ coup}[2], \text{ coup}[2'], \text{ coup}[2'']) \)
In relating $N_{\text{eff}}$ with the particle number of the system, we take $N_{\text{eff}} - N_0$ to be solely contributed by the lowest-order rotational part of $N_{\text{eff}}$. Namely we put the conditions on $N_0$ and $N - N_0$ as follows:

$$N^{(0)}(\lambda_0, A_0) = 2 \sum_{\delta > 0} \nu_{\delta}(\lambda_0, A_0) = N_0, \quad (4.8)$$

$$J \delta \lambda = N - N_0 = M, \quad J = R_1(1 + r^2). \quad (4.9)$$

We must thereby require all the higher-order terms of $N_{\text{eff}}$ to vanish. Its second-order rotational part gives

$$c_4(\delta \lambda)^2 R_1 + c_4(\delta \Delta)^2 R_2 + \frac{3}{2} S_2(\delta \lambda)^2 + (-S_1 + 2S_3) \delta \lambda \delta \Delta + \left(-S_2 + \frac{1}{2} S_4\right) (\delta \Delta)^2 = 0. \quad (4.10)$$

Together with Eq. (4.4), this fixes two coefficients $c_4$ and $c_4$. From Eqs. (4.6a, b) for $\xi$ and $\eta$, the first-order vibrational part was shown in II to vanish

$$\xi R_{22}(\omega) - \eta S_{11}(\omega) = 0. \quad (4.11)$$

By putting the coefficients of $(b^+b^+ + bb), (b^+b + bb^+)$ and $(b^+b - bb^+)$ in the second-order vibrational part (vib[2'] + vib[2]) to be equal to zero, we get three equations

$$\xi_{20} R_{22}(2\omega) - \eta_{20} S_{11}(2\omega) + \frac{2\xi^2}{\omega} \left[-S_{32}(2\omega) + S_{32}(\omega) + \frac{1}{2} S_{34}(\omega)\right]$$

$$+ \frac{2\xi^2}{\omega} (R_{21}(2\omega) - R_{21}(\omega)) - \frac{\eta^2}{\omega} S_{12}(\omega) = 0, \quad (4.12a)$$
Equation (4·12a) involving \( \xi_{20} \) and \( \eta_{20} \) can be shown to be equivalent to Eq. (4·7b). We can thus determine \( \xi_{20} \) and \( \eta_{20} \) by using Eqs. (4·7a, b), \( \xi_{11} \) by Eqs. (4·7c), (4·12b) and \( \eta_{11} \) by Eqs. (4·7d), (4·12c). The sum of the coupling terms depending on \( b^+ + b \) (coup [2], coup[2′] and coup[2″]) is put to zero:

\[
\begin{align*}
0 &= \delta \xi R_{22}(\omega) - \delta \eta S_{11}(\omega) \\
&\quad + \frac{2 \xi}{\omega} \left\{ \delta \lambda [-S_{31}(\omega) + S_{33}(\omega) + \omega S_{3}^{\prime}(\omega)] \right. \\
&\quad \left. + \delta \Delta [S_{3}(\omega) - S_{3}(\omega) - \omega S_{3}(\omega)] + \delta \omega \frac{\omega}{2} R_{22}(\omega) \right\} \\
&\quad + \frac{2 \eta}{\omega} \left\{ \delta \lambda [-R_{22}(\omega) + R_{2} - \omega R_{22}(\omega)] \right. \\
&\quad \left. + \delta \Delta [-R_{22}(\omega) + R_{2} + \omega R_{22}(\omega)] - \delta \omega \frac{\omega}{2} S_{11}(\omega) \right\} = 0 .
\end{align*}
\]

With the aid of equations given in Appendix A, especially Eqs. (A14a, b), we can show that Eq. (4·12d) reduces to Eq. (4·7f). For the determination of \( \delta \xi \), \( \delta \eta \) and \( \delta \omega \), a supplementary equation to Eqs. (4·7e, f) will be derived in § 6.

In order to have some physical insight, we will rewrite these equations more transparently in the next section.

§ 5. Further discussion on \( \bar{N}_{\text{eff}} \) and \( \bar{P}_{\text{eff}} \)

5.1. Rotational part of \( \bar{N}_{\text{eff}} \) and \( \bar{P}_{\text{eff}} \)

In treating the system with the particle number \( N \), the energy gap \( \Lambda \) and the chemical potential \( \lambda \), we use the transformation coefficients \( u_{\alpha}, v_{\alpha} \) and the phonon frequency \( \omega \) associated with the basis system with \( N_{0}, \Lambda_{0} \) and \( \lambda_{0} \). In the BCS approximation, the particle number and the pair-creation operator have the following average values:

\[
\begin{align*}
N^{(0)}(\lambda, \Lambda) &= 2 \sum_{\alpha>0} u_{\alpha} v_{\alpha}(\lambda, \Lambda), \\
P^{(0)}(\lambda, \Lambda) &= \sum_{\alpha>0} u_{\alpha}(\lambda, \Lambda) v_{\alpha}(\lambda, \Lambda).
\end{align*}
\]

We expand these in powers of the difference \( x = \lambda - \lambda_{0} \) and \( y = \Lambda - \Lambda_{0} \). First-
second-order differences of \( n_a \) and \( m_a \) defined in Eq. (3·6a) are given as follows:

\[
\delta_{[1]} n_a = R_{2a} x + R_{3a} y ,
\]

(5·2a)

\[
\delta_{[1]} m_a = - R_{1a} x - R_{2a} y ,
\]

(5·2b)

\[
\delta_{[2]} n_a = \left(-\frac{1}{2} S_{1a} + S_{3a}\right) x^2 + \left(-2 S_{2a} + S_{4a}\right) x y - \frac{3}{2} S_{3a} y^2 ,
\]

(5·3a)

\[
\delta_{[2]} m_a = - \frac{3}{2} S_{2a} x^2 + (S_{1a} - 2 S_{3a}) x y + \left(S_{2a} - \frac{1}{2} S_{4a}\right) y^2 .
\]

(5·3b)

After summing over \( a \), we have a relation for \( N(O) \) valid up to second-order

\[
N^{(0)}(\lambda_0 + \delta \lambda, \Delta_0 + \delta \Delta) = N^{(0)}(\lambda_0, \Delta_0) + \tilde{N}_{\text{eff}}(\text{rot}[1] + \text{rot}[2]) .
\]

(5·4)

By including the effect of \( c_1 \) and \( c_4 \), this relation is modified as

\[
N^{(0)}(\lambda_0', \Delta_0') = \tilde{N}_{\text{eff}} \text{ (up to second-order) .}
\]

(5·5)

Here \( \lambda_0' \) and \( \Delta_0' \) are defined by

\[
\lambda_0' = \lambda_0 + \delta \lambda + c_1 (\delta \lambda)^2 ,
\]

(5·6a)

\[
\Delta_0' = \Delta_0 + \delta \Delta + c_4 (\delta \Delta)^2 .
\]

(5·6b)

As the particle numbers \( N_0 \) and \( N \) are equated to the zeroth- and first-order terms of \( N^{(0)}(\lambda_0', \Delta_0') \), respectively,

\[
N_0 = N^{(0)}(\lambda_0, \Delta_0) ,
\]

(5·7a)

\[
N = N^{(0)}(\lambda_0 + \delta \lambda, \Delta_0 + \delta \Delta) \text{ (up to first-order) ,}
\]

(5·7b)

we must necessarily conclude that the second-order term of \( N^{(0)}(\lambda_0', \Delta_0') \) vanishes.

By a similar argument, the relation

\[
\tilde{P}_{\text{eff}} = P^{(0)}(\lambda_0', \Delta_0')
\]

(5·8)

is valid up to second-order. Writing this equation explicitly, we have

\[
GP^{(0)}(\lambda_0, \Delta_0) = \Delta_0 ,
\]

(5·9a)

\[
GP^{(0)}(\lambda_0 + \delta \lambda, \Delta_0 + \delta \Delta) = \Delta_0 + \delta \Delta ,
\]

(5·9b)

\[
GP^{(0)}(\lambda_0', \Delta_0') = \Delta_0' .
\]

(5·9c)

With the aid of the relation \( 2/G - R_3 = R_1 \), Eqs. (4·4) and (4·10) can be cast into the form

\[
\delta c_2 = -\frac{3}{2} S_2 + r\left(\frac{3}{2} S_1 - 3 S_3\right) + r^2 \left(3 S_2 - \frac{3}{2} S_4\right) + r^3 \frac{3}{2} S_5 ,
\]

(5·10a)

\[
\delta c_4 = - r\left(S_2 - \frac{1}{4} S_4\right) + (S_1 - \frac{7}{2} S_3) - \frac{1}{r} \left(\frac{3}{2} S_2 + 2 S_3 - S_4\right) - \frac{1}{r^2} \left(\frac{1}{2} S_1 - S_3\right) .
\]

(5·10b)

For an arbitrary function \( F \) of \( \lambda \) and \( \Delta \), we write the difference as \( \delta F \) and the
partial differences as $\delta_i F$ and $\delta_\Delta F$,

$$
\delta F = \frac{\partial F}{\partial \lambda} \delta \lambda + \frac{\partial F}{\partial \Delta} \delta \Delta = \delta_i F + \delta_\Delta F.
$$

(5.11)

In case that $F$ depends also on $\omega$, we add $\delta_o F = (\partial F/\partial \omega) \delta \omega$ to the right-hand side of Eq. (5.11). Appendix B lists the partial derivatives for some of the functions used in the present paper.

We can then express the rot[2] parts of $\tilde{\Pi}_{\text{eff}}$ and $\tilde{\Pi}_{\text{eff}}^*$, (4.4) and (4.10), as

$$
\tilde{\Pi}_{\text{eff}}(\text{rot[2]}) = \frac{1}{4} (\delta \lambda \delta R_2 + \delta \Delta \delta R_3),
$$

(5.12a)

$$
\tilde{\Pi}_{\text{eff}}^*(\text{rot[2]}) = \frac{1}{2} (\delta \lambda \delta R_1 + \delta \Delta \delta R_2).
$$

(5.12b)

By using these expressions, Eqs. (5.10a, b) can be rewritten as

$$
c_o \delta \lambda = -\frac{\delta \mathcal{J}}{2 \mathcal{J}}, \quad c_\Delta \delta \Delta = -\frac{\delta \mathcal{J}'}{2 \mathcal{J}'}, \quad \mathcal{J}' = R_2(1 + r^{-2}).
$$

(5.13)

Now let us confine ourselves to a simple case of a degenerate single-$j$ configuration. The energy gap and the chemical potential are then given by

$$
\Delta = \frac{G}{2} \sqrt{N(2 \Omega - N)}, \quad \lambda = -\frac{G}{2} (\Omega - N). \quad (\Omega = j + \frac{1}{2})
$$

(5.14)

Expansion of these in powers of $M = N - N_0$ yields

$$
\Delta = \Delta_0 + \delta \Delta + c_\Delta (\delta \Delta)^2 + \cdots,
$$

$$
\lambda = \lambda_0 + \delta \lambda + c_\lambda (\delta \lambda)^2 + \cdots.
$$

(5.15)

Here the differences $\delta \Delta$ and $\delta \lambda$ are given by

$$
\delta \Delta = \frac{G n_2}{2 n_1} M, \quad \delta \lambda = \frac{G}{2} M = \frac{1}{\mathcal{J}} M
$$

(5.16)

with

$$
n_1 = \sqrt{N_0(2 \Omega - N_0)}, \quad n_2 = \Omega - N_0, \quad \mathcal{J} = 2/G.
$$

(5.17)

The coefficients $c_\Delta$ and $c_\lambda$ are given by

$$
c_\Delta = -\frac{\Omega^2}{G n_1 n_2}, \quad c_\lambda = 0.
$$

(5.18)

Solutions of Eqs. (5.10a, b) in this single-$j$ limit reduce, indeed, to the above results.

5.2. Pair rotation-vibration coupling terms of $\tilde{\Pi}_{\text{eff}}$ and $\tilde{\Pi}_{\text{eff}}^*$

Terms of $\tilde{\Pi}_{\text{eff}}$ and $\tilde{\Pi}_{\text{eff}}^*$, expressing the effect of coupling between the pair rotation and vibration, involve complicated combinations of $R_i$ and $S_j$ functions. However, if we rewrite these in terms of the differences $\delta R_i$ and $\delta S_j$, we can obtain quite transparent expressions.

The first-order vibrational part of $\tilde{\Pi}_{\text{eff}}$ contains $R_{22}(\omega)$ and $S_{11}(\omega)$. With the aid
of their derivatives given in Appendix B, we find
\[ \tilde{N}_{\text{eff}}(\text{coup}[2] + \text{coup}[2'] + \text{coup}[2'\prime]) = (b^* + b) \delta(\xi R_{2\alpha}(\omega) - \eta S_{1\alpha}(\omega)). \] (5·19)

Thus, the coefficient of \((b^* + b)\) in the sum of Eq. (5·19) with \(\tilde{N}_{\text{eff}}(\text{vib}[1])\) becomes the first-order approximation of
\[ \xi_R R_{2\alpha}(\omega_R, \lambda_R, \Delta_R) - \eta_R S_{1\alpha}(\omega_R, \lambda_R, \Delta_R), \] (5·20)
in which
\[ \xi_R = \xi + \delta \xi, \quad \eta_R = \eta + \delta \eta, \quad \omega_R = \omega + \delta \omega, \]
\[ \lambda_R = \lambda_0 + \delta \lambda, \quad \Delta_R = \Delta_0 + \delta \Delta. \] (5·21)

It is quite plausible to put Eq. (5·20) to be zero, because it means that we extend the vibrational part of the lowest-order number equation (4·11) to this order of approximation.

We next consider the coupling part of \(\tilde{P}^{t\dagger}_{\text{eff}}\). We recall that \(\tilde{P}^{t\dagger}_{\text{eff}}(\text{vib}[1])\) contains \(R_{2\alpha}(\omega), S_{1\alpha}(\omega)\) and \(R_0(\omega)\). By using their derivatives, it is easily found that
\[ \tilde{P}^{t\dagger}_{\text{eff}}(\text{coup}[2] + \text{coup}[2'] + \text{coup}[2'\prime]) = \frac{1}{2} (b^* + b) \delta(\xi R_{2\alpha}(\omega) - \eta S_{1\alpha}(\omega)) + \frac{1}{2} (b^* - b) \delta(\xi S_{1\alpha}(\omega) - \eta R_{0}(\omega)). \] (5·22)

The coupling part of the self-consistency condition can then be cast into the following equations, valid up to first-order:
\[ \xi_R R_{2\alpha}(\omega_R, \lambda_R, \Delta_R) - \eta_R S_{1\alpha}(\omega_R, \lambda_R, \Delta_R) = \frac{2}{G} \xi_R, \] (5·23a)
\[ \xi_R S_{1\alpha}(\omega_R, \lambda_R, \Delta_R) - \eta_R R_{0}(\omega_R, \lambda_R, \Delta_R) = -\frac{2}{G} \eta_R. \] (5·23b)

By using Eqs. (A1) and (A7) of II and admitting that the self-consistency condition is now given by Eq. (5·9b), we can show Eq. (5·23b) to coincide with the equation obtained by setting Eq. (5·20) to be zero.

Since we can combine \(\delta A_{\text{vib}[1]}(2·9)\) with \(\delta A_{\text{coup}[2·11]}\) and include \(\delta \omega\) in the defining equation (2·4), it meets our expectation to have the renormalized parameters \(\xi_R, \eta_R\) and \(\omega_R\). Only after the above manipulations, however, we can see that together with \(\lambda_R\) and \(\Delta_R\), they constitute the set of equations determining the phonon frequency - with higher-order corrections. Note that we can replace \(\lambda_R\) and \(\Delta_R\) by \(\lambda'_R\) and \(\Delta'_R\) of Eqs. (5·6a, b) up to the second-order approximation.

§ 6. Third-order effective Hamiltonian

In constructing the effective Hamiltonian, we put \((\tilde{P}^{t\dagger}_{\text{eff}}\tilde{P}^{t}_{\text{eff}})\) under the Hartree approximation and use the self-consistency condition
\[ \tilde{H}_{\text{eff}} = \tilde{H}_{0,\text{eff}} + \lambda_0 \tilde{N}_{\text{eff}} + \frac{\Delta_0^2}{G} - \frac{1}{G} (\Delta - \Delta_0)(\Delta' - \Delta_0). \] (6·1)
Collective Pair Excitations in Dynamical Nuclear Field Theory. II

In third-order, $\tilde{H}_{0,\text{eff}}$ and the last term of Eq. (6·1) are split into the rotational part, two kinds of coupling terms (linear and quadratic in phonons) and the vibrational part (cubic in phonons). We will not go into discussion on the vibrational part, because of the difficulty in having a simple physical insight about it. Effective forms of an arbitrary one-body operator $\hat{O}$ and the basis Hamiltonian $H_0$ up to third-order in the perturbation $V$ are given in Appendix C.

6.1. Rotational part

In third-order, the rotational part of $\tilde{H}_{0,\text{eff}}$ is composed of the term arising from $V_r[1]$ acting three times and that involving $V_r[1]$ and $V_r[2]$ each acting once. They will be expressed schematically as $(V_r[1])^3$ and $V_r[1] \otimes V_r[2]$ and are denoted as rot$[3\_1]$ and rot$[3\_2]$, respectively, in the following equations:

$$\tilde{H}_{0,\text{eff}}(\text{rot}[3]) = \text{rot}[3\_1] + \text{rot}[3\_2]$$

with

$$\text{rot}[3\_1] = S_2(\delta \lambda)^3 + ( - S_1 + 2S_3)(\delta \lambda)^2 \delta \lambda + ( - 2S_2 + S_4) \delta \lambda (\delta \lambda)^2 - S_5 (\delta \lambda)^3,$$

$$\text{rot}[3\_2] = c_1(\delta \lambda)^3 (R_1 \delta \lambda + R_2 \delta \lambda) + c_2(\delta \lambda)^2 (R_2 \delta \lambda + R_3 \delta \lambda).$$

The first term can be reduced into the simple form

$$\text{rot}[3\_1] = - \frac{2}{3} \gamma c_1(\delta \lambda)^3 = \frac{1}{3} \gamma (\delta \lambda)^2.$$

On the other hand, $G \bar{P}^+ \bar{P}$ term contributes to

$$\text{rot}[3\_3] = - \frac{2}{G} c_4(\delta \lambda)^3.$$

By adding this with rot$[3\_2]$ (6·4), we have

$$\text{rot}[3\_2] + \text{rot}[3\_3] = \gamma c_1(\delta \lambda)^3.$$

Thus the rotational part of $\tilde{H}_{\text{eff}}$ becomes

$$\tilde{H}_{\text{eff}}(\text{rot}[3]) = \frac{1}{3} \gamma c_1(\delta \lambda)^3$$

or

$$\tilde{H}_{\text{eff}}(\text{rot}[2] + \text{rot}[3]) = \frac{1}{2} \left( \gamma - \frac{1}{3} \delta \gamma \right) (\delta \lambda)^2.$$

In order to help our understanding, we consider the ground state energy of $H$ in the BCS approximation

$$E^{(0)}(\lambda, \Delta) = \sum_{\delta \geq 0} 2e_\delta \nu_\delta^2(\lambda, \Delta) - \Delta^2 / G,$$

which can be decomposed as

$$E^{(0)}(\lambda, \Delta) = E^{(0)}(\lambda_0, \Delta_0) + \lambda_0 M + C(\lambda, \Delta) - (\Delta - \Delta_0)^2 / G,$$
By use of the first- and second-order differences of \( m_a \) and \( n_a \) given in Eqs. (5·2a, b) and (5·3a, b), we find for the corresponding differences of \( \mathcal{E} \)

\[
\delta_{[1]} \mathcal{E} = 0 , \quad \delta_{[2]} \mathcal{E} = \tilde{H}_{0,\text{eff}}(\text{rot}[2]) , \tag{6·13}
\]

and thus we recover the rotational energy for \( E^{(0)}(\lambda, \Delta) \)

\[
\delta_{[2]} \mathcal{E} - \frac{1}{G}(\delta \Delta)^2 = -\frac{1}{2} J (\delta \lambda)^2 . \tag{6·15}
\]

In third-order, we make use of the expansions

\[
\delta_{[3]} n_a = \left( -\frac{3}{2} T_{2a} + T_{4a} \right) x^3 + \left( T_{3a} - \frac{11}{2} T_{3a} + T_{3a} \right) x^2 y \\
+ \left( 3 T_{3a} - \frac{9}{2} T_{4a} \right) x y^2 + \left( 2 T_{3a} - \frac{1}{2} T_{3a} \right) y^3 , \tag{6·16a}
\]

\[
\delta_{[3]} m_a = \left( \frac{1}{2} T_{1a} - 2 T_{3a} \right) x^3 + \left( \frac{9}{2} T_{3a} - 3 T_{3a} \right) x^2 y \\
+ \left( - T_{1a} + \frac{11}{2} T_{3a} - T_{3a} \right) x y^2 + \left( - T_{2a} + \frac{3}{2} T_{4a} \right) y^3 , \tag{6·16b}
\]

in which the \( T_{ia} \)'s are defined in Eq. (A3) of Appendix A. We then get

\[
\delta_{[3]} \mathcal{E} = \tilde{H}_{0,\text{eff}}(\text{rot}[3]) . \tag{6·17}
\]

As in the discussion of § 5.1, we take into account the contribution from the \( c_i (\delta \lambda)^2 \), \( c_d (\delta \Delta)^2 \) terms in the expansions of \( \lambda - \lambda_0, \Delta - \Delta_0 \) and the one from the last term of Eq. (6·11). They correspond to the term \( \text{rot}[3_2] \) and \( \text{rot}[3'] \) of \( \tilde{H}_{\text{eff}} \), respectively. Therefore the relation

\[
E^{(0)}(\lambda', \Delta') = \tilde{H}_{\text{eff}}(\text{rot}) \tag{6·18}
\]

is valid up to third-order.

Suppose that we require the self-consistency to first-order only, and regard the whole rotational part of \( \tilde{N}_{\text{eff}} \) to be equal to \( M = N - N_0 \). Then we need not include the auxiliary interaction depending on \( c_i \) and \( c_d \). Expansions of \( \tilde{N}_{\text{eff}} \) and \( \tilde{H}_{\text{eff}} \) give the relations

\[
M = J \delta \lambda + C (\delta \lambda)^3 , \tag{6·19}
\]

\[
\tilde{H}_{\text{eff}} = E^{(0)} + \lambda_0 M + \frac{1}{2} J (\delta \lambda)^2 + C' (\delta \lambda)^3 , \tag{6·20}
\]

in which

\[
C = \frac{3}{2} S_2 + r (-S_1 + 2 S_0) + \frac{r^2}{2} (-2 S_2 + S_4) , \tag{6·21a}
\]
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We cannot find any simple relationship between the expansion coefficients $C$ and $C'$. Although they are equal for the degenerate single-$j$ configuration, we cannot get the exact result for the energy. For this case we have

$$C = C' = \frac{2n_2}{G^2 n_1^2}. \quad (6.22)$$

Expressing $\delta \lambda$ in terms of $M$ by Eq. (6.19), we can expand $\tilde{H}_{\text{eff}}$ as

$$\tilde{H}_{\text{eff}} = E^{(0)} + \lambda_0 M + \frac{1}{2} \frac{C}{\gamma} M^2 + \ldots. \quad (6.23)$$

On the other hand, the exact energy can be expressed as

$$E = E^{(0)} + \lambda_0 M - \frac{1}{2} G M + \frac{1}{2} \frac{C}{\gamma} M^2 \quad (6.24)$$

with no terms higher than second-order. These results show a deficiency of postulating the lowest-order self-consistency on the pair rotation.

6.2. Pair rotation-vibration coupling linear in phonons

The part of the effective quasiparticle Hamiltonian, depending linearly on phonons in third-order, gets contributions expressed symbolically as $(V_r[1])^2 \otimes V_r[1]$, $V_r[1] \otimes V_v[2]$ and $V_r[2] \otimes V_v[1]$. We further take into account the change in $\omega$. In the following, these four terms are represented as l.c. $[3_i, i=1 \sim 4$, in succession:

$$\tilde{H}_{\text{eff,lin. coupl}[3]} = (b' + b) \sum_{i=1}^{4} \text{l.c.}[3_i]. \quad (6.25)$$

First, the term l.c.$[3_1]$ is rather lengthily expressed as

$$\begin{align*}
\text{l.c.}[3_1] &= \frac{2 \xi}{\omega}\left[ (S_{33}(\omega) + \omega S_{33}(\omega) - S_{31}(\omega)) (\delta \lambda)^2 \\
&+ (S_{34}(\omega) + \omega S_{34}(\omega) - 3S_{32}(\omega) - \omega S_{32}(\omega)) \delta \lambda \delta \Delta \\
&+ (-2S_{33}(\omega) - \omega S_{33}(\omega)) (\delta \Delta)^2 \\
&+ \frac{2 \eta}{\omega}\left[ (R_2 - R_{22}(\omega) - \omega R_{22}(\omega)) (\delta \lambda)^2 - (\delta \Delta)^2 \\
&+ (R_3 - R_{23}(\omega) - \omega R_{23}(\omega) - R_1 + R_{21}(\omega) + \omega R_{21}(\omega)) \delta \lambda \delta \Delta \right]. \quad (6.26)
\end{align*}$$

By use of the differences, this can be compactly summarized as

$$\begin{align*}
\text{l.c.}[3_1] &= \frac{\xi}{\omega}(\delta \lambda R_{22}(\omega) + \delta \Delta R_{22}(\omega)) \delta \lambda + (\delta \lambda R_{23}(\omega) + \delta \Delta R_{23}(\omega)) \delta \Delta \\
&- \frac{\eta}{\omega}(\delta \lambda S_{11}(\omega) + \delta \Delta S_{11}(\omega)) \delta \lambda + (\delta \lambda S_{12}(\omega) + \delta \Delta S_{12}(\omega)) \delta \Delta. \quad (6.27a)
\end{align*}$$

The terms l.c.$[3_2]$ and l.c.$[3_3]$ can be written in a similar manner

$$\begin{align*}
\text{l.c.}[3_2] &= \xi \left[ R_{22}(\omega) \delta \lambda + R_{23}(\omega) \delta \Delta \right] - \eta \left[ S_{11}(\omega) \delta \lambda + S_{12}(\omega) \delta \Delta \right], \quad (6.27b)
\end{align*}$$
The last equation can be obtained from the second-order coupling (3.12) of II by replacing $\delta \lambda$ and $\delta \Delta$ by $c_1(\delta \lambda)^2$ and $c_4(\delta \Delta)^2$, respectively. The effect of changing $\omega$ is given by

$$l.c.[3]= [\xi R_{22}(\omega) - \eta S_{11}(\omega)] c_1(\delta \lambda)^2 + [\xi R_{23}(\omega) - \eta S_{12}(\omega)] c_4(\delta \Delta)^2.$$

Finally, the pairing interaction $-G \tilde{P}^\dagger \tilde{P}$ yields $(b^\dagger + b)$ term with the following coefficient:

$$l.c.[3]=-\frac{2}{G}(\delta \xi \delta \Delta + \xi c_4(\delta \Delta)^2).$$

By summing Eqs. (6.27 a ~ e), the coefficient of $(b^\dagger + b)$ in $\tilde{H}_{\text{eff}}$ (lin. coupl. [3]) can be expressed as a linear combination of $\delta \lambda$, $\delta \Delta$, $c_1(\delta \lambda)^2$, and $c_4(\delta \Delta)^2$. The coefficients of $c_1$ and $c_4$ are zero because of the first-order self-consistency (4.6a, b) on $\delta \xi$ and $\delta \eta$. The remaining coefficient becomes

$$\delta \lambda (\xi R_{22}(\omega) - \eta S_{11}(\omega)) + \delta \Delta \left(\xi R_{23}(\omega) - \eta S_{12}(\omega) - \frac{2}{G} \xi\right).$$

The coefficients of $\delta \lambda$ and $\delta \Delta$ turn out to be zero if we use the second-order self-consistency (5.23a, b) on $\delta \xi$ and $\delta \eta$.

We can thus conclude that

$$\tilde{H}_{\text{eff}}(\text{lin. coupl}[3])= 0.$$

6.3. 

Pair rotation-vibration coupling quadratic in phonons

The part of the effective quasiparticle Hamiltonian, depending quadratically on phonons, gets contributions from $V_r[V_1][2]$, $V_r[V_1][2]$, $V_r[V_1][2]$, $V_r[V_1][2]$, and that due to the change in $\omega$. In this order, we write $\tilde{H}_{\text{eff}}$ as

$$\tilde{H}_{\text{eff}}(\text{quad. coupl}[3])= \sum_{i=1}^{4} \text{coupl}[3_i].$$

The contribution from the pairing interaction in $\tilde{H}_{\text{eff}}$ is denoted as coupl[3s]

$$\text{coupl}[3]= -\frac{\delta \Delta}{G} (\delta \Delta_{\text{vib}}[2] + \delta \Delta_{\text{vib}}[2])$$

$$-\frac{1}{G} (\delta \Delta_{\text{vib}}[1] \delta \Delta_{\text{coupl}} + \delta \Delta_{\text{coupl}} \delta \Delta_{\text{vib}}[1])$$

We now study the coefficients of $(b^\dagger b^\dagger + bb)$, $(1/2)(b^\dagger b + bb^\dagger)$ and $(1/2)(b^\dagger b - bb^\dagger)$ arising from the quadratic coupling. First, the coefficient of $(b^\dagger b^\dagger + bb)$ is composed of the following terms (the q.c. [3] term is meant to be derived from coupl[3i]):

$$\text{q.c.}[3]= -\frac{\xi^2}{\omega} \{[2S_{22}(2\omega) - S_{44}(\omega) - \omega S_{33}(\omega)] \delta \lambda + [2S_{33}(2\omega) + S_{22}(\omega) + \omega S_{33}(\omega)] \delta \Delta\}$$

$$+ \frac{2\xi \eta}{\omega} \{[R_{21}(2\omega) - R_{21}(\omega)] \delta \lambda + [R_{22}(2\omega) - R_{22}(\omega)] \delta \Delta\}$$
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\[ -\frac{\eta^2}{\omega} [S_{12}(\omega) + \omega S'_{12}(\omega)] \delta \lambda - [S_{11}(\omega) + \omega S'_{11}(\omega)] \delta \Delta, \]
\(6.31a\)

q.c.[31] = \[\xi_{20} R_{22}(2\omega) - \eta_{20} S_{11}(2\omega)] \delta \lambda + [\xi_{20} R_{23}(2\omega) - \eta_{20} S_{12}(2\omega)] \delta \Delta, \]
\(6.31b\)

q.c.[33] = \[\xi \delta \xi_{23} R_{23}(\omega) - \eta \delta \eta R_0(\omega), \]
\(6.31c\)

q.c.[34] = \[\frac{1}{2} [\xi^2 \delta \omega R_{23}(\omega) - \eta^2 \delta \omega R_0(\omega)] \]
\(= \frac{1}{2} [\xi^2 \delta \omega R_{23}(\omega) - \eta^2 \delta \omega R_0(\omega)] + q.c.[34'], \]
\(6.31d\)

q.c.[35] = \[-\frac{2}{G} (\xi_{20} \delta \Delta + \xi \delta \xi - \eta \delta \eta). \]
\(6.31e\)

In the sum

q.c.[31] + q.c.[33] + q.c.[34']

= \delta \lambda \left[ \xi_{20} R_{22}(2\omega) - \eta_{20} S_{11}(2\omega) - \frac{2 \xi^2}{\omega} [S_{32}(2\omega) - S_{32}(\omega) - \frac{1}{2} S_{34}(\omega)] \right]

+ \frac{2 \xi \eta}{\omega} \left[ R_{33}(2\omega) - R_{33}(\omega) \right] - \frac{\eta^2}{\omega} S_{12}(\omega) \right]

+ \delta \Delta \left[ \xi_{20} R_{23}(2\omega) - \eta_{20} S_{12}(2\omega) - \frac{2 \xi^2}{\omega} \left[ 2 S_{33}(2\omega) - S_{33}(\omega) \right] \right]

+ \frac{2 \xi \eta}{\omega} \left[ R_{22}(2\omega) - R_{22}(\omega) \right] + \frac{\eta^2}{\omega} S_{11}(\omega) \right], \]
\(6.32\)

the coefficients of \(\delta \lambda\) and \(\delta \Delta\) can be shown to be equal to 0 and \((2/G) \xi_{20}\), respectively, if we use Eqs. (4.12a) and (4.7a). Then the coefficient of \((b^\dagger b + b b^\dagger)\) in \(\hat{H}_{\text{eff}}\) can be written in the form

\[ \delta \left[ \frac{1}{2} (\xi^2 R_{23}(\omega) - \eta^2 R_0(\omega)) - \frac{1}{G} (\xi^2 - \eta^2) \right]. \]
\(6.33\)

The corresponding coefficient of \(\hat{H}_{\text{eff}}\) in second-order, Eq. (3.25) of II, is just the square bracket of Eq. (6.33), which turns out to be zero because of the first-order self-consistency. The same condition in second-order brings the above difference to zero.

In a similar manner, we obtain the coefficient of \((1/2)(b^\dagger b + b b^\dagger)\) in \(\hat{H}_{\text{eff}}\) (quad. coup[3]) as

\[ \delta \left[ \omega (\xi^2 R_{23}(\omega) - 2 \xi \eta S_{12}(\omega) + \eta^2 R_0(\omega)) \right] = \delta \omega, \]
\(6.34\)

if we extend the normalization condition on \(\xi\) and \(\eta\) given in Eq. (3.29) of II to higher-order so as to

\[ \xi_{20}^2 R_{23}(\omega) - 2 \xi_{20} \eta S_{12}(\omega) + \eta^2 R_0(\omega) = 1. \]
\(6.35\)

This constitutes the third equation for determining \(\delta \xi, \delta \eta\) and \(\delta \omega\) in addition to Eqs. (4.7e, f) or Eqs. (5.23a, b).

Finally, the coefficient of \((1/2)(b^\dagger b - b b^\dagger)\) in \(\hat{H}_{\text{eff}}\) (quad. coup[3]) becomes
The corresponding coefficient of the second-order $\hat{H}_{\text{eff}}$ is equal to the content of the square bracket of the above equation. Summing up with the second-order term, the total coefficient of $(1/2)(b^{\dagger}b - bb^{\dagger})$ can be expressed as

$$
\delta\left\{ \xi_{b}^{2}[S_{23}(\omega_{b}) + \omega S_{23}(\omega)] - 2\xi_{b}\eta_{R}[R_{12}(\omega_{R}) + \omega R_{12}(\omega_{R})] \right. \\
+ \eta_{R}^{2}[S_{0}(\omega_{R}) + \omega S_{0}(\omega)] + \frac{4}{G}\xi_{b}\eta_{R} \right\}. \tag{6.36}
$$

We thereby obtain the result that

$$
\hat{H}_{\text{eff}}(\text{vib}[2]) + \hat{H}_{\text{eff}}(\text{quad. coupl}[3]) = E_{v}(\lambda_{R}, \Delta_{R}, \xi_{R}, \eta_{R}, \omega_{R}) + \omega_{R}b^{\dagger}b, \tag{6.38}
$$

in which $E_{v}$ is the zero point energy of the vibration given by Eq. (3.30) of II.

§ 7. Summary and conclusion

We have applied higher-order DNFT to the study of collective excitations of a nuclear system with the pairing force. We expand the collective parameters prescribing the deformed-field Hamiltonian up to second-order in the amplitudes of the pair rotation or vibration. In the expansion of the chemical potential, we have assumed the constant phase velocity, which is one of the basic ingredients of DNFT. Consequently, the residual interaction between particles and collective modes decomposes into the first- and second-order parts. Effective operators are calculated for the pair-creation and number operators up to second-order, and the Hamiltonian up to third-order.

Pair-rotational parts of $\hat{N}_{\text{eff}}$, $\hat{P}_{\text{eff}}$ and $\hat{H}_{\text{eff}}$ correspond to the expansions of $N^{(0)}$, $P^{(0)}$ and $E^{(0)}$ with the renormalized $\lambda$ and $\Delta$ as the arguments. But we cannot express the difference $E^{(0)}(\lambda_{R}, \Delta_{R}) - E^{(0)}(\lambda_{0}, \Delta_{0})$ in the form of the rotational energy with the moment of inertia specified by $\lambda_{R}$ and $\Delta_{R}$ (cf. Eq. (6.9)). In this sense we must treat the second-order corrections in the rotational parts of $(\lambda - \lambda_{0})$ and $(\Delta - \Delta_{0})$, different from other renormalizable parameters. The coefficients $c_{1}$ and $c_{2}$ are determined by the rotational parts of the number equation and the self-consistency condition. Pair rotation-vibration coupling terms of $\hat{N}_{\text{eff}}$ and $\hat{P}_{\text{eff}}$ can be regarded as renormalizing their first-order vibrational parts, by using the parameters $\xi_{R}$, $\eta_{R}$ and $\omega_{R}$ in addition to $\lambda_{R}$ and $\Delta_{R}$. In a similar way, two kinds of third-order coupling terms of $\hat{H}_{\text{eff}}$ can be included in the lower-order terms by using the renormalized variables. The improved frequency $\omega_{R}$ is determined by the first-order dispersion equation expressed by the renormalized parameters $\xi_{R}$ and $\eta_{R}$ instead of $\xi$ and $\eta$.

Thus the higher-order corrections can also be treated in such a way that we disregard the second-order terms in the vibrational and coupling parts of $(\lambda - \lambda_{0})$ and $(\Delta - \Delta_{0})$ and the $\delta \omega$ term, and omit all the diagrams expressing the coupling of the rotation with vibration, and determine the renormalized $\xi_{R}$, $\eta_{R}$ and $\omega_{R}$ from $\lambda_{R}$ and
Collective Pair Excitations in Dynamical Nuclear Field Theory. II

Δ' by using Eqs. (5·23a, b) and (6·35) as in the lowest-order approximation. With this procedure, we can considerably simplify the treatment of the higher-order expansion. We recall that for the quadrupole vibration we can prove the equivalence of DNFT with the boson expansion theory by using the lowest-order self-consistency, if we omit diagrams expressing the renormalization effect.4)

From the second- and third-order calculations presented here, we can infer the collective behavior of the system interacting by the pairing force in more higher-order approach. The pair rotational energy is expanded as a series in power of δλ, with the coefficients referring to the No system, which can also be expressed by using those of the N system. The pair vibrational mode yields a higher-order RPA equation. Its renormalized coefficients correspond to the N system. Coupling of the rotation with vibration does not appear explicitly. It is absorbed in renormalizing the collective parameters and the coefficients of the higher-order RPA.

Appendix A

--- Definitions and Some Useful Relations on Ri(x) and Si(x) ---

First, Ri(i=0~3), Si(j=1~4) and Tk(k=1~5) are defined as

\[ R_i = \sum_{\alpha>0} R_{ia}, \quad R_{0a} = 1/E_a, \quad R_{ia} = n_{a}^{i-1}m_{a}^{i-1}/E_a, \quad (i=1~3) \]  
\[ S_j = \sum_{\alpha>0} S_{ja}, \quad S_{ja} = n_{a}^{j-1}m_{a}^{j-1}/E_a^{2}, \quad (j=1~4) \]  
\[ T_k = \sum_{\alpha>0} T_{ka}, \quad T_{ka} = n_{a}^{k-1}m_{a}^{k-1}/E_a^{3}, \quad (k=1~5) \]

in which

\[ n_{a} = 2u_{a}v_{a} = \Delta_{0}/E_a, \quad m_{a} = u_{a}^{2} - v_{a}^{2} = \varepsilon_{a}/E_a. \]  

In a similar way, Ri(x) and Si(x) are defined as follows:

\[ R_{0a}(x) = \sum_{\alpha>0} R_{ija}(x), \quad R_{0a}(x) = (2E_a - x)^{-1} + (2E_a + x)^{-1}, \]  
\[ R_{ia}(x) = n_{a}^{i+1}m_{a}^{i-1}R_{0a}(x), \]  
\[ S_{ij}(x) = \sum_{\alpha>0} S_{ija}(x), \quad S_{ja}(x) = (2E_a - x)^{-1} - (2E_a + x)^{-1}, \]  
\[ S_{ja}(x) = n_{a}^{j+1}m_{a}^{j-1}S_{0a}(x). \quad (i=1~3, j=1~(i+1)) \]

The first and second derivatives of Ri(x) with respect to x are denoted as R_i'(x) and R_i''(x), respectively. Similar definitions hold for S_i(x). We can express R_{ia} and S_{ia} as

\[ R_{ia} = R_{2ia}(0), \quad S_{ia} = \frac{1}{2} S_{3ia}(0). \]

Next we show some simple relations

\[ R_1 + R_2 = \frac{G}{\Delta_{0}}, \quad S_1 + S_3 = \frac{R_1}{\Delta_{0}}, \quad S_2 + S_4 = \frac{R_2}{\Delta_{0}}. \]
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\[ R_{21}(x) + R_{32}(x) = R_0(x), \]  
\[ S_{21}(x) + S_{32}(x) = S_0(x). \]  

By using the equation

\[ n_o R_{0a}(x) = \frac{2\Delta_0}{x} S_{0a}(x), \]  

we obtain

\[ R_i(x) = \frac{2\Delta_0}{x} S_{i-1,0}(x) \quad (i=1 \sim 3, j=1, 2) \]  

in which \( S_0(x) \) is regarded to be equivalent to \( S_0(x) \). Similarly, the relation

\[ n_o S_{0a}(x) = \frac{2\Delta_0}{x} \left( R_{0a}(x) - \frac{1}{E_o} \right) \]  

leads to the result

\[ S_i(x) = \frac{2\Delta_0}{x} (R_{i-1,0}(x) - R_{i-1,0}(0)), \quad (i=1 \sim 3, j=1, 2) \]  

in which \( R_0(x) \) is considered to be equal to \( R_0(x) \). Differentiation of Eqs. (A11a, b) gives

\[ n_o R'_{0a}(x) = \frac{2\Delta_0}{x^2} (xS_{0a}(x) - S_{0a}(x)), \]  
\[ n_o S'_{0a}(x) = \frac{2\Delta_0}{x^2} [xR'_{0a}(x) - (R_{0a}(x) - R_{0a}(0))], \]  

which yield the relations

\[ R_i(x) + xR'_i(x) = 2\Delta_0 S_{i-1,0}, \]  
\[ S_i(x) + xS'_i(x) = 2\Delta_0 R_{i-1,0}. \quad (i=1 \sim 3, j=1, 2) \]  

We also have

\[ 2E_o R'_{0a}(x) = S_{0a}(x) + xS'_{0a}(x), \]  
\[ 2E_o S'_{0a}(x) = R_{0a}(x) + xR'_{0a}(x). \]  

**Appendix B**

*Derivatives of \( F(\lambda, \Delta, \omega) \) with Respect to \( \lambda \) and \( \Delta \)*

| Table A. |
|-----------------|-----------------|-----------------|
| \( F \)       | \( \partial F/\partial \lambda \) | \( \partial F/\partial \Delta \) |
| \( R_1 \)     | \( 3S_2 \)      | \( -S_1 + 2S_3 \) |
| \( R_2 \)     | \( -S_1 + 2S_3 \) | \( -2S_3 + S_1 \) |
| \( R_3 \)     | \( -2S_2 + S_4 \) | \( -3S_3 \) |
Table B.

<table>
<thead>
<tr>
<th>F</th>
<th>$\frac{\omega \cdot \frac{\partial F}{\partial \lambda}}{2}$</th>
<th>$\frac{\omega \cdot \partial F}{\partial \lambda}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$R_0(\omega)$</td>
<td>$\omega S_1(\omega)$</td>
<td>$-\omega S_1(\omega)$</td>
</tr>
<tr>
<td>$S_0(\omega)$</td>
<td>$\omega R_1(\omega)$</td>
<td>$-\omega R_1(\omega)$</td>
</tr>
<tr>
<td>$R_{12}(\omega)$</td>
<td>$-S_{21}(\omega) + \omega S_{22}(\omega)$</td>
<td>$-S_{21}(\omega) - \omega S_{22}(\omega)$</td>
</tr>
<tr>
<td>$S_{11}(\omega)$</td>
<td>$R_{22}(\omega) - R_{12} + \omega R_{21}(\omega)$</td>
<td>$R_{22}(\omega) - R_{12} - \omega R_{21}(\omega)$</td>
</tr>
<tr>
<td>$S_{12}(\omega)$</td>
<td>$-R_{12}(\omega) + R_{21} + \omega R_{21}(\omega)$</td>
<td>$R_{21}(\omega) - R_{12} + \omega R_{21}(\omega)$</td>
</tr>
<tr>
<td>$R_{21}(\omega)$</td>
<td>$2S_{21}(\omega) + \omega S_{21}(\omega)$</td>
<td>$2S_{21}(\omega) - \omega S_{21}(\omega)$</td>
</tr>
<tr>
<td>$R_{22}(\omega)$</td>
<td>$S_{22}(\omega) + \omega S_{22}(\omega)$</td>
<td>$S_{22}(\omega) - \omega S_{22}(\omega)$</td>
</tr>
<tr>
<td>$R_{31}(\omega)$</td>
<td>$2S_{31}(\omega) - \omega S_{31}(\omega)$</td>
<td>$2S_{31}(\omega) + \omega S_{31}(\omega)$</td>
</tr>
<tr>
<td>$S_{21}(\omega)$</td>
<td>$-2(R_{21}(\omega) - R_{21}(0)) + \omega R_{21}(\omega)$</td>
<td>$-2(R_{21}(\omega) - R_{21}(0)) - \omega R_{21}(\omega)$</td>
</tr>
</tbody>
</table>

Table C.

<table>
<thead>
<tr>
<th>F</th>
<th>$\frac{\omega^2 \cdot \frac{\partial F}{\partial \lambda}}{2}$</th>
<th>$\frac{\omega^2 \cdot \partial F}{\partial \lambda}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$R_0(\omega)$</td>
<td>$\omega^2 S_{11}(\omega)$</td>
<td>$-\omega^2 S_{11}(\omega)$</td>
</tr>
<tr>
<td>$S_0(\omega)$</td>
<td>$\omega^2 R_{11}(\omega)$</td>
<td>$-\omega^2 R_{11}(\omega)$</td>
</tr>
<tr>
<td>$R_{12}(\omega)$</td>
<td>$S_{12}(\omega) + \omega S_{21}(\omega) + \omega^2 S_{22}(\omega)$</td>
<td>$S_{12}(\omega) - \omega S_{21}(\omega) - \omega^2 S_{22}(\omega)$</td>
</tr>
<tr>
<td>$S_{11}(\omega)$</td>
<td>$R_{21}(\omega) - R_{12} - \omega R_{21}(\omega) + \omega^2 R_{21}(\omega)$</td>
<td>$R_{21}(\omega) + R_{12} - \omega R_{21}(\omega) + \omega^2 R_{21}(\omega)$</td>
</tr>
<tr>
<td>$R_{21}(\omega)$</td>
<td>$2S_{21}(\omega) - 2\omega S_{21}(\omega) + \omega^2 S_{21}(\omega)$</td>
<td>$2S_{21}(\omega) + 2\omega S_{21}(\omega) - \omega^2 S_{21}(\omega)$</td>
</tr>
<tr>
<td>$R_{22}(\omega)$</td>
<td>$2(R_{31}(\omega) - R_{31}(0)) - \omega R_{31}(\omega)$</td>
<td>$2(R_{31}(\omega) - R_{31}(0)) + \omega R_{31}(\omega)$</td>
</tr>
</tbody>
</table>

Appendix C

--- Third-Order Effective Operator ---

We have given in I the effective form of an arbitrary one-body operator $\hat{\mathcal{O}}$ up to second-order in the perturbation $V$. We here write down its third-order term:

$$p_1 \hat{\mathcal{O}}_{\text{eff}[3]} p_2 = \sum_{\mathcal{H}} \left[ -\gamma_{i1} \gamma_{j1} \gamma_{\lambda1}(p_1 V_q, V_j, V_k) q_\lambda \right]$$

$$+ \frac{1}{2} (\gamma_{\lambda1} + \gamma_{i1}) \gamma_{i1} \gamma_{j1} (p_1 V_q, V_j, V_k) q_\lambda [p_1 V_q, V_j, V_k, \hat{\mathcal{O}}_q]$$

$$+ (\gamma_{\lambda1} + \gamma_{j1}) \gamma_{j1} \gamma_{\lambda1} (p_1 V_q, V_j, V_k) q_\lambda - \gamma_{i1} \gamma_{j1} \gamma_{\lambda1} (p_1 V_q, V_j, V_k, \hat{\mathcal{O}}_q)$$

$$- \frac{1}{2} \gamma_{i1} \gamma_{j1} \gamma_{\lambda1} (p_1 V_q, V_j, V_k) q_\lambda - \frac{1}{2} \gamma_{i1} \gamma_{j1} \gamma_{\lambda1} (p_1 V_q, V_j, V_k, \hat{\mathcal{O}}_q)$$

$$- \gamma_{i1} \gamma_{j1} \gamma_{\lambda1} (p_1 V_q, V_j, V_k, \hat{\mathcal{O}}_q, V_p) + \frac{1}{2} \gamma_{i1} \gamma_{j1} \gamma_{\lambda1} (p_1 V_q, V_j, V_k, \hat{\mathcal{O}}_q, V_p)$$

$$+ \gamma_{i1} \gamma_{j1} \gamma_{\lambda1} (p_1 V_q, V_j, V_k, \hat{\mathcal{O}}_q, V_p)$$

The corresponding terms in which each factor $V, \hat{\mathcal{O}}, q_i, p_i,$ originally standing in the order from left to right, is now arranged in the reverse order, accompanied by the transposition $1 \leftrightarrow 2$ in their coefficients $\gamma$.

(C1)

In the special case of the basic Hamiltonian $H_0$, the above expression becomes very simple,
\[ p_1 H_{\text{eff}}[3] p_2 \]
\[ = \frac{1}{2} \sum_{\alpha, \beta} \left[ -((E_{41} + E_{42})\gamma_{31}\gamma_{41}\gamma_{42} + (E_{31} + E_{32})\gamma_{31}\gamma_{32}\gamma_{42})(p_1 Vq_3 Vq_4 Vp_2) \\
+ ((E_{41} + E_{42})\gamma_{41}\gamma_{42}\gamma_{43})(p_1 Vq_3 Vp_4 Vp_2) \\
+ ((E_{41} + E_{42})\gamma_{41}\gamma_{42}\gamma_{43})(p_1 Vp_3 Vq_4 Vp_2) \right]. \quad (C2) \]

Here \( E_{ij} = E(i) - E(j) \) is the difference of the quasiparticle energies in the states \( i \) and \( j \).

References