Combined condensation of neutral and charged pions at high-density neutron matter is studied in an approach based on the chiral symmetry. Energy density in the combined $\pi^0-\pi^+$ condensed phase to be considered as most energetically favored is derived in a realistic calculation, where we take into account the isobar $A(1232)$ degrees of freedom, baryon-baryon short-range correlations described in terms of the Landau-Migdal parameter $g'$, and form factors in the $\pi$-baryon vertex. Characteristic features of this phase are discussed in comparison with those of the pure $\pi^0$ or the pure $\pi^+$ condensation. The combined $\pi^0-\pi^+$ condensed phase sets in at baryon density $(3~5)$ times the nuclear density $\rho_0$ depending on $g'$ after the appearance of the pure $\pi^+$ condensed phase.

§ 1. Introduction

Recent progress in the observations of neutron star phenomena has been revealing many characteristic aspects of neutron stars and stimulating a strong interest on the phenomena such as cooling behavior$^{11,23}$ glitches,$^3$ and the X-ray$^{44,54}$ and γ-ray bursts.$^6$ Accumulated data have provided us with the information on the equation of state (EOS)$^4$ and thermal properties of neutron star matter, and seem to suggest some phase transitions at high density. First of all, pion condensation is very important and interesting among possible phase transitions, since as a result of the investigations from various points of view its phase transition has been shown to occur at the density $\rho \gtrsim (2~3)\rho_0$ ($\rho_0 = 0.17$ fm$^{-3}$ being the nuclear density)$^7$–$^{21}$

Two kinds of pion condensations have been preferentially studied hitherto among various possibilities; the neutral pion ($\pi^0$) one$^8$ with the standing-wave type and the charged pion ($\pi^+$) one with the running-wave type. The $\pi^0$ condensation affects neutron star phenomena mainly through the drastic change of baryon-state function due to the phase transition and the effect of softening EOS at high densities: It brings about a liquid-crystal like structure of baryonic systems called the Alternating-Layer-Spin (ALS) structure with a specific spin-isospin order. On the other hand, the charged pion ($\pi^+$) condensation has little effects on the EOS, while it enhances the neutrino emissivity by causing the extra cooling mechanism (the quasi-particle URCA process) and consequently accelerates the cooling of neutron stars.$^{14,22}$ These features of pion condensations bear important implication in the light of current observations of neutron stars.

Although the density regions of the $\pi^0$ and $\pi^+$ condensations have been shown to be overlapping, they have been mostly discussed independently and only few papers

$^*$ As for $\pi^0$ condensation, its implication has been also studied in connection with precritical phenomena in finite nuclei.$^9$
have been published on their coexistence yet.\(^{13,14}\) Prior to our work, Tamiya and Tamagaki first suggested the possibility of their coexistence through comparison of energies calculated in a simple model with use of the \(\pi\)-nucleon \(p\)-wave interaction only.\(^ {13}\) Following their idea, one of the authors (T.T.) studied this coexistence problem in the chiral symmetry approach.\(^ {14,*)\) The picture of the CPC adopted in these papers is shortly described as follows; the \(\pi^0\) condensation first develops in one dimension along the condensed momentum of \(\pi^0\) field, and the \(\pi^c\) condensation with the condensed momentum perpendicular to that of the \(\pi^0\) occurs in the remaining two dimensional plane without serious interference. We also adopt here the same viewpoint of coexistent \(\pi^0\) and \(\pi^c\) condensations as these.

In this paper, we fully study the coexistence of \(\pi^0\) and \(\pi^c\) condensations in a realistic situation beyond the simple model by taking into account the isobar \(\Delta\) (1232) degrees of freedom, baryon-baryon short-range correlations described in terms of the Landau-Migdal parameter \(g'\), and form factors in the \(\pi\)-baryon vertex.

Our formulation is based on the chiral symmetry approach\(^ {10-12,14}\) and the constituent quark model. Its original form has been already presented by one of the authors (T.T.)\(^ {12}\) in the context of the pure \(\pi^c\) condensation. One of the advantages in this formalism is to make it possible to calculate the ground state energy to extract the EOS governed by strong interaction and the neutrino emissivity due to weak interaction under this phase, systematically on the basis of the chiral symmetry. Another advantage is that we can easily involve the above-mentioned realistic effects. Especially we can treat nucleons and isobars in a unified fashion owing to the use of the constituent quark model.

This paper is the first in a series of our papers concerning the coexistence of \(\pi^0\) and \(\pi^c\) condensations and its influence on the neutron star phenomena in a realistic situation. It is arranged as follows. In § 2, we present our formalism to calculate the ground state energy. Numerical results and consequently the EOS of the CPC phase are given in § 3. There is also given the EOS for the pure \(\pi^0\) condensation separately in the same framework, which has not been studied realistically within the chiral symmetry approach yet. Section 4 is devoted to summary and concluding remarks.

§ 2. Formalism

2.1. The effective Hamiltonian

Here we briefly survey our procedure to study the CPC phase. Within the framework based on the chiral symmetry, the pion condensed states |P.C.\> are generally constructed by operating a unitary operator \(\hat{U}\) on the normal state |Normal\>,

\[
|\text{P.C.}\> = \hat{U}(\alpha, \beta)|\text{Normal}\> \quad (2.1)
\]

with

\(^{*)\) Shortly we denote this as the combined \(\pi^0,\pi^c\) condensation and abbreviate it to CPC.
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\[ \tilde{U}(\alpha, \beta) = \exp \left\{ i \left( \int \alpha \cdot V^0 d^3 x + \int \beta \cdot A^0 d^3 x \right) \right\}, \]  
(2.2)

where \( \int V^0 d^3 x \) and \( \int A^0 d^3 x \) are the generators of the chiral symmetry, and the gauge parameters \( \alpha \) and \( \beta \) are suitably specified in accordance with a type of pion condensed phase under consideration. To describe the CPC phase, we adopt the following product ansatz for the operator \( \tilde{U} \):  

\[ |\pi^0, \pi^\pm\rangle = \tilde{U}_{\pi^0, \pi^\pm}(\varphi, \chi, \theta)|\text{Normal}\rangle \]  
(2.3)

with  

\[ \tilde{U}_{\pi^0, \pi^\pm}(\varphi, \chi, \theta) = \tilde{U}_{\pi^0}(\varphi) \cdot \tilde{U}_{\pi^\pm}(\chi, \theta), \]  
(2.4)

\[ \tilde{U}_{\pi^0}(\varphi) = \exp \left( i \int A_0^0 \varphi d^3 x \right), \]  
(2.5a)

\[ \tilde{U}_{\pi^\pm}(\chi, \theta) = \exp \left( i \int V_0^0 \chi d^3 x \right) \cdot \exp \left( i \int A_0^0 \varphi d^3 x \right), \]  
(2.5b)

where \( \chi, \theta \), and \( \varphi \) are parameters to be specified.\(^{\ast)}\) Then classical pion fields in the CPC phase are followed as

\[ \langle \pi^0, \pi^\pm | \tilde{H} | \pi^0, \pi^\pm \rangle = \langle \text{Normal} | \tilde{U}_{\pi^0, \pi^\pm} | \tilde{U}_{\pi^0, \pi^\pm} | \text{Normal} \rangle = \frac{1}{\sqrt{2}} f_\pi \sin \theta \cdot e^{\pm ix}, \]  
(2.6a)

\[ \langle \pi^0, \pi^\pm | \tilde{H} | \pi^0, \pi^\pm \rangle = f_\pi \cos \theta \sin \varphi, \]  
(2.6b)

with \( f_\pi = 93 \text{ MeV} \) being the pion decay constant.

The effective Hamiltonian \( \tilde{H}_{\text{eff}} \), where the charge neutrality condition should be taken into account, consists of the two parts, the baryonic one \( \tilde{H}_{\text{eff}}^B \) and the mesonic one \( \tilde{H}_{\text{eff}}^M \),

\[ \tilde{H}_{\text{eff}} = \tilde{H}_{\text{eff}}^B + \tilde{H}_{\text{eff}}^M \]  
(2.7)

with

\[ \tilde{H}_{\text{eff}}^B = \tilde{H} + \mu_{\pi} \rho_\pi^c \]  
and  \[ \tilde{H}_{\text{eff}}^M = \tilde{H} - \mu_{\pi} \rho_\pi^c, \]  
(2.8)

where \( \rho_\pi^c(\rho_\pi^c) \) means a charge density of the baryonic field (the pion field) and \( \mu_{\pi} \) is the pion chemical potential. By using Eqs. (2.4)~(2.8), the energy density \( \varepsilon_{\text{p.c.}} \) is given as

\[ \varepsilon_{\text{p.c.}} = \langle \pi^0, \pi^\pm | \tilde{H}_{\text{eff}} | \pi^0, \pi^\pm \rangle = \varepsilon_{\text{eff}}^B + \varepsilon_{\text{eff}}^M \]  
(2.9)

with

\[ \varepsilon_{\text{eff}}^B = \langle \text{Normal} | \tilde{H}_{\text{eff}}^B | \text{Normal} \rangle, \quad \tilde{H}_{\text{eff}}^B = \tilde{U}_{\pi^0, \pi^\pm} \tilde{H}_{\text{eff}}^B \tilde{U}_{\pi^0, \pi^\pm}, \]  
(2.10a)

\(^{\ast)}\) Generally, these parameters have space-time dependence, and their relevant forms are given in § 2.2.
where $\mathcal{H}_{\text{eff}}^\text{ch}$ and $\mathcal{H}_{\text{eff}}^\text{m}$ are the chiral-transformed effective Hamiltonians.

Following Ref. 12), we, hereafter, take the $SU(2) \times SU(2)$ version of the $SU(3) \times SU(3)$ $\sigma$ model, and use the $SU(4)$ constituent quark model in order to treat nucleons and isobars ($\Delta$) in a unified way. Over the densities we consider, $\rho \leq 6\rho_0$, $\mathcal{H}_{\text{eff}}^\text{ch}$ can be treated non-relativistically, and is equivalent to the following form in terms of the $SU(4)$ constituent quark model,

$$
\mathcal{H}_{\text{eff}}^\text{ch} = T^a + q^a \left[ -\frac{1}{2} \sigma^a \left( x_0 \cos \theta \cdot \mathbf{P} + x_0 \sin \theta \cdot \mathbf{P} \chi + x_0 \mathbf{P} \psi \right) 
- x_0 \mu_2 \sin^2 \theta + \frac{1}{2} \mu_1 (1 + x_0) \right] q ,
$$

where $T^a$ being the kinetic term of quarks. It is to be noted that we utilize the $SU(4)$ quark model for each baryonic state and $\pi$-baryon vertex, and never consider any quark exchange effects between baryons. The fermion nature of baryonic states is taken into account on the level of quasi-nucleons explained later on. The mesonic Hamiltonian density concerning the classical fields can be written,

$$
\mathcal{H}_{\text{eff}}^\text{m} = \frac{1}{2} f_\pi^2 \left[ \partial^2 + (\mathbf{P} \cdot \mathbf{P}) + \cos^2 \theta \left( \mathbf{x}^2 + (\mathbf{P} \cdot \mathbf{x})^2 \right) + \sin^2 \theta \left( \mathbf{x}^2 + (\mathbf{P} \cdot \mathbf{x})^2 \right) \right]
- f_\pi^2 m_\pi^2 \cos \theta \cos \varphi - f_\pi^2 \mu_2 \sin^2 \theta
$$

with $m_\pi$ being the pion mass. In the following sections we consider only $\mathcal{H}_{\text{eff}}^\text{m}$ in the form Eq. (2·12) within the mean-field approximation.

2.2. Quasi-baryonic states and model wavefunction of the CPC state

As mentioned in § 1, we consider the following configuration for the CPC phase; the $\pi^0$ condensation with one-dimensional order along the $z$-axis and the $\pi^c$ one developed in the other 2-dimensional $r_\perp$ plane perpendicular to the $z$-axis. The gauge parameters suitable for this picture are given as

$$
\chi = \mathbf{k}_\perp \cdot r_\perp + \mu_1 \ell , \quad \theta = 0
$$

and

$$
\varphi = A \sin k_0 z ,
$$

where $\mathbf{k}_\perp$ is the momentum of the condensed $\pi^c$ and $A$ is an amplitude for $\pi^0$, whose condensed momentum is $\mathbf{k}_0 = k_0 \mathbf{z}$, $\mathbf{z}$ being the unit vector along the $z$-axis. For each selection of Eq. (2·13a) with $\varphi = 0$ or Eq. (2·13b) with $\chi = \theta = 0$, we recover the usual $\pi^c$ condensation in the running-wave mode and the $\pi^0$ one in the standing-wave mode, respectively. Especially, such choice of $\varphi$ as Eq. (2·13b) has been shown to be consistent with the Alternating-Layer-Spin (ALS) model, where baryons are localized in one dimension with a specific spin-isospin order. This configuration has been already shown to be the most favorable one among various possible versions for the CPC state, because of no essential interference between both condensations, within the simple model calculations (SMC).
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Substituting Eqs. (2·13a, b) into Eq. (2·11), we get
\[
\mathcal{M}_{\text{eff}} = q^2 \left[ K + \frac{3}{5} b (\sigma_+ \tau_+ \vec{k}_- + \sigma_- \tau_- \vec{k}_+) - \sigma_+ \tau_- \vec{k}_- - \sigma_- \tau_+ \vec{k}_+ \right]
\]
\[
+ \frac{3}{5} \alpha \sigma^a \tau^a + c \tau^a \right] q
\]
with \( K = \frac{1}{2} \mu_+ + \delta M \cdot P_d \) and \( \vec{k}_\pm = (k_1 \pm ik_2) / |k_\perp| \), where \( P_d \) is the projection operator into the \( \Delta \) space, and the \( N-\Delta \) mass difference \( \delta M \approx 290 \text{ MeV} \) is introduced. Here we put
\[
a = -\bar{f}_\pi \cos \theta k_0 A \cos k_0 z , \tag{2·15a}
\]
\[
b = \bar{f}_\pi k_\perp \sin \theta \tag{2·15b}
\]
and
\[
c = \frac{1}{2} \mu_\pi \cos \theta . \tag{2·15c}
\]

In Eq. (2·14), the axial-vector coupling strength in the quark model \((5/3)\) is corrected by the experimental value \( g_A \approx 1.25 \) and further \( g_A \) is replaced by \( 2 \bar{f}_\pi \) with \( \bar{f} = f_{\pi NN} / m_\pi \) (\( f_{\pi NN} \) is the \( \pi-N \) coupling constant) by using the Goldberger-Treiman relation.

For the mesonic part Eq. (2·12), we get from Eqs. (2·13a, b)
\[
\mathcal{M}_{\text{eff}} = \frac{1}{2} f_\pi^2 \sin^2 \theta (k_\perp^2 - \mu_\pi^2) + \frac{1}{2} f_\pi^2 k_0^2 A^2 \cos^2 \theta \cos^2 k_0 z
\]
\[
- f_\pi^2 m^2 \cos \theta \left[ J_0(A) + \sum_{n=1}^{\infty} 2 \cos 2n k_0 z \cdot J_{2n}(A) \right] , \tag{2·16}
\]
where \( J_{2n}(A) \) \((n \geq 0)\) is the Bessel function of the 1st kind.

Following the previous method,\(^ {12} \) we prepare quasi-baryonic states for diagonalizing the effective Hamiltonian of baryonic part composed of Eq. (2·14), in the \((N+\Delta)\) space. For example, a quasi-proton state is described as
\[
|\mathcal{P}_{1/2}\rangle = N^{-1/2}\left[ |\mathcal{P}_{1/2}\rangle + i y_1 \left( \frac{1}{2} \vec{k}_+ - |\mathcal{A}_{1/2}^+\rangle \right) - \frac{\sqrt{3}}{2} \vec{k}_- - |\mathcal{A}_{3/2}^+\rangle \right] \tag{2·17}
\]
\[
+ i y_2 \left( \frac{1}{2} \vec{k}_+ - |\mathcal{A}_{1/2}^0\rangle \right) - \frac{\sqrt{3}}{2} \vec{k}_- - |\mathcal{A}_{3/2}^0\rangle \right] \right]
\]
with \( N = 1 + y_1^2 + y_2^2 \), where
\[
|\mathcal{P}_{1/2}\rangle = N_0^{-1/2}(|\mathcal{P}_{1/2}\rangle + x |\mathcal{A}_{1/2}^0\rangle) , \quad |\mathcal{A}_{1/2}^0\rangle = N_0^{-1/2}(-x |n_{-1/2}\rangle + |\mathcal{A}_{1/2}^0\rangle) \tag{2·18}
\]
with \( N_0 = 1 + x^2 \). In Eq. (2·17), the numerical factor \( 1/2 \) or \( \sqrt{3}/2 \) in front of each \( \Delta \) state stems from the Clebsch-Gordan coefficient for the transition to \( \Delta \). \( y_1 \) and \( y_2 \) are \( \Delta \)-mixing parameters due to the \( \pi^c \) condensation, and \( x \) is the \( \pi^0 \) one.

The superscript (subscript) attached to each baryonic state shows the charge state (the spin state). The baryonic states other than \( |\mathcal{P}_{1/2}\rangle \) are given in a similar way.
and their explicit forms are separately given in Appendix A.

It should be noted that Eq. (2.17) is somewhat different in form from Eq. (2.7) in Ref. 12, because we choose the direction of momentum of the condensed $\pi^c$ such that $k_1 \perp \bar{z}$, whereas the one in Ref. 12 is taken along $\bar{z}$. However, they are essentially equivalent to each other (see Appendix A).

We divide the quasi-baryonic space into two sets

$$\vec{\varphi} = \begin{pmatrix} \varphi_+ \\ \varphi_- \end{pmatrix}$$

(2.19)

with

$$\vec{\varphi}_\pm \equiv \begin{pmatrix} \vec{p}_{\pm 1/2} \\ \vec{n}_{\pm 1/2} \end{pmatrix}.$$

(2.20)

Then the representation of $\bar{H}_{\text{eff}}$ composed of Eq. (2.14) within these quasi-baryonic bases is given in the block diagonalized matrix form $\bar{H}_{\text{eff}}$,

$$\bar{H}_{\text{eff}} = \begin{pmatrix} \bar{H}_{\text{eff}} & 0 \\ 0 & \bar{H}_{\text{eff}} \end{pmatrix},$$

(2.21)

where

$$\bar{H}_{\text{eff}} = \begin{pmatrix} \langle \vec{p}_{\pm 1/2}|H|\vec{p}_{\pm 1/2} \rangle & \langle \vec{n}_{\pm 1/2}|H|\vec{n}_{\mp 1/2} \rangle \\ \langle \vec{n}_{\mp 1/2}|H|\vec{p}_{\pm 1/2} \rangle & \langle \vec{n}_{\mp 1/2}|H|\vec{n}_{\pm 1/2} \rangle \end{pmatrix}$$

(2.22)

and the eigenvalue equations are reduced into

$$\bar{H}_{\text{eff}} \vec{\varphi}_\pm = E_\pm \vec{\varphi}_\pm.$$ 

(2.23)

The matrix elements in Eq. (2.22) are easily calculated and lead to the following results,

$$\bar{H}_{\text{eff}} = \begin{pmatrix} -\nabla^2/2m \pm N^{-1}A_1 a + \frac{1}{2} \mu_\pi + N^{-1}(\delta M \cdot R + a_3^{-1}c - B_2 b) & i\bar{k}_\pm N^{-1}(B_1 b \mp A_2 a) \\ -i\bar{k}_\pm N^{-1}(B_1 b \mp A_2 a) & -\nabla^2/2m \pm N^{-1}A_1 a + \frac{1}{2} \mu_\pi + N^{-1}(\delta M \cdot R - a_3^{-1}c - B_2 b) \end{pmatrix}$$

(2.24)

with $m$ being the nucleon mass, and

$$A_1 = N_0^{-1} \left( \frac{1}{5} x^2 + \frac{8 \sqrt{2}}{5} y_2 + 1 \right) (1 - y_2^2/4) + \frac{6}{5} y_1^2 - \frac{3}{20} y_2^2,$$

(2.25a)

$$A_2 = -\frac{4}{5} y_2 N_0^{-1}(y_2 x - 1)(x + \sqrt{2}),$$

(2.25b)

$$R = N - 1 + (1 - y_2^2/4) (1 - N_0^{-1}),$$

(2.25c)

$$a_3^{-1} = 1 + 3y_1^2 - y_2^2,$$

(2.25d)
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Here we make a minor approximation by putting $y_2 = 0$ on the following reasoning. Since the terms proportional to $y_2$ in Eq. (2.17) result from the crossed diagram of $\pi N$ scattering, their contribution to the probability of $A^0$ and $A^+$ states is expected to be much smaller than that from the uncrossed diagram ($y_1$), i.e., $|y_1| \gg |y_2|$. Indeed, it has been shown that the probability ratio, $P_{y_2} = N^{-1}y_2^2$ to $P_{y_1} = N^{-1}y_1^2$, is very small ($\leq 10\%$) all over the densities of interest in the pure $\pi^c$ condensed case. Since each diagonal matrix element in Eq. (2.24) has $z$-dependence through $a \cos k_0 z$, as given in Eq. (2.15a), single-particle eigenstates can be written in the form,

$$
\psi_{\pm, a}(r) = \tilde{\phi}_{\pm, a}(r).
$$

Then the eigenvalue problem can be separated into two parts. One is the diagonalization of the matrix Eq. (2.24) in the spin-isospin space ($\tilde{\phi}_{\pm}$), and the other is to solve the following Schrödinger equation for the space-dependent function $\phi_{\pm, a}(r)$,

$$
(-\hbar^2/2m + N^{-1}A_1 \int f_{\pm} \cos \theta k_0 A \cos k_0 z)\phi_{\pm, a}(r) = E_{\pm, a}\phi_{\pm, a}(r).
$$

As is seen from Eq. (2.27), there appears a periodic potential for baryons in the $z$-direction due to the $\pi^c$ condensation. Therefore, the wavefunction $\phi_{\pm, a}(r)$ with respect to $z$ becomes the Bloch-type, while its $r_\perp$-dependence is still that of the 2-dim. plane wave,

$$
\phi_{\pm, a}(r) = e^{i p_{\perp} \cdot r_\perp} \sqrt{Q_\perp} \phi_{\pm, p_\perp}(z),
$$

where $p_{\perp}(p_\perp)$ represents the momentum in the $r_\perp$ plane ($z$), and $Q_\perp$ is the 2-dim. normalization volume. $\phi_{\pm, p_\perp}(z)$ is a Bloch-type wavefunction, and further represented by the superposition of the Wannier-type wavefunctions $\phi_{\pm, w}(z)$, which correspond to the localization picture in the ALS model,

$$
\phi_{\pm, p_\perp}(z) = k_0^{-1/2} \sum_l \phi_{\pm, w}(z - 2dl) e^{2idl \cdot p_\perp},
$$

where $l$ is the number of layer site and $d = \pi/k_0$ is an inter-layer distance. Instead of solving Eq. (2.27) by substituting (2.29), we adopt a variational approach by approximating the Wannier function $\phi_{\pm, w}(z)$ to a Gaussian form,

$$
\phi_{\pm, w}(z) = (a/\pi)^{1/4} \exp \left[ -\frac{1}{2} a (z - d + d/2)^2 \right]
$$

with a variational parameter $a$. Thus the relevant wavefunction is reduced to the ALS model one. This approximation has been proved to work well in the $\pi^c$ condensation.
After diagonalizing Eq. (2.24) together with $E_{\pm,a}$, we get the following sets of eigenvalues and eigenspinors,

$$E_{\pm,a}^{(Q)} = E_{\pm,a} + \frac{1}{2} \mu \pi + N^{-1}[\delta M \cdot R - B_3 b + (a_3^{-2} c^2 + B_1^{-2} b^2)^{1/2}],$$  
(2.31a)

$$|\xi_{\pm,a}\rangle = \cos \phi \cdot |\bar{n}_{\pm 1/2}\rangle - i \bar{\kappa}_{\pm} \sin \phi \cdot |\bar{n}_{\pm 1/2}\rangle,$$  
(2.31b)

and

$$E_{\pm,a}^{(N)} = E_{\pm,a} + \frac{1}{2} \mu \pi + N^{-1}[\delta M \cdot R - B_3 b - (a_3^{-2} c^2 + B_1^{-2} b^2)^{1/2}],$$  
(2.32a)

$$|\eta_{\pm,a}\rangle = \cos \phi \cdot |\bar{n}_{\pm 1/2}\rangle - i \bar{\kappa}_{\pm} \sin \phi \cdot |\bar{n}_{\pm 1/2}\rangle,$$  
(2.32b)

where $\phi$ is the mixing angle defined by

$$\tan \phi = B_3 b / [a_3^{-1} c + (a_3^{-2} c^2 + B_1^{-2} b^2)^{1/2}].$$  
(2.33)

The ground state of the baryonic system is constructed by taking the usual one Fermi-sea approximation in which baryons occupy only the lower-energy quasi-particle states $|\eta_{\pm,a}\rangle$ in Eq. (2.32b),

$$|\text{Normal}\rangle = \prod_{\text{occ}} \eta_{\pm,a} \eta_{\pm,a}^\dagger |0\rangle,$$  
(2.34)

where $\eta_{\pm,a}^\dagger$ are the creation operators of the $\eta_{\pm,a}$ states, and “occ” means the product over the occupied states. The Fermi sea for this ground state becomes almost cylindrical as is shown in Fig. 1, because of the large depth of the potential due to the $\pi^0$ condensation.

2.3. Introduction of short-range correlations and form factors

In the same way as in Ref. 12), we take into account the baryon-baryon short-range correlations in the pion channel by introducing the minimal interaction in the usual form,

$$\mathcal{H}' = \frac{1}{2} g' (q^\dagger \sigma^a \tau^a q) (q^\dagger \sigma^a \tau^a q)$$  
(2.35)

in terms of the constituent quark model. Here $g'$ is the Landau-Migdal parameter. In the Hartree approximation, Eq. (2.35) is reduced to

\[^{(*)}\text{This approximation corresponds to the lowest band one. Strictly speaking, the single particle energy in the lowest band becomes slightly larger than the second-band energy in some densities in our case. However, there appears little difference for the ground state energy and other physical quantities, even if we fill the second band instead of the lowest band approximation.}\]
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\[ \mathcal{H}' = g' \bar{\rho} \rho (q^+ \sigma^q k^+ \sigma_+ q) + g' \bar{\rho} \rho (q^+ \sigma^q k^- \sigma_+ q) + g' \bar{\rho}_3 (q^+ \sigma^q r^q q) - \frac{1}{2} g'(\bar{\rho}_3^2 + \bar{\rho}_3 \rho) \]  
\hspace{1cm} (2.36)

where

\[ \bar{\rho}_3 = 2 \bar{k}_z \langle q^+ \sigma_3 r^q q \rangle \cdot \frac{3}{5} \]
\hspace{1cm} (2.37)

with

\[ \rho_0(\Gamma) = (\pi/\Gamma)^{1/2} \]
\[ \rho_2(\Gamma) = 2(\pi/\Gamma)^{1/2} \exp\left[ - (2n)^2 \pi^2 / 4 \Gamma \right] \text{ for } n \geq 1 \]  
\hspace{1cm} (2.38)

and

\[ \tilde{\rho}_3 = \langle q^+ \sigma_3 r^q q \rangle \cdot \frac{3}{5} \]
\hspace{1cm} (2.39)

with

\[ f_{2n+1}(\Gamma) = 2(\pi/\Gamma)^{1/2} \exp\left[ - (2n+1)^2 \pi^2 / 4 \Gamma \right] \]  
\hspace{1cm} (2.40)

The parameter \( \Gamma = ad^2 \) signifies the measure of localization of the baryons compared to the layer-distance. \( \tilde{\rho}(\tilde{\rho}_3) \) is the spin-isospin density of baryons induced by the \( \pi^c(\pi^0) \) condensation. It is to be noted that \( \tilde{\rho} \) becomes almost uniform and \( \tilde{\rho}_3 \) almost proportional to \( \cos k_0 z \) in accordance with the choice of parameters given in Eq. (2.13).

Concerning the parameter \( g' \), the values indicated in nuclear matter calculations are \( 0.5 \sim 0.6 f^2 \), and their density-dependence is rather small. The values inferred from the analysis of the Gamow-Teller giant resonance in finite nuclei are almost in the same magnitude. However, there still exist some uncertainties, such as the differences among \( g'_{NN} \) for nucleon-nucleon particle-hole states, \( g'_{ND} \) for delta-nucleon ones, and \( g'_{DD} \) for delta-delta ones. In the present study, we lay aside this uncertainty, and put \( g'_{NN} = g'_{ND} = g'_{DD} = g' \). Letting the universal value of \( g' \) have some varieties, we take two typical values \( 0.5 f^2 \) and \( 0.6 f^2 \) in this paper.

Next we introduce the following form factors at the pion-baryon (\( \pi-B \)) vertex as a vertex renormalization,

\[ F_\pi = (\Lambda^2 - m_\pi^2) / (\Lambda^2 + k_\pi^2 - \mu_\pi^2) \]  
\hspace{1cm} (2.41a)

and

\[ F_0 = (\Lambda^2 - m_0^2) / (\Lambda^2 + k_0^2) \]  
\hspace{1cm} (2.41b)
As for the cutoff factor, we choose the value of 1.2 GeV, referring to the theory of OBEP.\(^{34}\)

Then the two realistic effects above-mentioned are introduced together into the original effective Hamiltonian Equation (2-14) in the following way:

1. Replace \(a\) and \(b\) of Eqs. (2-15a, b) by

\[
\begin{align*}
\alpha &\rightarrow \alpha' = (-\tilde{f}_s k_0 A \cos \theta \cos k_0 z + g' F_0 \tilde{b}_3) F_0, \\
\beta &\rightarrow \beta' = (-\tilde{f}_s k_0 \sin \theta - g' F_c \tilde{b}) F_c.
\end{align*}
\] (2.42a, 2.42b)

2. Add the terms,

\[
-\frac{1}{2} g' F_0^2 \tilde{b}_3^2 - \frac{1}{2} g' F_c^2 \tilde{b}^2,
\] (2.43)

which correspond to the last term in Eq. (2.36).

2.4. Energy expressions and constraints

The energy density of the system is given by using Eqs. (2-9), (2-16) and (2-32a),

\[
\varepsilon^{\pi^-} = \varepsilon^{\pi^0} + \varepsilon^{\pi^+} + \varepsilon^{\pi^0 - \pi^-}
\] (2.44)

with

\[
\varepsilon^{\pi^0} = \frac{3}{5} g \kappa^2 \rho + \frac{k_0^2}{4 \pi^2 m} \rho,
\] (2.45a)

\[
\bar{\varepsilon}^{\pi^0} = (1 - N_0^{-1}) \delta M \cdot \rho - \tilde{f}_s F_0 N_0^{-1} \left( \frac{1}{5} x^2 + \frac{8}{5} y + 1 \right) + \frac{1}{2} g' F_0^2 \tilde{b}_3^2,
\] (2.45b)

\[
\bar{\varepsilon}^{\pi^+} = (1 - N^{-1}) \delta M \cdot \rho + \frac{1}{2} \mu_{\pi^0} - N^{-1} \left[ a_2 b' + (a_2^{-2} + a_1^{-2} b^2)^{1/2} \right] \rho
\]

\[
+ \frac{1}{2} f_\pi^2 \sin^2 \theta(k_1^2 - \mu_2^2) - f_\pi^2 m_\pi^2 (\cos \theta - 1) - \frac{1}{2} g' F_c^2 \tilde{b}^2
\] (2.45c)

and

\[
\varepsilon^{\pi^0 - \pi^-} = [N^{-1}(1 - y_2^2/4) - 1] (1 - N_0^{-1}) \delta M \cdot \rho
\]

\[
- N^{-1} (B_2 - a_3) b' \rho - N^{-1} \left[ (a_3^{-2} c^2 + B_1^{-2} b^2)^{1/2} - (a_3^{-2} c^2 + a_1^{-2} b^2)^{1/2} \right] \rho
\]

\[
- f_\pi F_0 k_0 A e^{-\pi/4 \tau} \left[ N^{-1} A \cos \theta - N_0^{-1} \left( \frac{1}{5} x^2 + \frac{8}{5} y + 1 \right) \right] \rho
\]

\[
+ \frac{1}{4} f_\pi k_0 A^2 (\cos^2 \theta - 1) - f_\pi^2 m_\pi^2 (\cos \theta - 1) [J_0(A) - 1],
\] (2.45d)

where \(a_1^{-1} = 1 + 2/5 \cdot 2y_2 (\sqrt{3} y_1 - y_2)\), \(a_2 = 4\sqrt{2}/5 \cdot (\sqrt{3} y_1 + y_2)\) and \(a_3^{-1} = 1 + 3y_2^2 - y_2^2\) are the same expressions as those in Ref. 12), and \(\tilde{b}_3^2\) means the spatial average of \(\tilde{b}_3^2\). In Eq. (2.45a), \(k_\pi = (3 \pi^2 \rho)^{1/3}\) is the Fermi momentum and \(E_\pi = k_\pi^2 / 2m\) the Fermi kinetic energy. \(\varepsilon^{\pi^0}\) represents the kinetic energy density of baryons, which is modified from
the 3-dim. Fermi gas one by the deformation of the Fermi surface due to the \( \pi^0 \) condensation. \( \bar{\varepsilon}^{\pi^0} \) represents the contribution from the \( \pi^0 \) condensation, \( \bar{\varepsilon}^{\pi^0} \) from the \( \pi^\pm \) one,\(^{12}\) and the last term \( \varepsilon^{\pi^0-\pi^\pm} \) is brought about by the \( \pi^0-\pi^\pm \) interactions.

The gauge parameters \( \chi, \theta \) and \( \varphi \) must satisfy the classical field equations derived from \( \mathcal{L} = \langle \text{Normal} | \bar{\mathcal{U}}_{\pi^0} \mathcal{L} \sigma \cdot \mathcal{U}_{\pi^0} | \text{Normal} \rangle \) and the resultant equations play the role of dynamical constraints on these parameters. General forms of the field equations in terms of \( \chi, \theta \) and \( \varphi \) are given in Appendix B. Here we concretely show the expressions by substituting Eqs. (2.13a, b) into them.

For the parameter \( \chi \), we get a trivial equation implying the conservation of the isospin current (see Appendix B).

For the parameter \( \theta \), from Eq. (B.6), we get

\[
f_x^2 \sin \theta \cos \left( \frac{1}{2} k_0^2 A^2 + \mu_x^2 - k_\perp^2 \right) - f_x^2 m_x^2 \sin \theta [J_0(A) + \sum_{n=1}^{\infty} 2 \cos 2n k_0 z \cdot J_{2n}(A)]
\]

\[= - \int f_\pi F_0 k_0 A \cdot \rho (N^{-1} A_1) (\Gamma / \pi)^{1/2} \sum_{n=0}^{\infty} f_{2n+1}(\Gamma) \cos [(2n+1) k_0 z] \cdot \cos k_0 z \]

\[+ \left[ \int f_\pi F_0 k_1 \rho \cos \theta N^{-1} (B_2 + B_1 \sin \phi) \right]
\]

\[= 0, \quad (2.46)\]

where we have used Eqs. (2.37) and (2.39). Finally for the parameter \( \varphi \), from Eq. (B.7), we get

\[f_{n0} k_0^2 A \cos \theta \sin k_0 z + 2 f_{n0} m_0^2 \sum_{n=0}^{\infty} \sin [(2n+1) k_0 z] \cdot J_{2n+1}(A)\]

\[= \int F_0 k_0 (N^{-1} A_1) \cdot \rho (\Gamma / \pi)^{1/2} \sum_{n=0}^{\infty} (2n+1) f_{2n+1}(\Gamma) \sin [(2n+1) k_0 z], \quad (2.47)\]

where we have used Eq. (2.39).

Self-consistency in our formalism under the assumption of Eq. (2.13b) demands the condition \( f_1(\Gamma) > f_3(\Gamma) \), i.e., \( \Gamma \approx 15 \) as is seen from Eq. (2.47), which has been already mentioned in the context of the pure \( \pi^0 \) condensation and leads to the lowest harmonics approximation (LHA).\(^{9,11,17-20}\) In fact, the realistic approach including short-range correlation effects to the \( \pi^0 \) condensation in the potential model approach has shown that \( \Gamma \) appears in the above-mentioned region.\(^{17,18}\) Under the LHA, Eq. (2.47) reads

\[f_{n0} k_0^2 A \cos \theta + 2 f_{n0} m_0^2 J_1(A) = 2 \int F_0 k_0 (N^{-1} A_1) \rho e^{-\pi^0 \Gamma}. \quad (2.48)\]

The field equation for \( \theta \), Eq. (2.46), is rewritten in the same manner. Terms \( \propto \cos 2n k_0 z (n \geq 1) \) almost cancel each other and we get

\[f_x^2 \sin \theta \cos \left( \frac{1}{2} k_0^2 A^2 + \mu_x^2 - k_\perp^2 \right) - f_x^2 m_x^2 \sin \theta \cdot J_0(A)\]

- $\bar{f}_\pi F_0 k_0 A \sin \theta \cdot \rho (N^{-1} A_1) e^{-x^2/4r} + \bar{f}_\pi F_0 k_\perp \rho \cos \theta \cdot N^{-1} (B_2 + B_1 \sin 2\phi) \nonumber \\
- \frac{1}{2} \mu_\pi \rho (N^{-1} a_3^{-1}) \sin \theta \cos 2\phi \nonumber \\
= 0. \quad (2.49)$

It is to be noted that Eq. (2.49) is equivalent to $\partial \varepsilon^{x_0, x_0} / \partial \theta = 0$, i.e., the energy-extremum condition with respect to $\theta$, while Eq. (2.48) can never be derived from such a condition, since it comes from the local constraint.

Finally, the ground state energy is obtained by minimizing $\varepsilon^{x_0, x_0}$ (Eq. (2.44)) with respect to parameters $k_0, x, \mu_\pi$, $k_\perp$ and $\gamma_1$ under the constraints Eqs. (2.48) and (2.49) and the charge neutrality condition:

$$\frac{1}{2} (1 - N^{-1} a_3^{-1} \cos \theta \cos 2\phi) \rho - \mu_\pi f_\pi^2 \sin^2 \theta = 0. \quad (2.50)$$

§ 3. Numerical results and discussion

3.1. Pure $\pi^0$ condensation

We begin with the pure $\pi^0$ condensed phase within the present formalism in order to extract its intrinsic features from the properties of the CPC phase. Since several realistic investigations concerning the pure $\pi^0$ condensation in neutron matter have been done,17)-21) we also compare our results with some of them in the following.

By setting the quantities concerning the $\pi^c$ condensate to be zero, the energy expression Eq. (2.44) reduces to the one of the pure $\pi^0$ condensed case,

$$\varepsilon^{x_0} / \rho = \frac{3}{5} E_\pi \frac{10 k_\pi}{9 k_0} + \frac{k_\pi^2 \Gamma}{4 \pi^2 m} + (1 - N_0^{-1}) \delta M - \bar{f}_\pi F_0 G(x) k_0 A e^{-x^2/4r} \nonumber \\
+ \frac{1}{4} f_\pi k_\pi^2 A^2 / \rho - f_\pi m_\pi^2 [J_0(A) - 1] / \rho + \frac{1}{8} \gamma_1 \rho_0^2 \bar{\rho}_N^2 / \rho, \quad (3.1)$$

where the last term in Eq. (3.1) is given by

$$\frac{1}{2} \gamma_1 \rho_0^2 \bar{\rho}_N^2 / \rho = \gamma_1 \rho_0^2 \rho (G(x))^2 e^{-x^2/8r} \quad (3.2)$$

and

$$G(x) = N_0^{-1} \left( \frac{1}{8} x^2 + \frac{8 \sqrt{2}}{5} x + 1 \right). \quad (3.3)$$

Here the variational parameters are $k_0, A, \Gamma$ and $x$, constrained by Eq. (2.48),

$$f_\pi k_\pi^2 A + 2 f_\pi m_\pi^2 J_1(A) = 2 \bar{f}_\pi F_0 G(x) \rho e^{-x^2/4r} \quad (3.4)$$

by setting $\theta = \gamma_1 = 0$. Furthermore, the available range of $x$ is restricted by the extremum condition $\partial \varepsilon^{x_0} / \partial x = 0$, which gives

$$x G(x) / [(1 - \sqrt{2} x)(\sqrt{2} + x)] = 4 k_\pi^2 \Gamma^2 / (5 \pi^4 m \delta M). \quad (3.5)$$

*) As for the pure $\pi^c$ condensate within the same formalism, see Ref. 12).
Since the r.h.s. of Eq. (3·5) is always positive, the condition for the positivity of the l.h.s. yields $0 < x < 1 / \sqrt{2}$.

Our numerical results show that it is possible to get an appropriate energy minimum which satisfies the self-consistency condition mentioned at the end of § 2, only over the limited density region. For example, in the case $g' = 0.5 \tilde{f}^2(0.6 \tilde{f}^2)$, the energy minima disappear for $\rho \gtrsim 3.5 \rho_0 (\rho \gtrsim 4.5 \rho_0)$, and only the false minima with large $\Gamma(\gg 15)$ exist beyond the LHA. It also turns out to be clear that the higher-order terms of $A$ in Eqs. (3·1) and (3·4), which are brought about by the self interactions of pions,\textsuperscript{*} control the global behaviour of the energy density with respect to the parameters $A$ and $\Gamma$ in such densities. As a result, the minima relevant to the physically meaningful situation are concealed by being suffered from the large $A$ and $\Gamma$ region, and then the self-consistency breaks down. We, however, expect that in a more refined treatment such false minima are clean swept and only the physically meaningful minima with small $A$ and $\Gamma$ survive there.

From the above arguments we must draw out physically meaningful minima by suppressing only large $A$ and $\Gamma$ effects and recover the self-consistency in our formalism. To this end we apply the following “conditional lowest harmonics approximation” (CLHA). First we rewrite the energy expression Eq. (3·1) by using Eq. (3·4) and successively divide it into two parts,

$$
\varepsilon^{\pi_3}/\rho = \varepsilon^{\pi_3}_{\text{CLHA}}/\rho + \delta \varepsilon^{\pi_3}/\rho .
$$

(3·6)

The first term on the r.h.s. is the energy per baryon in the CLHA which is obtained by truncating $J_0(A)$ and $J_1(A)$ into the lowest order of $A, J_0(A) \rightarrow 1 - A^2/4$ and $J_1(A) \rightarrow A/2$, 

$$
\varepsilon^{\pi_3}_{\text{CLHA}}/\rho = \varepsilon^{\pi_3}_{\text{CLHA}}(\text{kin})/\rho + \varepsilon^{\pi_3}_{\text{CLHA}}(\text{pot})/\rho (3·7)
$$

where

$$
\varepsilon^{\pi_3}_{\text{CLHA}}(\text{kin})/\rho = \frac{3}{9} E_F \frac{10 k_F}{9 k_0} + \frac{k_0^2 \Gamma_0}{4 \pi^2 m} + (1 - N_0^{-1}) \delta M
$$

(3·8)

and

$$
\varepsilon^{\pi_3}_{\text{CLHA}}(\text{pot})/\rho = -\frac{1}{4} f_\pi^2 A^2 \omega_0^2 \left(1 - \frac{g'}{f_\pi^2} \frac{\omega_0^2}{k_0^2} \right)/\rho .
$$

(3·9)

Here $\Gamma_0$ is defined by the following equation,

$$
A = \frac{2 f F_0}{f_\pi} \frac{k_0}{\omega_0^2} G(x) \rho e^{-\pi^2 \sigma_0^2}
$$

(3·10)

with

$$
\omega_0^2 = k_0^2 + m_\pi^2.
$$

Then the energy expression (3·7) has essentially the same form as the conventional one taken by Kunihiro and Tamagaki\textsuperscript{17}. The remaining term $\delta \varepsilon^{\pi_3}/\rho$ contains higher-

\textsuperscript{*} It is to be noted that the existence of such higher-order self interactions and the description of them within the $\sigma$ model is somewhat obscure now.
order terms of $A$ than $O(A^4)$, whose concrete form is not concerned hereafter. We minimize only $\varepsilon_{\text{LHA}}^{\text{eff}}/\rho$ and finally check up that resultant parameters satisfy the self-consistency condition and the contribution from $\delta \varepsilon^{\text{eff}}/\rho$ is small. Here we put a criterion that $\delta \varepsilon^{\text{eff}} / \varepsilon_{\text{LHA}}^{\text{eff}}(\text{pot}) \lesssim 0.1$. It is to be noted that this criterion is analogous to the one taken for the LHA. As is seen later, the CLHA works well in the density region we consider. We can also find the efficacy of the CLHA in the densities where the full expression Eq. (3.1) has the relevant minima satisfying the self-consistency condition. The energy minima which we have got by using the CLHA for such densities are almost the same as those within the full expression, and $\delta \varepsilon^{\text{eff}} / \varepsilon_{\text{LHA}}^{\text{eff}}(\text{pot}) = 0.030(0.026)$ for $g' = 0.5 \tilde{f}^2(0.6 \tilde{f}^2)$ at the typical density $\rho = 3.5 \rho_0(4 \rho_0)$ (see Table I).

In Fig. 2, we show the energy differences under the CLHA, $\Delta \varepsilon^{\text{eff}} / \rho = \varepsilon_{\text{LHA}}^{\text{eff}} / \rho - \varepsilon_{\text{FG}} / \rho$ as a function of $\rho$, where $\varepsilon_{\text{FG}} / \rho = 3/5 E_F$ is the 3-dim. Fermi gas energy. The critical density $\rho_c$ for the pure $\pi^0$ condensation where $\Delta \varepsilon^{\text{eff}} / \rho = 0$ is $\rho_c \sim 3.0 \rho_0(\sim 5.2 \rho_0)$ for $g' = 0.5 \tilde{f}^2(0.6 \tilde{f}^2)$. It is to be noted that $\rho_c$ is strongly dependent on $g'$, since this feature reflects the subtleness of the localization to the short-range correlations. The critical point obtained by Tamiya and Tamagaki based on the reaction matrix theory is somewhat lower than our result; $\rho_c \sim 2 \rho_0$. However, these two results should not be considered as inconsistent with each other within the allowance of $g'$. Indeed we can get $\rho_c \sim 2 \rho_0$ by choosing $g' / \tilde{f}^2 = 0.4$ which is estimated from their result, $g' / \tilde{f}^2 = 0.43$ (denoted as $g_L$ in Ref. 19) at $\rho = 3 \rho_0$, based on the assumption of weak $\rho$ dependence of $g'$.

For comparison, we also give the critical density $\tilde{\rho}_c$ obtained by the threshold condition for the pion Green function $D$,

$$D^{-1}(\omega, k_0; \rho) \Big|_{\rho=0} = 0,$$  

$$\partial D^{-1}(\omega, k_0; \rho) / \partial k_0 \Big|_{\rho=0} = 0,$$  

where $D^{-1}(\omega, k_0; \rho)$ is given by

$$D^{-1}(\omega, k_0; \rho) = \omega^2 - \omega_0^2 + \tilde{f}^2 F_0^2 k_0^2 U(\omega, k_0; \rho)$$

$$/[1 + g' F_0^2 U(\omega, k_0; \rho) + \frac{8}{9} (\tilde{f}^2 / \tilde{f}) U(\omega, k_0; \rho) \cdot \cdot \cdot]$$

Fig. 2. The energy gain per baryon of the $\pi^0$ condensed phase from the 3-dim. Fermi gas one: $\Delta \varepsilon^{\text{eff}} / \rho = \varepsilon_{\text{LHA}}^{\text{eff}} / \rho - \varepsilon_{\text{FG}} / \rho$. The bold (thin) dashed line shows the result with $g' = 0.5 \tilde{f}^2(0.6 \tilde{f}^2)$. The bold and thin arrows show the corresponding critical density $\tilde{\rho}_c$ obtained by the threshold condition for the pion Green function. The cutoff factor $\Lambda$ is commonly chosen as 1.2 GeV. See the text for details.
On the Existence of Combined Condensation

Table I. The density and $g'$ dependence of the parameters in the pure $\pi^0$ condensed phase within the CLHA. $k_0$ is the momentum of the condensed $\pi^0$, $A$ the amplitude of the $\pi^0$ field (Eq. (2-13b)), $\Gamma$ the localization parameter of baryons, and $P_x$ the $\Delta$-mixing probability. The energy gain in the $\pi^0$ condensed phase, $\Delta e^{\pi^0}/\rho = e^{\pi^0}/\rho - E_{\pi^0}/\rho$, and the ratio $\delta e^{\pi^0}/E^{\pi^0}_{\text{CLHA}}(\text{pot})$ ($\delta e^{\pi^0}$ and $E^{\pi^0}_{\text{CLHA}}(\text{pot})$ are defined by Eqs. (3-6~9)) are also listed. The dashed lines are inserted to show the boundary of $\rho = \rho_c$. The numerical values in parentheses are those obtained by minimizing the full energy expression in the densities where they satisfy the self-consistency condition.

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<th>$A$</th>
<th>$\Gamma$</th>
<th>$P_x$ (%)</th>
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<th>$\delta e^{\pi^0}/E^{\pi^0}_{\text{CLHA}}(\text{pot})$ ($\times 10^{-2}$)</th>
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In Eq. (3·15), $U_N(U_d)$ is the Lindhard function for the nucleon-nucleon particle-hole ($\Delta$-nucleon particle-hole), and $T_\pi = T_{\pi N}/m_\pi$ is the $\pi N \Delta$ coupling constant. $\tilde{\rho}$ is indicated in Fig. 2 by an arrow for each $g'$, $\tilde{\rho} \sim 2.1 \rho_0(\sim 3.2 \rho_0)$ and the critical momentum $k_c \sim 2.1 \text{fm}^{-1}(\sim 2.2 \text{fm}^{-1})$ for $g' = 0.5 \tilde{f}^2(0.6 \tilde{f}^2)$. There is rather large difference between $\rho_c$ and $\tilde{\rho}_c$. For $g' = 0.5 \tilde{f}^2(0.6 \tilde{f}^2)$, $\rho_c - \tilde{\rho}_c \sim 0.9 \rho_0(\sim 2.0 \rho_0)$. This discrepancy is due to the circumstance that the deformation of the Fermi surface near the critical point cannot be completely pursued within the ALS model.15,16) We briefly sketch this process between $\rho_c$ and $\tilde{\rho}_c$: The $\pi^0$ condensation sets in at the lower density $\rho_c$, and as density increases, deformation of the Fermi surface proceeds very slowly accompanying the 1-dim. localization of the baryonic system. At the higher density near $\rho_c$, the Fermi surface is well deformed into a cylindrical form, where the ALS structure is well developed (see Table I). These features near $\rho_c$ are qualitatively the same as those of Ref. 19) and other works on the ALS structure.9,11,17,20)

Concerning the energy gain $\Delta e^{\pi^0}/\rho$, it increases rapidly with $\rho$, and depends strongly on $g'$ (see also Table I). These features are in contrast with the case of the pure $\pi^c$ condensation, where the energy gain is moderate with increase of $\rho$, and not so strongly dependent on $g'$.12)

The resultant values of the parameters are listed in Table I. The ratio $\delta e^{\pi^0}/E^{\pi^0}_{\text{CLHA}}(\text{pot})$ at each density is also given there. The parameters increase monotonically with $\rho$ as a whole. Among them, the increase in $k_0$ is very moderate, or nearly constant. The $\Delta$-mixing probability $P_x$ is 9% (13%) for $g' = 0.5 \tilde{f}^2(0.6 \tilde{f}^2)$ around $\rho = \rho_c$, and 21% (16%) for $g' = 0.5 \tilde{f}^2(0.6 \tilde{f}^2)$ around $\rho = 6 \rho_0$. It is also found that the values of $A$ and $\Gamma$ are moderately small enough to satisfy the self-consistency
condition, and the ratio \( \partial \varepsilon^{\pi^0}/\varepsilon_{\text{CLHA}}(\text{pot}) \) is less than 0.1 over the relevant density region. Thus we can conclude that the CLHA works well.

3.2. Combined \( \pi^0-\pi^c \) condensation

In the following, we concentrate on the case of the combined \( \pi^0-\pi^c \) condensation. We also find that the energy minima, which satisfy the self-consistency condition, do not exist over some densities due to the breaking down effects originating from the \( \pi^0 \) condensation (see § 3.1). Therefore, we hereafter apply the CLHA in the same spirit as in § 3.1,

\[
\varepsilon^{\pi^0-\pi^c}/\rho = \varepsilon^{\pi^0-\pi^c}_{\text{CLHA}}/\rho + \partial \varepsilon^{\pi^0-\pi^c}/\rho,
\]

where

\[
\varepsilon^{\pi^0-\pi^c}_{\text{CLHA}}/\rho = \varepsilon^{\pi^0-\pi^c}_{\text{CLHA}}(\text{kin})/\rho + \varepsilon^{\pi^0-\pi^c}_{\text{CLHA}}(\text{pot})/\rho
\]

with

\[
\varepsilon^{\pi^0-\pi^c}_{\text{CLHA}}(\text{kin})/\rho = \frac{3}{5} E_{\pi} \frac{10 k_F}{9 k_0} + \frac{k_0^2 \Gamma_1}{4 \pi^2 m} + N^{-1} R \delta M,
\]

\[
\varepsilon^{\pi^0-\pi^c}_{\text{CLHA}}(\text{pot})/\rho = \frac{1}{\pi} \mu_\pi - N^{-1} \left[ B_2 b' + (a_2 c^2 + B_1^2 b'^2)^{1/2} \right] + \frac{1}{\pi} f_\pi^2 \sin^2 \theta (k_L^2 - \mu^2) \rho + f_\pi^2 m_\pi^2 (1 - \cos \theta)/\rho
\]

\[
- \frac{1}{4} f_\pi^2 A^2 (k_0^2 \cos \theta + m_\pi^2) \left[ \cos \theta - \frac{g'}{f_\pi^2} \frac{k_0^2 \cos \theta + m_\pi^2}{k_0^2} \right]/\rho
\]

\[
- \frac{1}{2} g' F_0 \bar{\rho}^2/\rho
\]

with \( \Gamma_1 \) defined by

\[
A = \frac{2 \bar{f} F_0}{f_\pi^2} \frac{k_0}{k_0^2 \cos \theta + m_\pi^2} (N^{-1} A_1) \rho e^{\pi^0 A_1}.
\]

As is seen later, the CLHA works well over the relevant densities also in this case.

In Fig. 3, we show the energy per baryon in the CPC phase obtained under the CLHA. Results for the pure \( \pi^0 \) condensed phase given in § 3.1 and the pure \( \pi^c \) one\(^{12}\) are drawn together there. It is found that the \( \pi^c \) condensed phase firstly appears at the lower density; \( \rho_c^{\pi^c} \sim 1.5 \rho_0 (\sim 2.2 \rho_0) \) for \( g' = 0.5 \bar{f}^2(0.6 \bar{f}^2) \),\(^{12}\) and later the CPC phase is realized at the higher density. The energy per baryon of the pure \( \pi^0 \) condensed phase (dashed line) is always higher than that of the pure \( \pi^c \) condensed phase (dotted line) or the CPC phase (solid line). Thus the pure \( \pi^0 \) condensed phase can never appear solely over the relevant densities. This is qualitatively different from the case of the SMC, where the \( \pi^c \) condensation is preceded by the \( \pi^0 \) one, and the pure \( \pi^c \) condensed phase does not appear solely there.\(^{13,14}\)

The critical density \( \rho_c^{\pi^0-\pi^c} \) where the energy difference from the pure \( \pi^c \) condensed phase gets into zero is, \( \rho_c^{\pi^0-\pi^c} \sim 3.5 \rho_0 (\sim 5.5 \rho_0) \) for \( g' = 0.5 \bar{f}^2(0.6 \bar{f}^2) \). Here we can
also see the strong $g'$-dependence of $\rho_{\pi^0,\pi^c}$ due to the $\pi^0$ condensation as in § 3.1. It is to be noted that $\rho_{\pi^0,\pi^c}$ defined in this way is expected to be somewhat larger than that determined with use of the pion Green function on the same reason as that of the pure $\pi^0$ case (see § 3.1).

In Fig. 4, $\Delta\varepsilon_{\pi^0,\pi^c}/\rho$, the energy gain per baryon from the 3 dim. Fermi gas for each pion condensation is given as the function of $\rho$. It is to be noted that the simple additivity of the energy gain shown in Refs. 13) and 14), $\Delta\varepsilon_{\pi^0,\pi^c}/\rho = \Delta\varepsilon_{\pi^0}/\rho + \Delta\varepsilon_{\pi^c}/\rho$, does not hold even near the onset density $\rho_{\pi^0,\pi^c}$. The repulsive effects coming from the $\pi^0\pi^c$ interaction considerably reduce the energy gain from the simple sum of those of $\pi^0$ and $\pi^c$ condensation. This is another qualitatively different feature from the one in the SMC.10,14) Main contributions to the repulsive effects come from the net effect of the two terms in Eqs. (3·16) and (3·17),

\begin{align*}
(1) & \quad -\frac{k_0^2}{4\pi^2 m_\pi} \cdot \frac{2\eta_0^2}{2\omega_0^2} \sin^2 \theta \simeq -30 \text{ MeV} (-18 \text{ MeV}) \quad \text{for } g' = 0.5 \bar{f}^2 (0.6 \bar{f}^2) \\
(2) & \quad \frac{1}{8} \pi^2 A^2 \sin^2 \theta \cdot \left( 2k_0^2 + m_\pi^2 - 2 \frac{g'}{f_2} \omega_0^2 \right) / \rho \simeq 41 \text{ MeV} (26 \text{ MeV}) \quad \text{for } g' = 0.5 \bar{f}^2 (0.6 \bar{f}^2)
\end{align*}

at $\rho \sim \rho_{\pi^0,\pi^c}$, where we keep only the 2nd order of $\sin \theta \propto \langle \pi^0, \pi^c | \hat{\pi}^0 | \pi^0, \pi^c \rangle$ because of small values of $\theta = 0.4 \sim 0.5$. Thus we find that the sum of (1) and (2) is nearly equal to the energy difference $\Delta\varepsilon_{\pi^0,\pi^c}/\rho - (\Delta\varepsilon_{\pi^0}/\rho + \Delta\varepsilon_{\pi^c}/\rho)$ indicated in Fig. 4.

The numerical value of $\Delta\varepsilon_{\pi^0,\pi^c}/\rho$ and its contribution to the pressure, $\Delta P(\rho) = \rho^2 \cdot \delta/(\Delta\varepsilon_{\pi^0,\pi^c}/\rho)$, are listed in Table II. Due to the $\pi^0$ condensation, the energy gain is large, and grows almost linearly to $\rho - \rho_{\pi^0,\pi^c}$. We can also find that the energy gain depends sensitively on $g'$ due to the same reason for $\rho_{\pi^0,\pi^c}$.

The resultant values of the parameters and the ratio $\delta\varepsilon_{\pi^0,\pi^c}/\varepsilon_{\pi^0,\pi^c}(\text{pot})$ are given in Table II together with those for the minima obtained with the full expression Eq. (2·44) once they satisfy the self-consistency condition. The values and behaviour of the parameters referring to the $\pi^0$ condensate are not so changed (see Table I), while those referring to $\pi^c$ one show somewhat different features. The chiral angle $\theta$ and the $\Delta$-mixing probabil-

---

* The pressure of the system obtained by the following ansatz $P(\rho) = P_0(\rho) + \Delta P(\rho)$ where $P_0(\rho)$ is the one of the normal phase, always remains positive even in the case of a soft $P_0(\rho)$ such as obtained from the Reid soft core potential. This point will be shown in detail in a subsequent paper.
ity \( P_x \) turn to decrease at a certain density due to the \( \pi^0 - \pi^\circ \) interactions. Since the energy gain due to the \( \pi^0 \) condensation is much larger than that of the \( \pi^\circ \) one, the former controls the behaviour of the whole energy gain once the \( \pi^0 \) condensation appears at the higher density. Hence, the quantities with respect to the \( \pi^\circ \) condensation are gradually suppressed as the density increases beyond \( \rho_c \), while those of the \( \pi^0 \) condensation increase monotonically.

\( \Delta \)-mixing probability \( P_x = x^2 / (1 + x^2) \) due to the \( \pi^0 \) condensation is 10% and 19% at \( \rho = 3.5 \rho_0 \) and 6.0 \( \rho_0 \), respectively, while \( P_y \) is 7% and 4% for the corresponding densities with \( \rho' = 0.5 \bar{f}^2 \). Thus the former \((P_x)\) is somewhat larger than the latter \((P_y)\) over the density region, \( \rho > \rho_c \). The same feature also can be seen in the case of \( \rho' = 0.6 \bar{f}^2 \).

\( \Gamma \) and \( \rho \) referring to the \( \pi^0 \) condensation stay in magnitude within the limit of the self-consistency, and the ratio \( \delta \varepsilon^{\pi^0 - \pi^\circ} / \varepsilon^{\pi^0 - \pi^\circ}_{\text{CLHA}}(\text{pot}) \) is less than 0.1 over the relevant density region. See Table I. It is to be noted that the results on the CLHA reproduce similar values to those given from the full energy expression for such densities where they satisfy the self-consistency. Thus the efficacy of the CLHA is proved to be well over the relevant density region also in this case.

### § 4. Summary and concluding remarks

We have discussed the EOS of the combined \( \pi^0 - \pi^\circ \) condensed (CPC) state in neutron matter by doing a realistic calculation, where isobars \( \Delta \) (1232), baryon-baryon short-range correlations and form factors in \( \pi^- B \) vertex are included, and found that the pure \( \pi^\circ \) condensation occurs first at relatively low densities \((1.5 \rho_0 \sim 2.2 \rho_0)\) and the \( \pi^\circ \) condensed phase is gradually superseded by the CPC phase as density goes higher. It is also to be noted that the region of the pure \( \pi^0 \) condensation alone never appears.

Table II. The density and \( \rho' \) dependence of the parameters and energy and pressure gains in the CPC within the CLHA. The ratio \( \delta \varepsilon^{\pi^0 - \pi^\circ} / \varepsilon^{\pi^0 - \pi^\circ}_{\text{CLHA}}(\text{pot}) \) (\( \delta \varepsilon^{\pi^0 - \pi^\circ} \) and \( \varepsilon^{\pi^0 - \pi^\circ}_{\text{CLHA}}(\text{pot}) \) are defined by Eqs. (3.14 ~ 17)) are also listed at each density. \( \theta \) is the chiral angle, \( \mu_c \) the chemical potential, \( k_b \) the momentum of the condensed \( \pi^\circ \), \( \bar{\rho} \) the spin-isospin density, and \( P_o \) the \( \Delta \)-mixing probability. The meaning of other parameters are the same as in Table I. \( \delta \varepsilon^{\pi^0 - \pi^\circ}_p/\rho = \varepsilon^{\pi^0 - \pi^\circ}_p - \varepsilon^{\pi^\circ}_p \) and \( \Delta P = P^{\pi^0 - \pi^\circ}_p - P_o \) mean the energy and pressure gain, respectively. The numerical values in parentheses are those obtained by minimizing the full energy expression in the same manner as in Table I.
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Fig. 4. The energy gain per baryon of the CPC from the 3-dim. Fermi gas one. As in Fig. 3, the other cases are also given for comparison. The case of the pure \( \pi^0 \) condensation is the same as that of Fig. 2. The dash-dotted line represents the simple sum of energy gain in \( \pi^0 \) and \( \pi^e \) condensed phase: \( \Delta \varepsilon_{\pi^0} / \rho + \Delta \varepsilon_{\pi^e} / \rho \), and repulsive effects due to \( \pi^0-\pi^e \) interactions are indicated by arrows (see the text).

The energy gain per baryon of the CPC from the 3-dim. Fermi gas one. As in Fig. 3, the other cases are also given for comparison. The case of the pure \( \pi^0 \) condensation is the same as that of Fig. 2. The dash-dotted line represents the simple sum of energy gain in \( \pi^0 \) and \( \pi^e \) condensed phase: \( \Delta \varepsilon_{\pi^0} / \rho + \Delta \varepsilon_{\pi^e} / \rho \), and repulsive effects due to \( \pi^0-\pi^e \) interactions are indicated by arrows (see the text).

all over the densities. These features have not been seen within the simple model calculations (SMC) where the pure \( \pi^0 \) condensation precedes others and the pure \( \pi^e \) condensed phase cannot exist solely at any density.

The critical densities for the combined \( \pi^0-\pi^e \) condensation are scattered around \((2 \sim 5) \rho_0\) depending on the value of the Landau-Migdal parameter \( g' \) in contrast with the pure \( \pi^e \) condensed case. It is caused by the strong \( g' \)-dependence of the critical density for the \( \pi^0 \) condensation, reflecting the subtleness of the effects of short-range correlations on the localization. The large energy gain is also due to the \( \pi^0 \) condensate. As for the additivity of energy gains coming from both \( \pi^0 \) and \( \pi^e \) condensations, we have found that the simple additivity does not hold even around the critical density because of the repulsive interaction among pions, especially \( \pi^0-\pi^e \) interactions. This feature is also different from the one within the SMC.

The conditional lowest harmonics approximation (CLHA) proposed in this paper has been needed to ensure the self-consistency among approximations involved in our model and proved to work well at the relevant densities \(( \lesssim (5 \sim 6) \rho_0) \). The importance of the self-consistency is to be emphasized in connection with the \( \pi^0 \) condensation or even the ALS model.

The results given in this paper provide a foundation of our later studies. For example, the quantities characterizing the global structure of neutron stars, the mass-radius relation, the moment of inertia and so on, can be calculated by combining them with the dynamical equations derived from the general relativity. The cooling curve of neutron stars can be also calculated with them. Our results on these subjects will be reported elsewhere.

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Here we present the explicit forms of the quasi-baryonic states,

\begin{align}
|\bar{p}_{1/2}\rangle &= N^{-1/2} \left[ |\bar{p}_{1/2}\rangle + i\gamma_1 \left( \frac{1}{2} \vec{k}_+ |\Delta^+_{1/2}\rangle - \frac{\sqrt{3}}{2} \vec{k}_- |\Delta^0_{1/2}\rangle \right) \right. \\
& \quad \quad \quad \quad \quad \left. + i\gamma_2 \left( \frac{1}{2} \vec{k}_+ |\Delta^0_{1/2}\rangle - \frac{\sqrt{3}}{2} \vec{k}_- |\Delta^+_{1/2}\rangle \right) \right] \\
|\bar{n}_{-1/2}\rangle &= N^{-1/2} \left[ |\bar{n}_{-1/2}\rangle + i\gamma_1 \left( \frac{\sqrt{3}}{2} \vec{k}_+ |\Delta^0_{-1/2}\rangle - \frac{1}{2} \vec{k}_- |\Delta^+_{1/2}\rangle \right) \right. \\
& \quad \quad \quad \quad \quad \left. + i\gamma_2 \left( \frac{\sqrt{3}}{2} \vec{k}_+ |\Delta^+_{-1/2}\rangle - \frac{1}{2} \vec{k}_- |\Delta^0_{1/2}\rangle \right) \right] \\
|\bar{n}_{1/2}\rangle &= N^{-1/2} \left[ |\bar{n}_{1/2}\rangle + i\gamma_1 \left( \frac{1}{2} \vec{k}_+ |\Delta^+_{1/2}\rangle - \frac{\sqrt{3}}{2} \vec{k}_- |\Delta^0_{1/2}\rangle \right) \right. \\
& \quad \quad \quad \quad \quad \left. + i\gamma_2 \left( \frac{1}{2} \vec{k}_+ |\Delta^0_{1/2}\rangle - \frac{\sqrt{3}}{2} \vec{k}_- |\Delta^+_{1/2}\rangle \right) \right] \\
|\bar{p}_{-1/2}\rangle &= N^{-1/2} \left[ |\bar{p}_{-1/2}\rangle + i\gamma_1 \left( \frac{\sqrt{3}}{2} \vec{k}_+ |\Delta^0_{-1/2}\rangle - \frac{1}{2} \vec{k}_- |\Delta^+_{1/2}\rangle \right) \right. \\
& \quad \quad \quad \quad \quad \left. + i\gamma_2 \left( \frac{\sqrt{3}}{2} \vec{k}_+ |\Delta^+_{-1/2}\rangle - \frac{1}{2} \vec{k}_- |\Delta^0_{1/2}\rangle \right) \right] \\
|\bar{n}_{1/2}\rangle &= N^{-1/2} \left[ |\bar{n}_{1/2}\rangle + i\gamma_1 \left( \frac{1}{2} \vec{k}_+ |\Delta^+_{1/2}\rangle - \frac{\sqrt{3}}{2} \vec{k}_- |\Delta^0_{1/2}\rangle \right) \right. \\
& \quad \quad \quad \quad \quad \left. + i\gamma_2 \left( \frac{1}{2} \vec{k}_+ |\Delta^0_{1/2}\rangle - \frac{\sqrt{3}}{2} \vec{k}_- |\Delta^+_{1/2}\rangle \right) \right] \\
\text{(A-1)}
\end{align}

with \( N = 1 + y_1^2 + y_2^2 \) and

\begin{align}
|\bar{p}_{1/2}\rangle &= N_0^{-1/2} (|p_{1/2}\rangle + x |\Delta^+_{1/2}\rangle) , \\
|\bar{p}_{-1/2}\rangle &= N_0^{-1/2} (|p_{-1/2}\rangle - x |\Delta^+_{1/2}\rangle) , \\
|\bar{d}_{-1/2}\rangle &= N_0^{-1/2} (|\bar{d}_{-1/2}\rangle + x |\Delta^0_{-1/2}\rangle) , \\
|\bar{d}_{1/2}\rangle &= N_0^{-1/2} (|\bar{d}_{1/2}\rangle - x |\Delta^+_{1/2}\rangle) , \\
|\bar{n}_{-1/2}\rangle &= N_0^{-1/2} (|n_{-1/2}\rangle + x |\Delta^0_{-1/2}\rangle) , \\
|\bar{n}_{1/2}\rangle &= N_0^{-1/2} (|n_{1/2}\rangle - x |\Delta^0_{1/2}\rangle) , \\
|\bar{d}_{0,1/2}\rangle &= N_0^{-1/2} (|\bar{d}_{0,1/2}\rangle + x |\Delta^0_{0,1/2}\rangle) , \\
|\bar{d}_{0,1/2}\rangle &= N_0^{-1/2} (|\bar{d}_{0,1/2}\rangle - x |\Delta^0_{0,1/2}\rangle) , \\
\text{(A-2)}
\end{align}

with \( N_0 = 1 + x^2 \). In the above expressions, we take the \( \pi^0 \) condensed momentum \( \vec{k}_0 \) perpendicular to \( \vec{k} \). To relate them to the previous forms given in Ref. 12), we rewrite each baryonic state Eq. (A-1). We consider the case, \( x = 0 \) for simplicity. For the quasi-baryonic state \( |\bar{n}_{1/2}\rangle \), we get

\begin{align}
|\bar{n}_{1/2}\rangle &= N^{-1/2} \left[ |n_{1/2}\rangle + i\gamma_1 \sum_i |\Delta^i \rangle <\Delta^i| \tau^+ (\vec{k}_+ \sigma^- + \vec{k}_- \sigma^+) |n_{1/2}\rangle \frac{1}{2\sqrt{6}} \right. \\
& \quad \quad \quad \quad \quad \left. - i\gamma_2 \sum_i |\Delta^i \rangle <\Delta^i| \tau^- (\vec{k}_+ \sigma^- + \vec{k}_- \sigma^+) |n_{1/2}\rangle \frac{3}{2\sqrt{6}} \right] . \\
\text{(A-3)}
\end{align}
We define the following space-rotation operator,
\[ \hat{L} = \exp\left( i \frac{\pi}{4} J_3 \right) \cdot \exp\left( i \frac{\phi}{2} J_3 \right) \]  
(A.4)

with \( \tan \phi = k_2/k_1 \) to satisfy the relation,
\[ \hat{L}(k_1 \cdot \sigma \cdot \sigma) \hat{L}^{-1} = \hat{L}(\hat{k}_1 \cdot \sigma \cdot \sigma + \hat{k}_2 \cdot \sigma \cdot \sigma) \hat{L}^{-1} \]
\[ = \hat{k}_3 \sigma \cdot \sigma . \]  
(A.5)

Operating \( \hat{L} \) on Eq. (A.3), we get
\[ \hat{L}|\bar{n}_{1/2}\rangle = N^{-1/2}\left[ |\bar{n}_{1/2}\rangle \cdot \mathcal{L}_i (|\bar{n}_{1/2}\rangle \cdot \mathcal{L}_i) \cdot \frac{3}{2\sqrt{2}} - i\gamma \sum_t \mathcal{L}_t (|\bar{n}_{1/2}\rangle \cdot \mathcal{L}_t) \cdot \frac{3}{2\sqrt{2}} \right] , \]  
(A.6)

where Eq. (A.5) has been used. Furthermore, we redefine the new baryonic states, \( |\bar{n}_{1/2}\rangle = N^{-1/2}[|n_{1/2}\rangle \cdot \mathcal{L}_i (|n_{1/2}\rangle \cdot \mathcal{L}_i) \cdot \frac{3}{2\sqrt{2}} \]  
(A.7)

Thus we get the same expression as Eq. (2.7) of Ref. 12). The other states can be also shown to be the same forms in a similar manner.

Appendix B

Field equations for \( \chi, \theta \) and \( \varphi \) in their general form are derived from the chiral-transformed \( \sigma \) model Lagrangian density, \( \mathcal{L}_{\sigma - \text{model}} = \hat{U}_{\sigma - \pi} \mathcal{L}_{\sigma - \text{model}} \hat{U}^\dagger_{\pi - \pi} \) with
\[ \mathcal{L}_{\sigma - \text{model}} = \frac{1}{2} \left( \partial_\mu \sigma \partial^\mu + \partial_\mu \pi \cdot \partial^\mu \pi \right) + \frac{1}{2} m_0^2 (\sigma^2 + \pi^2) \]
\[ - \frac{m_0^2}{4} (\pi^2 + \pi^2)^2 + \bar{q} \left[ \gamma^\alpha \gamma^\beta \partial_\alpha \sigma + \sigma + i \pi^a \cdot \gamma r \gamma^5 \right] q + f_\pi m_\pi^2 \sigma . \]  
(B.1)

For the parameter \( \chi \), we get
\[ \frac{\partial}{\partial t} \left( \frac{q^+ z_q}{2} \cos \theta \cdot q \right) + \frac{3}{5} f_{\pi} F_c \cdot \langle q^+ \sigma^a z_q^a \sin \theta \cdot q \rangle \]
\[ + f_\pi^2 \partial^\alpha (\sin^2 \theta \cdot \partial_\alpha \chi) = 0 . \]  
(B.2)

By defining the isospin current,
\[ j_\chi = \left( \frac{q^+ z_q}{2} \cos \theta \cdot q \right) + f_\pi^2 \sin^2 \theta \cdot \frac{\partial \chi}{\partial t} , \]  
(B.3)

\[ j_{ls} = f_{\pi} F_c \langle q^+ \sigma^a z_q^a \sin \theta \cdot q \rangle - f_\pi^2 \sin^2 \theta \cdot \varphi \chi , \]  
(B.4)

Eq. (B.2) reads in the form,
This is merely an equation of conservation for isospin ($I_3$) current. For the parameter $\theta$, we get

\[ f_\pi^2 \partial_\phi \alpha_\theta + \frac{3}{5} \widehat{f}_\pi F_\pi \varphi <q^+ \sigma^a \tau^a q> \]

\[ = -\frac{1}{2} f_\pi^2 \sin 2\theta (\partial_\phi \varphi^a - \partial_\alpha \varphi^a) - f_\pi^2 m_\pi^2 \sin \theta \cos \varphi \]

\[ - \frac{3}{5} \widehat{f}_\pi F_0 <q^+ \sigma^a \tau^a q> \varphi \sin \theta + \frac{3}{5} \widehat{f}_\pi F_\pi <q^+ \sigma^a \tau^a q> \varphi \cos \theta \]

\[ - \left< q^+ \frac{\tau^a}{2} \frac{1}{q} \right> \frac{\partial \chi}{\partial t} \sin \theta + \left< q^+ \frac{\tau^a}{2} \frac{1}{q} \right> \frac{\partial \varphi}{\partial t} \cos \theta. \]  

Finally, for the parameter $\varphi$, we get

\[ f_\pi^2 \partial_\alpha (\cos^2 \theta \partial_\varphi \varphi) + f_\pi^2 m_\pi^2 \cos \theta \sin \varphi \]

\[ = -\frac{3}{5} \widehat{f}_\pi F_\pi \varphi \left< q^+ \sigma^a \tau^a q \right> \cos \theta - \frac{\partial}{\partial t} \left\{ \left< q^+ \frac{\tau^a}{2} \frac{1}{q} \right> \sin \theta \right\}. \]
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