A systematic method is presented for studying the stability of the solution of a quantum system with both zero temperature and non-zero temperature. It employs the generalized action functional and its first and second derivatives. The zero eigenvalue and corresponding eigenvector of the matrix constructed by the second derivative of the action functional determine the eigenmode of the spectrum. It gives the generalized on-shell condition applicable to the space-time translation non-invariant case also.

The method provides at the same time a general formalism of deriving exact bound state equation.

§ 1. Introduction

The purpose of the present paper is to give a general formalism to discuss the stability of any given solution of the quantum mechanical system. We give our arguments step by step in accordance with the complexities of the problem; we first discuss an N-particle system and then a system described by the second quantized field. These systems can be both non-relativistic and relativistic and are first assumed to be in the zero temperature. Therefore our solution refers in this case to the ground state expectation value of any quantum operators. Discussions are then generalized to the non-zero temperature where we deal with the statistical average of the quantum variables.

Our formalism turns out to provide a systematic derivation of exact bound state equations composed of any number of particles. For example the Bethe-Salpeter (B-S) equation (and its generalization to the non-zero temperature case) of the relativistic field theory emerges as a special case.

The minimum property of the vacuum energy and its relation to the B-S equation have been discussed by Kugo in the lowest approximation for the special type of interaction. This was generalized to full order and to any type of interaction by the author. The present paper is elaboration of the latter and studies the dynamical stability of any quantum system by using the generalized action functional. Note that the vacuum energy is obtained as a special case of the effective action and we cannot discuss the time dependent dynamical behavior by the study of the vacuum energy only.

In order to clarify what we are going to do in the following, let us consider a classical N-particle system described by the particle co-ordinate \( q_i(t) \) \((i=1 \sim N)\) and the Lagrangian \( L(q_i(t), \dot{q}_i(t), t) \). Here \( \dot{q}_i(t) \) is the time derivative \( \dot{q}_i(t) = dq_i(t)/dt \). Defining the action \( I \), a functional of \( q_i(t) \), by
the Lagrange equation of motion is given by the functional derivative of $I$:

$$0 = \frac{\delta I}{\delta q_i(t)} - \frac{d}{dt} \frac{\delta L}{\delta q_i(t)} - \frac{\partial L}{\partial q_i(t)}.$$

Let a solution to this equation be $q_i^{(0)}(t)$, which is assumed to be a bounded function for the interval of $t$ we are interested in. It can be a static or a stationary solution as a special case. In order to discuss the stability of $q_i^{(0)}(t)$, we look for another solution $q_i(t)$ to Eq. (2) near $q_i^{(0)}(t)$ by setting $q_i(t) = q_i^{(0)}(t) + \Delta q_i(t)$ and assume $\Delta q_i(t)$ to be small. Then up to the first order in $\Delta q_i(t)$, we get

$$0 = - \int dt' \left( \frac{\delta^2 I}{\delta q_i(t) \delta q_j(t')} \right)_0 \Delta q_j(t')$$

$$= - \frac{d}{dt} \left( \left( \frac{\partial^2 L}{\partial q_i(t) \partial q_j(t')} \right)_0 \Delta q_j(t) \right) - \left( \frac{\partial L}{\partial q_i(t) \partial q_j(t)} \right)_0 \Delta q_j(t).$$

Here $(\cdots)_0$ denotes the value of $(\cdots)$ at $q_i(t) = q_i^{(0)}(t)$. We have used and will use in the following the Einstein convention that, unless otherwise stated, the repeated indices are summed over. Thus in Eqs. (3) and (4), $\sum_{i=1}^{N}$ is implied.

Equation (3) means that $\Delta q_i(t)$ is an eigenvector or an eigenmode corresponding to the zero eigenvalue of the matrix $I_{ij}^{(0)}(t, t') = (\delta^2 I / \delta q_i(t) \delta q_j(t'))_0$. In terms of the Fourier coefficients $\Delta q_i(\omega) = \int dt e^{i\omega t} \Delta q_i(t)$, we have

$$\int d\omega I_{ij}^{(0)}(\omega, -\omega') \Delta q_j(\omega) = 0.$$

If $q_i^{(0)}(t)$ is a static solution, then $I_{ij}^{(0)}(t, t')$ is a function of $t - t'$ and $I_{ij}^{(0)}(\omega, -\omega')$ is diagonal in $\omega$. In this case we write $I_{ij}^{(0)}(\omega, -\omega') = \delta(\omega - \omega') I_{ij}^{(0)}(\omega)$ and Eq. (5) becomes

$$I_{ij}^{(0)}(\omega) \Delta q_j(\omega) = 0.$$

Let eigenvalues of $I_{ij}^{(0)}(\omega)$ be written by $\lambda^{(k)}(\omega)$ then the solution is given, using Dirac's $\delta$-function, by the form,

$$\Delta q_i(\omega) = \sum_k C_{ik}(\omega) \delta(\lambda^{(k)}(\omega)),$$

where the matrix $C_{ik}(\omega)$ is the eigenvector of $I_{ij}^{(0)}(\omega)$ with the eigenvalue $\lambda^{(k)}(\omega)$. $\Delta q_i(t)$ is given now by

$$\Delta q_i(t) = \sum_k C_{ik} e^{-i\omega^{(k)} t}$$

with some constants $C_{ik}$. Here $\omega^{(k)}$ is the solution of $\lambda^{(k)}(\omega) = 0$. Our criterion of the stability of $q_i^{(0)}$ is that

all $\omega^{(k)}$s are real.

In case $\omega^{(k)}$ contains the imaginary part, $\Delta q_i(t)$ blows up for $t = +\infty$ or $-\infty$. Such a solution, however, cannot directly be found as a well defined solution of Eq. (3)
Stability Conditions in Quantum System

so that we follow here the usually adopted procedure: Let us assume that the Lagrangian $L$ contains a set of parameters $\rho_k (k=1\sim n)$ then by solving Eq. (3) we determine the stable region $\Omega$ in the parameter space in which all the eigenmodes are stable modes. Then boundary of $\Omega$ fixes the critical values of $\rho_k$ beyond which the solution $q_i^{(0)}(t)$ becomes unstable. Our task is thus to find the region $\Omega$.

In the case where $q_i^{(0)}(t)$ depends on $t$, we have to diagonalize Eq. (3) or (5) in two indices $(i, \omega)$. We can write in this case as

$$\Delta q_i(\omega) = \sum_k \int d\omega' U_{i\omega, k\omega'} \delta(\lambda_k^{(0)}(\omega')),$$

where $U_{i\omega, k\omega'}$ diagonalizes $I_{ij}^{(2)}(\omega, -\omega')$. Therefore

$$\Delta q_i(t) = \sum_k \int d\omega \frac{1}{\sqrt{2\pi}} e^{-i\omega t} U_{i\omega, k\omega}(k) \times \vert C(k) \vert^{-1},$$

where $C(k) = \Delta \lambda_k^{(0)}(\omega)/d\omega|_{\omega=\omega'k}$. Stability in this case depends on both $\lambda_k^{(0)}$ and the matrix $U$, which depend in turn on the $t$-dependence of $q_i^{(0)}(t)$.

**Example** We take $L=(m/2)\sum_i \dot{q}_i^2 - V(q_i)$ where $m$ is the mass. The equation of motion is

$$0 = -\frac{\delta I}{\delta q_i(t)} = m\ddot{q}_i(t) + \delta V/\delta q_i(t).$$

Let us consider the static solution $q_i^{(0)}$ which satisfies $\partial V/\partial q_i^{(0)} = 0$. Then, by defining $V_i = (\partial^2/\partial q_i \partial q_i)_0$, we obtain

$$I_{ij}^{(2)}(\omega, -\omega') = (-m\omega^2 \delta_{ij} + V_i) \delta(\omega - \omega').$$

Therefore Eq. (5) is now

$$(-m\omega^2 \delta_{ij} + V_i) \Delta q_i(\omega) = 0.$$    \tag{9}

Let us first assume that the parameters in $V$ are in the range where $V_i$ is positive definite. Then the solution of Eq. (9) is

$$\Delta q_i(\omega) = \sum_k C_{ik} \delta \left( \omega^2 - \frac{v_k^{(0)}}{m} \right),$$

where $v_k^{(0)}>0$ is the eigenvalue of $V_i$ and $C_{ik}$ is some constant. Now we get, by introducing $\omega^{(k)} = \sqrt{v_k^{(0)}/m}$,

$$\Delta q_i(t) = \sum_k C_{ik} e^{\omega^{(k)} t} + \text{c.c.}$$

(c.c. implies complex conjugation). The stable region $\Omega$ of the parameter space is defined by positive definiteness of $V_i$ — a well-known fact.

We discuss the same problem for any quantum mechanical system and for quantum statistical system. Our main observation in this paper is the utility of the generalized action functional $\Gamma[\phi]$ written in terms of the expectation value $\phi_i$ of the operator. In terms of $\Gamma$, there is a complete parallelism between the classical and the quantum system:
\[ I \to \frac{\delta I}{\delta q_i(t)} = 0 \to \int dt' (\frac{\delta^2 I}{\delta q_i(t) \delta q_j(t')} \partial q_j(t')) = 0 , \]

\[ \Gamma \to \frac{\delta \Gamma}{\delta \phi_i(t)} = 0 \to \int dt' (\frac{\delta^2 \Gamma}{\delta \phi_i(t) \delta \phi_j(t')} \partial \phi_j(t')) = 0 . \]

If our system is described by a quantum field theory, \( \Gamma \) is usually called the effective action in particle physics,\(^5\) so that we call \( \Gamma \) effective action in the following. For the quantum statistical system, the generalized action functional \( \Gamma \) does not exist in the strict sense but the above formal processes work through with a minor modifications (see below).

We work in the following mainly in the path integral or the functional integral representation for notational convenience but all the discussions can be done in the operator formalism.

\section{2. Quantum N-particle system}

Since \( q_i(t) \) is now an operator, we have to define the expectation value. For that purpose, we introduce artificial sources \( j_i(t) (i=1 \sim N) \) into the theory and set \( j_i(t) = 0 \) at the end of calculations. Now let us define the generating functional \( W[j] \) by the path integral expression:

\[ \exp \frac{i}{\hbar} W[j] = \int [dq] \exp \frac{i}{\hbar} \left\{ \int_{-\infty}^{\infty} dt L(t) + \int_{-\infty}^{\infty} dt j_i(t) q_i(t) \right\} , \quad (10) \]

where \( L(t) = L(q_i(t), q_j(t), t) \) and \( j[ dq \) is the functional integral. The derivatives of \( W[j] \) at \( j=0 \) are known to give the correlation functions evaluated in the ground state:

\[ \langle \frac{\delta W[j]}{\delta q_i(t)} \rangle = \langle q_i(t) \rangle , \]

\[ \langle \frac{\delta^n W[j]}{\delta j_{i_1}(t_1) \delta j_{i_2}(t_2) \cdots \delta j_{i_n}(t_n)} \rangle = \langle T q_{i_1}(t_1) q_{i_2}(t_2) \cdots q_{i_n}(t_n) \rangle , \]

where \( T \) symbolizes the chronological ordering. We now make the Legendre transform defining \( \phi_i(t) \) for non-zero \( j_i(t) \) by

\[ \phi_i(t) = \frac{\delta W[j]}{\delta j_i(t)} . \quad (11) \]

The definition of the effective action is

\[ \Gamma[\phi] = W[j] - \int_{-\infty}^{\infty} dt j_i(t) \frac{\delta W[j]}{\delta \phi_i(t)} . \quad (12) \]

Since by the property of the Legendre transform \( \frac{\delta \Gamma[\phi]}{\delta \phi_i(t)} = -j_i(t) \), the equation of motion of \( \phi_i(t) \) is obtained in parallel with Eq. (2),

\[ \frac{\delta \Gamma[\phi]}{\delta \phi_i(t)} = 0 . \quad (13) \]

A solution of Eq. (13) is written as \( \phi^{(0)}(t) \) and we look for another solution of the form \( \phi_i(t) = \phi_i^{(0)}(t) + \Delta \phi_i(t) \), then we get

\[ \int dt' (\Gamma_{\mu', \nu'}) \partial \Delta \phi_j(t') = 0 , \quad (14) \]
where we have defined
\[ \Gamma_{it,jt'} = \delta^a \Gamma_{ij}(t) \delta \phi_j(t') \]
and \((-\cdot)_0\) means the value at \(\phi = \phi^{(0)}\). In the Fourier component this reads
\[ 0 = \int d\omega' \Gamma_{i\omega,j-\omega'} \Delta \phi_{i}(\omega') . \]  
Equations (12), (13), (14) (or (15)) are the solutions to our problem.

We see that Eqs. (14) and (15) determine the eigenmode of the spectrum of our system. The "generalized on-shell condition" is obtained which is applicable to the time translation non-invariant case also. It reads as
\[ \text{det}(\Gamma_{it,jt'}) = 0 \quad \text{or} \quad \text{det}(\Gamma_{i\omega,j-\omega'}) = 0 , \]
where the determinant is taken regarding indices \(it(i\omega)\) and \(jt'(j-\omega')\) as indices of rows and columns.

Let us investigate the structure of (14) more closely. Again by the property of the Legendre transform, we have an identity
\[ -\delta_{ij}(t-t') = \int dt'' \Gamma_{it,kt'} W_{kt',jt'} \]
\[ = \int dt'' W_{it,kt'} \Gamma_{kt',jt'} , \]  
where
\[ W_{it,kt'} = \delta^a \delta f_i(t) \delta f_k(t'') . \]
Since \((\Gamma_{it,jt'})_0\) is the minus of the inverse of \((W_{it,kt'})_0\), \(\Delta \phi_i(t)\) satisfying Eq. (14) is the eigenvector of the pole of \(\langle Tq_i(t)q_j(t') \rangle\): We write the eigenvector of \((W_{it,kt'})_0\) by \(\xi_i^s(t)\) and the eigenvalue by \(1/\lambda_s\). They satisfy
\[ \int dt' (W_{it,jt'})_0 \xi_i^s(t') = 1/\lambda_s \xi_i^s(t) . \]  
Then \(\xi_i^s(t)\) is the eigenvector of \(\Gamma_{it,jt'}\) with the eigenvalue \(-\lambda_s\)
\[ \int dt (\Gamma_{it,jt'})_0 \xi_i^s(t) = -\lambda_s \xi_i^s(t) . \]
Thus we conclude that \(\Delta \phi_i(t)\) is proportional to the eigenvector \(\xi_i^0(t)\) corresponding to \(\lambda_s = 0\) — the pole of \(W\). Since the eigenvectors corresponding to the pole of \(\langle Tq_i(t)q_j(t') \rangle\) represent the one particle eigenmode, the solution to Eq. (14) is the one particle eigenmode. This is the reason why \(\Gamma_{it,jt'}\) comes in when we study the stability of a solution.

For the time translation invariant case, we can relate \(\xi_i^0(t)\) to the residue of the pole of \(W_{it,jt'}\), which is now a function of \(t-t'\). In the Fourier representation, (15) becomes diagonal in \(\omega\) and is written as
\[ W_{it}(\omega) \xi_i^s(\omega) = 1/\lambda_s \xi_i^s(\omega) , \]
where
\[ W_{\delta}(\omega) = \int e^{i(wt-t')l} \langle T(q_i(t)q_j(t'))d(t-t') \].

The zero of \( \lambda\) as a function of \( \omega \) is seen to correspond to the pole of \( W_{\delta}(\omega) \). Let the state of the one particle eigenstate in question be \( |\omega_0\rangle \) where \( \hbar \omega_0 \) is the energy of the state. Then near \( \omega = \omega_0 \), we have
\[
W_{\delta}(\omega) \approx \frac{if_i(\omega_0)f_j(\omega_0)}{\omega - \omega_0 + i\epsilon} - \frac{i\overline{f}_i(\omega_0)f_j(\omega_0)}{\omega + \omega_0 - i\epsilon},
\]
where \( \epsilon \) is an infinitesimal positive constant and
\[
\overline{f}_i(\omega_0) = \langle 0|q_i(0)|\omega_0\rangle, \quad f_j(\omega_0) = \langle \omega_0|q_j(0)|0\rangle.
\]
We have assumed the energy of the ground state to be zero. Therefore \( \lambda \) becomes zero at \( \omega = \pm \omega_0 \). For \( \omega = +\omega_0 \), \( \xi_i(\omega) \propto \delta(\omega - \omega_0) \times \overline{f}_i(\omega_0) \) and for \( \omega = -\omega_0 \), \( \xi_i(\omega) \propto \delta(\omega + \omega_0)f_i(\omega_0) \). The stability condition is that \( \omega_0 \) is real.

The functional \( \Gamma[\phi] \) is usually called the effective action by the property (13). When the time independent solution \( \phi^{(0)}(t) \) is inserted, \( \Gamma[\phi^{(0)}] \) becomes \( -i\int d\tau \gamma(\phi^{(0)}) \) where \( /dt \) is the whole infinite time interval and \( \gamma(\phi^{(0)}) \) is the energy of the ground state in which the expectation value of \( q_i(t) \) is \( \phi_i^{(0)} \). This property is also common to the action. The diagrammatic rule to calculate \( \Gamma[\phi] \) is known: If the system is described by the quantum Lagrangian \( L(q_i(t), \dot{q}_i(t)) \) then we insert \( q_i(t) = \phi_i(t) + q_i(t) \) into \( L \) and omit the term linear in \( \dot{q}_i(t) \) or \( q_i(t) \). Then \( L \) becomes \( L(\phi_i(t), \dot{\phi}_i(t), t) + L' \), where \( L' \) depends on \( q_i(t), \dot{q}_i(t), \phi_i(t) \) and \( \dot{\phi}_i(t) \). \( \Gamma[\phi] \) is given by
\[
\Gamma[\phi] = \int dt L(\phi_i(t), \dot{\phi}_i(t), t) + \Delta \Gamma[\phi],
\]
where the first term represents the tree term and \( \Delta \Gamma[\phi] \) is the loop correction which is the sum of the one-particle irreducible vacuum diagrams generated by \( L' \).

We can generalize the above arguments to include the two particle operators \( q_i^2(t) \) or higher operators or operators of any function of \( q_i(t) \). These operators are written as \( O_i(t) \) and we introduce \( j_i(t) \) for each \( i \) to define \( W[j] \) and make Legendre transformation to obtain \( \Gamma[\phi] \) by (11) and (12). \( \Delta \phi_i(t) \) is the eigenvector of the pole of the Green's function matrix \( \langle TO_i(t)O_i(t') \rangle \). For the case \( O_i(t) = q_i^2(t) \) for instance it represents the two particle bound state pole. The diagrammatic rule for obtaining \( \Gamma[\phi] \) in the general case, however, is not known yet.

**Example** Consider the Lagrangian of one particle system \( L = (m/2)q^2 - (k/2)q^2 \). Then we have, apart from irrelevant constant
\[
\begin{align*}
W[j] &= \frac{1}{2} \int d\omega j(\omega)(-\omega)/(k-m\omega^2), \\
\Gamma[\phi] &= \frac{1}{2} \int dt (m\dot{\phi}(t)^2 - k\phi(t)^2).
\end{align*}
\]

The equation determining \( \phi(t) \) is \( 0 = \delta \Gamma/\delta \phi(t) = -m\dot{\phi}(t) - k\phi(t) \). Take a (trivial) solution \( \phi^{(0)}(t) = 0 \), then writing \( \phi(t) = \phi^{(0)}(t) + \Delta \phi(t) \), \( \Delta \phi(t) \) or \( \Delta \phi(\omega) \) is determined by
\((-m\omega^2 + k)\Delta\phi(\omega) = 0, \) so that \(\Delta\phi(\omega) \propto \delta(-m\omega^2 + k)\) and \(\Delta\phi(t) \propto \exp i\sqrt{k/m} t + c.c. \cdots.\)
The stability condition is \(k > 0.\) Since the Lagrangian is quadratic in \(q,\) we get the same equation for \(\phi\) as for \(q\) in the classical system.

§ 3. Quantum field theory—the case of local field

The system is now described by the quantized bosonic fields \(\phi^b_i(x, t)\) or fermionic field \(\phi^f_i(x, t)\). Here \(x\) is the space co-ordinate and the indices \(i, k, l\) specify different kinds of field or the spin components. We use the notation \(x\) for \(x\) and \(t\) and write these fields collectively as \(\phi_i(x).\) The Lagrangian \(L\) is given as \(L = \int d^3 x L(x)\) where \(L(x) = L(\phi_i(x), \dot{\phi}_i(x), \nabla\phi_i(x), t).\)

The expectation values of any operators composed of \(\phi_i(x)\) are usually called order parameters. In this section we consider as the order parameters only local operators such as \(\phi_i(x), \phi_i^*(x), \cdots, f(\phi_i(x))\) etc., where \(f\) is any local functions of \(\phi_i(x).\) These operators are denoted by \(O_k(x).\) In order to study the equation satisfied by the local order parameters, we define the effective action. All the arguments of the preceding section go through with the replacement \(\Sigma \rightarrow \sum \int_{-\infty}^{\infty} d^3 x\): By using the notation \(\int d^4 x \equiv \int_{-\infty}^{\infty} dt \int_{-\infty}^{\infty} d^3 x,\) we define \(W[j]\) by

\[
\exp \frac{i}{\hbar} W[j] = \int [d\phi] \exp \frac{i}{\hbar} \int d^4 x (L(x) + j(x) O_k(x)),
\]

where \([d\phi]\) is the functional integral over the field (Grassman integral in the case of fermions). Now the field theoretical Legendre transform is performed,

\[
\Gamma[\phi] = W[j] - \int d^4 x j(x) \delta W[j]/\delta j(x),
\]

where

\[
\phi_k(x) = \delta W[j]/\delta j_k(x).
\]

The equation of motion to determine \(\phi_k(x)\) is

\[
\delta \Gamma[\phi]/\delta \phi_k(x) = 0.
\]

The stability of one of the solutions \(\phi_k^{(0)}(x)\) of Eq. (24) is determined by

\[
\int d^4 y (\delta^2 \Gamma[\phi]/\delta \phi_k(x) \delta \phi_k(y)) \Delta \phi_i(y) = 0.
\]

\(\Delta \phi_k(x)\) is the eigenvector of the pole of the Green's function matrix

\[
\langle TO_k(x) O_i(y) \rangle \equiv G_k(x, y).
\]

This is a consequence of the relation, with obvious notations,

\[
-\delta_{kl} \delta^4(x - y) = \int d^4 x' \Gamma_{kk',xx'} W_{kk',xx'},
\]

\[
= \int d^4 x' W_{kk',xx'} \Gamma_{kk',xx'},
\]
where \( \delta^4(x-y) = \delta^3(x-y) \delta(t-t') \) and we have introduced \( x=(x, t), y=(y, t') \).
\( \Delta \phi_i(x) \) represents the eigenmode or the bound state which couples to \( O_k \) and \( O_l \).

§ 4. Quantum field theory—the case of bilocal field

We consider bilocal products of fields \( \phi^i(x)\phi^j(y), \phi^i_k(x)\phi^j(y)' \), \( \phi^i(x)\phi^j(y) \) etc., and the expectation values of these operators are taken as the order parameters. In the following we write any bilocal fields as \( \phi_i(x)\phi_j(y) \) collectively. By the straightforward generalization of the preceding procedure, we are naturally led to the B-S equation.

The conventional approach for the study of the stability of the chosen solution expands the effective action around \( P_\mu=0 \) where \( P_\mu \) is the total momentum of the two particle system.\(^{2,4,5} \) Our approach here does not employ the expansion and discusses the complete eigenmode equation.

We first define \( W[j] \) by

\[
\exp \frac{i}{\hbar} W[j] = \int [d\phi] \exp \left\{ \int d^4x L + \text{source term} \right\},
\]

where

\[
\text{source term} = \int d^4x \int d^4y j_\mu(x, y) \phi_i(x)\phi_j(y).
\]

Introducing

\[
\phi_\mu(x, y) = \delta W/\delta j_\mu(x, y),
\]

we make the Legendre transform:

\[
\Gamma[\phi] = W - \int d^4x \int d^4y j_\mu(x, y) \phi_\mu(x)\phi_\mu(y).
\]

At \( j=0 \), \( \phi \) becomes the propagators of the field \( \phi \)

\[
\phi_\mu(x, y) = \langle T\phi_i(x)\phi_j(y) \rangle.
\]

The equation of motion for \( \phi \) is given by

\[
\delta \Gamma[\phi]/\delta \phi_\mu(x, y) = 0,
\]

which is known to be the Schwinger-Dyson equation for \( \phi_\mu(x, y) \).

The stability of the solution \( \phi_\mu(x, y) \) to Eq. (32) is determined by solving

\[
\int d^4x' \int d^4y' (\Gamma_{ix,yj} r_{x',y'})_0 \Delta \phi_i(x') \phi_j(y') = 0,
\]

where

\[
\Gamma_{ix,yj} r_{x',y'} = \delta^4 \Gamma/\delta \phi_\mu(x, y) \phi_\mu(x', y').
\]

According to our observations just made, \( \Delta \phi_\mu(x, y) \) is expected to be the eigenvector of the residue of the two particle bound state pole of the Green's functions.
Using the notation
\[ W_{ix,jy; i'x',j'y} = \delta^2 W/\delta j_i(x, y) \delta j_{i'}(x', y'), \]
we note the following relations,
\[-\delta_{i'i'} \delta_{j'j} \delta^d(x-x') \delta^d(y-y')
= \int d^4z \int d^4z' \Gamma_{ix,jy; i'x',j'y} W_{ix',j'y'; i'x',j'y'}
= \int d^4z \int d^4z' W_{ix,jy; i'x',j'y'} \Gamma_{ix',j'y'; i'x',j'y'}. \]

Now the above \( W \) is written in terms of the four point Green's functions \( G^{(4)} \):
\[ (W_{ix,jy; i'x',j'y})_0 = \langle T \phi_i(x) \phi_j(y) \phi_{i'}(x') \phi_{j'}(y') \rangle \]
\[ = G^{(4)} \]
and the pole of \( G^{(4)} \) in two particle channel represents the two particle eigenmode or the two particle bound state. Let us write the eigenvalue of \( (W_{ix,jy; i'x',j'y})_0 \) by \( 1/\lambda_0 \) and corresponding eigenvector by \( \xi_0(x, y) \), then \( \Delta \phi_0(x, y) \) is proportional to \( \xi_0(x, y) \) which gives the residue of the pole \( (\lambda_0=0) \) of \( W_{ix,jy; i'x',j'y} \). This is called the B-S amplitude. To see this explicitly we write the eigenvalue equation as
\[ \int d^4x' d^4y' (W_{ix,jy; i'x',j'y})_0 \xi_0(x', y') = 1/\lambda_0 \xi_0(x, y). \]

For the time translation invariant solution \( \phi_0(x-y) \), we can extract the two particle pole of \( W \) as in (19): We define
\[ W_{i;j, i'j'}(P, q, q')_0 = \int d^4r d^4r' d^4X e^{ip \cdot X + iq \cdot r + iq' \cdot r'} (W_{ix,jy; i'x',j'y})_0, \]
where \( r = x - y, r' = x' - y', X = (1/2)(x+y)-(1/2)(x'+y') \), and we have defined \( q \cdot r = -q \cdot r + q_0 r_0 \) for \( q = (q, q_0), r = (r, r_0) \). Equation (36) is now diagonal in \( P = (P, \omega) \). The state of the two particle eigenmode with energy \( \hbar \omega_0 \) and the momentum \( P \) is denoted by \( |P, \omega_0 \rangle \). Then we have
\[ W_{i;j, i'j'}(P, q, q')_0 \approx \frac{i \tilde{g}_i \tilde{g}_{i'}(P, q)}{\omega + \omega_0 - \omega \pm i \epsilon} - \frac{i \tilde{g}_i \tilde{g}_{i'}(P, q)}{\omega + \omega_0 + \omega \pm i \epsilon}, \]
where
\[ \tilde{g}_i \equiv \tilde{g}_i(P, q) \]
\[ = \int \langle P, \omega_0 | T(\phi_i(x) \phi_j(y)) | 0 \rangle \ e^{iq \cdot (x-y)} d^4r \times e^{-ip \cdot (x+y)/2}, \]
\[ g_{i;j, i'j'}(P, q') \]
\[ = \int \langle 0 | T(\phi_{i'}(x') \phi_{j'}(y')) | P, \omega_0 \rangle e^{i \omega_0 (x'-y')} d^4r' \times e^{iP \cdot (x'+y')/2}. \]
Since Eq. (36) is written in Fourier space by
\[
\int \frac{d^4q}{(2\pi)^4} W_{ij,\nu,\rho}(P, q, q') \xi_i^{\nu}(q') = \frac{1}{\lambda_s} \xi_i^{\nu}(q),
\]
\(\lambda_s\) becomes zero at \(\omega = \pm \omega_0\). For \(\omega = +\omega_0\), we have \(\xi_i^{\nu}(q) \propto \delta(\omega - \omega_0) \bar{\phi}_{\nu,\rho}(P, q)\) while for \(\omega = -\omega_0\), \(\xi_i^{\nu}(q) \propto \delta(\omega + \omega_0) \bar{\phi}_{\nu,\rho}(P, q)\). Since \(\xi_i^{\nu}(x, y) \sim e^{\pm i\omega_0 X_0}\), the stability in the time coordinate \(X_0\) requires \(\omega_0\) to be real.

For the solutions without time translation invariance, we have to know the transformation matrix which diagonalize Eq. (33) in energy space, as has been done in Eq. (8).

We next write explicitly Eq. (33) and show that it exactly coincides with the B-S equation for the two particle bound state. The Lagrangian is assumed for simplicity to contain only bosonic field \(\phi_{\nu,\rho}(x)\) but the arguments apply to the case where fermionic fields are also present. The Lagrangian can in general be written in the form

\[
L = -\frac{1}{2} \int d^4x \int d^4y \psi_{\nu}(x) D_{\nu,\rho}(x, y) \psi_{\nu}(y) + L_I,
\]
where the first term is the free part and \(L_I\) represents the interaction term. We adopt the new notation \(D_{\nu,\rho}(x, y)\) for \(\phi_{\nu,\rho}(x, y)\). The explicit diagrammatic rule for \(\Gamma[D]\) is known, which is given, in the matrix notation \(D_{ix, iy}=Du(x, y)\), by

\[
\Gamma[D] = -\frac{\hbar}{2i} \text{Tr} \ln D - \frac{1}{2} \text{Tr} D_0^{-1} D + \gamma^{(2)}[D].
\]
Here the trace is over the indices \(i\) and \(x\), and \(\gamma^{(2)}[D]\) is the one and two particle irreducible vacuum graphs generated by the interaction \(L_I\). In these graphs internal propagator lines are replaced by \(D_{\nu,\rho}(x, y)\). The equation \(\delta \Gamma/\delta D_{\nu,\rho}(x, y) = 0\) gives the Schwinger-Dyson equation

\[
D_{\nu,\rho}^{-1}(x, y) - D_{\nu,\rho}^{-1}(x, x') D_{\nu,\rho}^{-1}(y', y) - \Sigma_{\nu,\rho}(x, y) = 0,
\]
\(\Sigma_{\nu,\rho}(x, y) = (2i/\hbar) \delta \gamma^{(2)}[D]/\delta D_{\nu,\rho}(y, x)\),

where we have introduced \(D_0^{-1} = (1/\hbar i) D_0^{-1}\) and used \(\delta \text{Tr} \ln D/\delta D_{\nu,\rho}(x, y) = D_{\nu,\rho}(y, x)\).

By Eq. (40) we see that \(\Sigma\) is just the proper self-energy. One of the solutions to Eq. (39) is denoted by \(D^{(0)}\) and we write \(D = D^{(0)} + \delta D\). By the formula \(\partial (A^{-1})_{\nu,\rho}/\partial A_{kl} = - (A^{-1})_{ik} (A^{-1})_{jl}\) for any matrix \(A\), we get

\[
\iint d^4x' d^4y' \left[- D^{(0)-1}_{\nu,\rho}(x, x') D^{(0)-1}(y', y) + T^{(4)}_{\nu,\rho}(x', y', y')\right] \delta D_k(x', y') = 0,
\]
where

\[
T^{(4)}_{\nu,\rho}(x', y') = -(\delta \Sigma_{\nu,\rho}(x, y)/\delta D_{kl}(x', y'))_0
\]
and \((\cdots)_0\) means that we set \(D = D^{(0)}\). \(T^{(4)}\) given in Eq. (42) is the two particle irreducible four point Green's function with the external lines (represented by the propagator \(D^{(0)}\)) amputated. This is precisely the B-S kernel and Eq. (41) is the B-S equation.

Example Consider the single component phonon field or the Higgs field Lagrangian in particle physics.
Stability Conditions in Quantum System

\[ L = \frac{1}{2} \int d^3x (\dot{\phi}(x)^2 - \phi(x)) - \frac{g}{4!} \int d^3x \phi^4(x). \]

Here \( g \) is the coupling constant and \( \omega^2(-P^2) \) determines the dispersion relation of the free \( \phi \) particle which need not be specified for the moment.

We are interested in the case of the attractive interaction \( (g < 0) \), where the instabilities are expected to occur in various channels. The method in this section enables us to study the instabilities in both one and two particle channels at the same time and we will see within the approximation adopted that as \( g \) is varied in such a way that the attractive force is increased, the stability in one particle channel is lost before the stability in the two particle channel. To the lowest order in \( g \), \( \gamma^{(2)} \) of Eq. (38) is given by the familiar two loop vacuum graph (which looks like "8"), and is calculated to be

\[ \gamma^{(2)}[D] = -\left(\frac{3g}{4!}\right) \int D(x, x)^2 d^4x. \]

Equation (39) is

\[ D^{-1}(x, y) - D_0^{-1}(x - y) = \left(\frac{ig}{2\hbar}\right) D(x, x) \delta^4(x - y). \]

The solution \( D^{(0)} \) depends on \( x - y \) only so we define \( D^{(0)}(p) \) by

\[ D^{(0)}(p) = \int d^4(x - y) e^{ip \cdot (x - y)} D(x - y), \]

where \( p \) stands for \( p, p_0 \) and \( p \cdot x = p_0 t - p \cdot x. \) It is given by

\[ D^{(0)}(p) = i\hbar /\left(p_0^2 - \omega^2(p^2) - m \right), \]

where \( m \) is determined by the gap equation

\[ m = \frac{g}{2} D^{(0)}(x, x) = \frac{\hbar g}{4} \int d^3p \frac{1}{\sqrt{\omega^2(p^2) + m}}. \]

We have assumed a small negative imaginary part for \( \omega(p^2) \).

Let us consider the attractive case \( (g < 0) \). As \(-g\) is increased, \(-m\) is increased accordingly and at some \( g = g_c^{(1)} (< 0) \), we have \( \omega(0) + m = 0 \). This \( g_c^{(1)} \) represents the critical coupling for the instability of the one particle channel. By Eq. (43), \( T^{(0)} \) given in Eq. (42) is now, in our approximation,

\[ T^{(0)}_{x, y} = -\left(\frac{ig}{2\hbar}\right) \delta^4(x - y) \delta^4(x - x') \delta^4(x - y'). \]

We take the Fourier transform of Eq. (41) which reduces to

\[ (p_0^2 - \omega^2(p^2) - m)(k_0^2 - \omega^2(k^2) - m) \Delta D(P, q) \]

\[ = i\hbar \frac{g}{2} \int \frac{d^4q}{(2\pi)^4} \Delta D(P, q). \]

Here we have defined the total momentum \( P \) and relative momentum \( q \) of the two particle system by \( p = (P/2) + q, k = (P/2) - q \) and
\[ \Delta D(P, q) = \int d^4 x d^4 y \Delta D(x, y) \exp \left\{ i \left[ \left( \frac{P}{2} + q \right)x + \left( \frac{P}{2} - q \right)y \right] \right\}. \]

Equation (45) is diagonal in \( P \), which is easily solved:
\[ \Delta D(P, q) = \text{const} \times (p_o^2 - \omega^2(p^2) - m)^{-1}(k_o^2 - \omega^2(k^2) - m)^{-1}. \]

\( P \) is determined by
\[ 1 = i\hbar \frac{g}{2} \int \frac{d^4 q}{(2\pi)^4} \frac{1}{(p_o^2 - \omega^2(p^2) - m)(k_o^2 - \omega^2(k^2) - m)}. \]

We put here \( P = 0 \) for simplicity then this takes the form
\[ 1 = -\frac{\hbar G}{8} \int \frac{d^3 q}{(2\pi)^3} \frac{1}{(\omega^2(q^2) + m - \frac{P_0^2}{4})} \frac{1}{\sqrt{\omega^2(q^2) + m}}. \]

Let us suppose that at some \( g = g_e(2) < 0 \) the energy of the bound state becomes zero. We assume that at \( g = g_e(2) \) the one particle channel is stable, i.e., \( \omega^2(0) + m^{(2)} > 0 \). Since Eq. (44) tells us that the function \( |m(g)| \) is increasing as \( |g| \) increases from 0 to \( |g_e(1)| \), we see that \( 0 > g_e(2) > g_e(1) \) and \( 0 > m^{(2)} > m^{(1)} \). Then we have
\[ 1 = -\frac{\hbar g_e(2)}{8} \int \frac{d^3 q}{(2\pi)^3} \frac{1}{(\omega^2(q^2) + m^{(2)})^{3/2}}. \]

Let us multiply \( m^{(1)} \) on both sides of this equation, then we get
\[ m^{(1)} = \frac{\hbar g_e(1)}{4} C \int \frac{1}{\sqrt{\omega^2(q^2) + m^{(1)}}} \frac{d^3 q}{(2\pi)^3}, \]

where
\[ C = \frac{g_e(2)}{2 g_e(1)} \int \frac{m^{(1)}}{\omega^2(q^2) + m^{(2)}} \frac{d^3 q}{(2\pi)^3} / \int \frac{1}{\sqrt{\omega^2(q^2) + m^{(1)}}} \frac{d^3 q}{(2\pi)^3}. \]

But since \( 0 > m^{(2)} > m^{(1)} \), it is easy to show that \( C < 1/2 \). Therefore Eq. (46) is seen to be inconsistent with Eq. (44). We have to abandon the assumption \( g_e(2) > g_e(1) \), leading to the conclusion that the instability of the one particle channel occurs first before the instability in two particle channel as we increase the magnitude of the attractive force.

§ 5. Quantum field theory at non-zero temperature — the case of local field

Next we consider a finite temperature system. The field variables are written as \( \psi_i(x) \) symbolically as in the previous case. Now the expectation value of any operator involves both the quantum and statistical average. We assume that the system is at some instant of time in a mixed state described by the density matrix \( \rho_H \) in the Heisenberg representation and the system develops in time according to the given Hamiltonian. Then the expectation value at the time \( t \) of any local operator \( A_H(t) \) in Heisenberg representation is given as (we suppress \( x \) for simplicity),
\[ \langle A_h(t) \rangle = \text{Tr} \rho_s A_h(t) = \text{Tr} \rho_s(t) A_h(t) , \]

where subscript \( s \) denotes the Schrödinger representation. The operator \( A_h(t) \) has been assumed to contain the explicit time dependence. It is convenient to go over to the interaction representation splitting the total Hamiltonian \( H_s(t) \) into \( H_0 + \Delta H_s(t) \) with \( H_0 \equiv H_s(-\infty) \) and \( \Delta H_s(-\infty) = 0 \). Our interaction representation is based on \( H_0 \). This choice of \( H_0 \) is convenient for the practical use. We introduce the operators in the interaction representation,

\[ A_1(t) = \exp \left( \frac{i}{\hbar} H_0 t \right) A_h(t) \exp \left( - \frac{i}{\hbar} H_0 t \right) , \]

\[ U(t', t) = T \exp \left( - \frac{i}{\hbar} \int_{t'}^{t} \Delta H_i(t'') dt'' \right) . \]

Then we have

\[ \langle A_h(t) \rangle = \text{Tr} \rho_1(-\infty) U(-\infty, t) A_1(t) U(t, -\infty) , \quad (47) \]

where \( \rho_1(-\infty) \) represents the statistical distribution at \( t = -\infty \). We assume as usual that it is an equilibrium governed by the Hamiltonian \( H_0 \) and assume \( \rho_1(-\infty) = \exp(-\beta H_0) \) with \( \beta^{-1} = kT \) where \( k \) is the Boltzmann constant and \( T \) is the temperature. Then the expectation values of any operators approach the equilibrium value for \( t = -\infty \) so that our study of the instability problem is confined to the behavior at \( t = +\infty \).

The path integral representation is easily obtained by the above expression as

\[ \langle A_h(t) \rangle = \int [d\phi^{(1)} d\phi^{(2)} d\phi^{(3)}] A_1(t) \]

\[ \times \exp \left\{ \frac{i}{\hbar} \int_{-\infty}^{t} L_1(t') dt' - \frac{i}{\hbar} \int_{-\infty}^{t} L_2(t') dt' - \frac{i}{\hbar} \int_{-\infty-i\beta \hbar}^{0} L_3(t') dt' \right\} . \quad (48) \]

Here \( L_1(t)(L_2(t), L_3(t')) \) is the total Lagrangian of the system written in terms of \( \phi^{(1)}(t) \), \( \phi^{(2)}(t) \), \( \phi^{(3)}(t) \). In Eq. (48), \( A_1(t) \) is expressed by \( \phi^{(1)}(t) \) but we can use \( \phi^{(2)}(t) \) also. The integral \( \int [d\phi^{(3)}] \) is restricted by the periodicity condition \( \phi^{(3)}(-\infty - i\beta \hbar) = \phi^{(3)}(-\infty) \). For the subsequent discussions, these integrals are not essential so that we write Eq. (48) symbolically as

\[ \langle A_h(t) \rangle = \text{Tr} \rho \int [d\phi^{(1)} d\phi^{(2)}] A_1(t) \exp \left( \frac{i}{\hbar} \int_{-\infty}^{t} dt' (L_1(t') - L_2(t')) \right) . \quad (49) \]

The functional integral is performed along the double time paths, the one from \(-\infty\) to \( t \) and the other from \( t \) to \(-\infty\).

If the temperature is absolute zero and if the ground state is not degenerate and is separated by a finite gap from the excited states, Eq. (49) becomes the well-known zero temperature formula:

\[ \langle A_h(t) \rangle = \langle 0 | U(\infty, t) A_1(t) U(t, -\infty) | 0 \rangle / \langle 0 | U(\infty, -\infty) | 0 \rangle . \quad (50) \]

Here \( | 0 \rangle \) denotes the ground state. We have used this formula in Eq. (10) where a single straight path from \(-\infty\) to \( \infty \) has been employed.
Let us consider any local operators $O_k(t)$ as order parameters and define the generalized effective action. For that purpose the generating functional $W[j^{(1)}, j^{(2)}]$ is introduced. By recovering spatial coordinate $x$ and using the notation $x$ for $x$ and $t$, it is defined by

$$\exp \frac{i}{\hbar} W[j^{(1)}, j^{(2)}] = \text{Tr}_\rho \int [d\phi^{(1)} d\phi^{(2)}] \exp \frac{i}{\hbar} \int d^4x \{ \mathcal{L}^{(1)}(x)$$

$$+ j^{(1)}_k(x) O_k^{(1)}(x) - \mathcal{L}^{(2)}(x) - j^{(2)}_k(x) O_k^{(2)}(x) \} .$$

(51)

$\mathcal{L}^{(1)}(x)$ is the Lagrangian density and $O_k^{(1)}(x)$ is $O_k(x)$ written in terms of $\phi^{(1)}(x)$. Generalized effective action $\Gamma[\phi^{(1)}, \phi^{(2)}]$ is given by

$$\Gamma[\phi^{(1)}, \phi^{(2)}] = W - \int d^4x \int d^4y \delta W/\delta j^{(1)}_k(x) \delta W/\delta j^{(2)}_k(x) ,$$

(52)

$$\phi^{(1)}_k(x) = \delta W/\delta j^{(1)}_k(x) , \quad \phi^{(2)}_k(x) = - \delta W/\delta j^{(2)}_k(x) .$$

(53)

The above $\Gamma$ is not, however, a well-defined quantity since for $j^{(1)} \neq j^{(2)}$ we are considering an unphysical system which does not correspond to any real system. In perturbation series of $\Gamma$, for example, there appear terms proportional to $\delta(0)$, the value of the $\delta$-function at the origin. If we set, however,

$$j^{(1)}_k(x) = j^{(2)}_k(x) \equiv j_k(x)$$

(54)

in $\delta W/\delta j^{(1)}_k(x)$ then it becomes well-defined. This is because it now coincides with the expectation value of $O_k(t)$ for the physical system where the Lagrangian density is given by $\mathcal{L}(x) + j_k(x) O_k(x)$. It is easy to see that when Eq. (54) is satisfied we have also

$$\phi^{(1)}_k(x) = \phi^{(2)}_k(x) \equiv \phi_k(x) .$$

(55)

This $\phi_k(x)$ is the well-defined physical quantity. Now the equation $\delta \Gamma/\delta \phi^{(1)}_k(x) = \delta \Gamma/\delta \phi^{(2)}_k(x) = 0$ is equivalent to

$$\delta \Gamma[\phi^{(1)}, \phi^{(2)}] / \delta \phi^{(1)}_k(x)|_{\phi^{(1)}=\phi^{(2)}=\phi} = 0$$

(56)

which is our equation of motion to determine $\phi_k(x)$. Equation (56) is not of the form of the functional derivative of the action functional: A system with non-zero temperature is not a conservative system and the equation of motion cannot be derived from a variational principle in a strict sense. It should be modified as Eq. (56).

Let one of the solution of Eq. (56) be $\phi^{(0)}(t)$ and write

$$\phi_k(t) = \phi^{(0)}_k(t) + \Delta \phi_k(t) .$$

Then the equation for $\Delta \phi_k(x)$ becomes

$$\int d^4y (\Gamma_{1k,1k'\gamma} + \Gamma_{1k,2k'\gamma}) \delta \phi_k(y) = 0 ,$$

(57)

where for $i, j = 1$ or $2$

$$\Gamma_{1k,1k'\gamma} = \delta^2 \Gamma/\delta \phi^{(j)}_k(x) \delta \phi^{(j)}_k(y) .$$
The subscript 0 in Eq. (57) implies that we set \( \phi^{(1)} = \phi^{(2)} = \phi^{(0)} \) there.

Now we have an identity

\[
\delta_{ij} \delta_{kk'} \delta^4(x - x')
\]

\[
= \int d^4y \Gamma^{i}_{kx, ijy} (-)^{i+1} W^{j}_{lij} \nu, x', x
\]

\[
= \int d^4y W^{i}_{kx, ijy} (-)^{i+1} \Gamma^{j}_{lij} \nu, x', x',
\]

(58)

where the summation over the indices \( j \) runs from 1 to 2 and

\[
W_{ikx, ijy} = \delta^3 \delta_{ij}^{(i)}(x) \delta_{ij}^{(j)}(y)
\]

The factor \((-)^{i+1}\) appears by the definition (53). Equation (57) is well defined if Eq. (54) or (55) is satisfied. Let us consider the case where Eq. (54) or (55) holds. We note first that

\[
W_{ikx, 1lj} = \langle T O_k(x) O_l(y) \rangle = G^R_{ikl}(x, y),
\]

\[
W_{ikx, 2lj} = - \langle O_l(y) O_k(x) \rangle = -G^R_{ikl}(x, y),
\]

\[
W_{2kx, 1lj} = - \langle O_k(x) O_l(y) \rangle = -G^R_{ikl}(x, y),
\]

\[
W_{2kx, 2lj} = \langle \overline{T} O_k(x) O_l(y) \rangle = G^R_{ikl}(x, y),
\]

(59)

where \( \overline{T} \) symbol means the anti-chronological ordering and the expectation value is taken for the Lagrangian \( \mathcal{L}(x) + j_h(x) O_k(x) \). We have not yet set \( j_h(x) = 0 \). There is an obvious identity

\[
\sum_{i,j=1}^{2} W_{ikx, ijy} = 0.
\]

(60)

By using Eqs. (58) and (60) it is relatively an easy matter to derive a relation

\[
\int d^4y (\Gamma^{i}_{kx, 1k'y} + \Gamma^{i}_{kx, 2k'y}) (W_{1k'y, 1lx'} + W_{1k'y, 2lx'})
\]

\[
= - \delta_{kk'} \delta^4(x - x').
\]

(61)

But since

\[
W_{1kx, 1k'x'} + W_{1kx, 2k'x'} = \theta(t - t') \langle [O_k(x), O_{k'}(x')] \rangle
\]

\[
= i G^R_{kk'}(x, x'),
\]

(62)

where \( G^R \) is the usual retarded function, we conclude that \( \Delta \phi_k(y) \) in Eq. (57) is proportional to the right eigenvector \( \xi_k(y) \) of the pole of the retarded Green's function. This corresponds to the one-particle eigenmode as is well known.

Consider the case where the time translation invariance holds. The appearance of the retarded Green's function here is reasonable since, as pointed out by Landau, it is \( G^R \), not \( G^e \) for instance, that is analytic in the complex \( \omega \)-plane: The Fourier component \( G^R_{kl}(\omega) = \int e^{i \omega t} G^R_{kl}(t) dt \) is analytic for \( \text{Im} \omega > 0 \) if \( G^R(t) \) does not blow up for \( t \to +\infty \).
Suppose the solution \( \phi^{(0)} \) is independent of \( t \) and \( x \) and suppose the eigenstate giving the pole of \( G^r \) is given by \( |P, \omega_0\rangle \) with the energy \( \hbar \omega_0 \) and the momentum \( P \). Then we have, with \( P=(P, \omega) \), near the pole

\[
iG_{kk'}(P) = \frac{\tilde{f}_{nk} f_{nk'}}{\omega - \omega_0 + i\epsilon - \omega + \omega_0 + i\epsilon},
\]

where

\[
\omega_0 = \omega_0 - \omega, \\
\tilde{f}_{nk} = \langle n|O_k(0)|P, \omega_0\rangle, \\
f_{nk} = \langle P, \omega_0|O_k(0)|n\rangle,
\]

and \( |n\rangle \) is the complete set of states of \( H_0 \) and \( \hbar \omega_0 \) is the energy of these states. The eigenvalue equation we have to consider is

\[
iG_{kk'}(P) \xi^k(P) = 1/\lambda_s \xi^k(P).
\]

We see that \( \lambda_s \) vanishes at \( \omega = \omega_0 \). For \( \omega = \omega_0 \), the eigenvector corresponding to the pole of \( G^r \) is \( \xi^k(P) \propto \delta(\omega - \omega_0) f_{nk} \) and for \( \omega = -\omega_0 \), \( \xi^k(\omega) \propto \delta(\omega + \omega_0) f_{nk} \). In the continuum limit where the volume of the system goes to infinity these poles coalesce into the cut.

Let us study the case where, due to interaction, \( G^r \) has an unstable pole in the complex \( \omega \)-plane.

**Example** We suppress the spatial coordinate and assume for \( G^r \), evaluated at \( \phi_k(t) = \phi_k^{(0)} \) (or at \( j_k(t)=0 \)), the following form

\[
G^r_{kk'}(\omega) = \frac{C_{kk'}}{\omega - \omega_0},
\]

with some constant matrix \( C_{kk'} \) and \( \text{Im} \omega_0 < 0 \). Then Eq. (63) becomes

\[
\frac{C_{kk'}}{\omega - \omega_0} \xi^k(\omega) = -\frac{1}{\lambda_s} \xi^k(\omega).
\]

Therefore we have \( \lambda_s = -(\omega - \omega_0) \lambda_s \xi^k(\omega) \propto \delta(\omega - \omega_0) \sum U_{hi} \xi^{(i)}_h \). Here \( \lambda_i \) and \( \xi^{(i)} \) are the eigenvalue and right eigenvector of \( C_{kk'} \) respectively and \( U_{hi} \) is some matrix. Thus \( \xi^k(t) \) has the form

\[
\xi^k(\omega) \sim C_{kk'} e^{-i\omega t}
\]

which decays at \( t \to +\infty \). Suppose the parameters contained in the theory are varied and \( \omega_0 \) moves into the region \( \text{Im} \omega_0 > 0 \). Then \( \xi^k(t) \) blows up for \( t \to +\infty \). Our criterion for the stability of \( \xi^{(0)}_k(t) \) is that the pole of \( G^r(\omega) \) remains in the lower half \( \omega \)-plane—a well-known fact.

In the zero temperature limit, we recover the results of previous sections. By using the general formula (50) in this limit, it can be seen that \( iG^r \) becomes identical
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The direct way to obtain this result is to observe that
\[ iG_{kk'}(x, x') = G_{kk'}(x, x') - G_{kk'}^-(x, x') . \]

Now, for time translation invariant case, the Fourier component of \( G_{kk'}^-(x-x') \) is, suppressing the space coordinate,
\[ G_{kk'}^-(\omega) = \sum_n \langle 0 | O_n(0) | n \rangle \langle n | O_n(0) | 0 \rangle \delta(\omega + E_n) . \]

If there is an energy gap above the ground state and if \( \langle 0 | O_n(0) | 0 \rangle = 0 \) and further if \( \omega \) is not so large \((|\omega| < E_n \) with \( E_n \) the energy of the excited states) then \( G^- = 0 \). If \( \langle 0 | O_n(0) | 0 \rangle \neq 0 \), we can redefine \( O_n \) such that it vanishes. These coincide with the well-known conditions for the recovery of the expression (50) in the zero temperature limit.

§ 6. Quantum field theory at non-zero temperature—the case of bilocal field

We proceed to study the case where the bilocal fields are taken as the order parameters. The main purpose of this section is the derivation of the generalized B-S equation at non-zero temperature. This is of practical use of course but here the general discussions are given, leaving the application to the real problems for the future publications.

The bilocal field we choose is written by \( \phi_i(x) \phi_j(y) \) in general. We have to generalize the method of previous section slightly and introduce four kinds of sources \( J_{ij}^{ab}(x, y) \) where \( a, b = 1 \) or 2. Define first \( W[J] \) by the path integral formula using the notation of Eq. (51):
\[
\exp \left\{ - \frac{i}{\hbar} \int d^4x \left( \mathcal{L}^{(1)}(x) - \mathcal{L}^{(2)}(x) \right) \right\},
\]
where
\[
\text{source term} = \int d^4x d^4y \left( - \right)^{a+b} J_{ij}^{ab}(x, y) \phi_i^{(a)}(x) \phi_j^{(b)}(y) \]
and the summation over \( a \) and \( b \) is implied. Effective action \( I[\phi] \) is introduced as usual.
\[
I[\phi] = W - \int d^4x d^4y J_{ij}^{ab}(x, y) \frac{\partial W}{\partial J_{ij}^{ab}}(x, y)
\]
with the relation
\[
\phi_i^{(a)}(x, y) = (-)^{a+b} \frac{\partial W}{\partial J_{ij}^{ab}}(x, y).
\]

The \( I \) defined above is not well defined so that we set in Eq. (66),
\[
J_{ij}^{ab}(x, y) = J_{ij}(x, y)
\]
for any \( a \) and \( b \). This is the physical condition which corresponds to Eq. (54) of the local field case. The reason why we get Eq. (67) is seen by rewriting the source term using Eq. (67) as

\[
\exp \frac{i}{\hbar} (\text{source term}) = \exp \frac{i}{\hbar} \int d^4 x \int d^4 y (\phi_i^{(1)}(x) - \phi_i^{(2)}(x))
\]

\[
\times \left( \phi_j^{(1)}(y) - \phi_j^{(2)}(y) \right) \eta_i(x, y).
\]

\[
\sim (\text{det } J)^{-1/2} \int [d\eta] \exp \left[ \frac{i}{\hbar} \int d^4 x \eta_i(x)(\phi_j^{(1)}(x) - \phi_j^{(2)}(x))
\right.
\]

\[
- \frac{i}{4\hbar} \int d^4 x \int d^4 y \eta_i(x)(J^{-1})_{ij}(x, y) \eta_j(y).
\]  

(68)

Therefore Eq. (66) can be written as a superposition (over \( \eta \)) of the physical correlation function determined by the Lagrangian \( L(x) + \eta_i(x)\phi_i(x) \). In such a case Eq. (66) gives well defined quantities. We obtain in this way under the constraint (67)

\[
\phi_{ij}^{(1)}(x, y) = \text{ave} \langle T\phi_i(x)\phi_j(y) \rangle_\eta = G^{(1)}_{ij}(x, y),
\]

\[
\phi_{ij}^{(2)}(x, y) = \text{ave} \langle \phi_i(x)\phi_j(y) \rangle_\eta = G^{(2)}_{ij}(x, y),
\]

\[
\phi_{ij}^{(3)}(x, y) = \text{ave} \langle \phi_j(y)\phi_i(x) \rangle_\eta = G^{(3)}_{ij}(x, y),
\]

\[
\phi_{ij}^{(4)}(x, y) = \text{ave} \langle T\phi_i(x)\phi_j(y) \rangle_\eta = G^{(4)}_{ij}(x, y),
\]

where \( \langle \cdots \rangle_\eta \) is the expectation value for the Lagrangian \( L(x) + \eta_i(x)\phi_i(x) \) and \( \text{ave} \) implies

\[
\text{ave} \langle \cdots \rangle_\eta = (\text{det } J)^{-1/2} \int [d\eta] \langle \cdots \rangle \exp \left[ \frac{i}{\hbar} \int d^4 x \int d^4 y \eta_i(x)(J^{-1})_{ij}(x, y) \eta_j(y) \right].
\]

There are three obvious relations among the \( \phi^{ab} \)'s:

\[
\phi_{ij}^{(1)}(x, y) = \phi_{ij}^{(1)}(y, x),
\]

\[
\phi_{ij}^{(2)}(x, y) = \theta(x_0 - y_0)\phi_{ij}^{(2)}(x, y) + \theta(y_0 - x_0)\phi_{ij}^{(2)}(x, y),
\]

\[
\phi_{ij}^{(3)}(x, y) = \theta(y_0 - x_0)\phi_{ij}^{(3)}(x, y) + \theta(x_0 - y_0)\phi_{ij}^{(3)}(x, y),
\]

\[
\phi_{ij}^{(4)}(x, y) = \theta(y_0 - x_0)\phi_{ij}^{(4)}(x, y) + \theta(x_0 - y_0)\phi_{ij}^{(4)}(x, y),
\]

(69)

where the time component of \( x \) or \( y \) is denoted by \( x_0 \) or \( y_0 \). Equation (69) can be inverted expressing \( \phi^{12} \) or \( \phi^{34} \) in terms of \( \phi^{11} \) and \( \phi^{22} \). These relations correspond to Eq. (55) and they imply that among four \( \phi^{ab} \) only one of them is independent. This is consistent with Eq. (67). Note an identity

\[
\sum_{a, b=1}^{2} \left( - \right)^{a+b}\phi_{ij}^{ab}(x, y) = 0.
\]  

(70)

Now we consider the properties of the second derivatives of the \( W \) or \( \Gamma \) under the constraint (67) or (69). We substitute Eq. (67) into Eq. (66) and expand Eq. (66) around \( J = 0 \). We use the fact that since Eq. (69) is valid for any \( J \), it holds for the expansion coefficients. Let us define

\[
\bar{W}^{ab}_{x, y; x', y'} = \left( \delta^a W / \delta J^{ab}(x, y) \right) \delta J^{ab}(x', y').
\]  

(71)
where \((\cdots)_0\) signifies the value at \(J=0\), then the desired relations are

\[
\begin{align*}
W^{11}_{i_1,j_1; i'x,j'y} &= W^{12}_{j_1,i_1; i'x,j'y}, \\
\left( \frac{W^{11}_{i_1,j_1; i'x,j'y}}{W^{12}_{i_1,j_1; i'x,j'y}} \right) &= \left( \frac{\theta(x_0-y_0)}{\theta(y_0-x_0)} \right) \left( \frac{\theta(x_0-y_0)}{\theta(y_0-x_0)} \right) W^{21}_{i_1,j_1; i'x,j'y}.
\end{align*}
\]  

(72)

In order to get the corresponding relation for \(I\) we take a solution \(\phi^{(0)}_{ab}(x, y)\) satisfying

\[
\delta \Gamma / \delta \phi^{(0)}_{ab}(x, y) = 0
\]

and insert \(\phi^{(I)}_{ab}(x, y) = \phi^{(0)}_{ab}(x, y) + \Delta \phi^{(I)}_{ab}(x, y)\) into the identity

\[
J^{ab}_{\alpha}(x, y) = \left( -\right)^{a+b+1} \delta \Gamma / \delta \phi^{(I)}_{ab}(x, y)
\]

(73)

and use Eq. (67). Expanding the right-hand side of Eq. (73) in \(\Delta \phi\) we get, with no summation on \(a\) and \(b\),

\[
J_{\alpha}(x, y) = \left( -\right)^{a+b+1} \sum_{a, b=1}^{2} \int d^4x' \int d^4y' \Gamma_{aix,bjy; a'x',b'y'} \Delta \phi_{a'b'}^{(I)}(x', y'),
\]

(74)

where \((\cdots)_0\) means the value at \(\phi = \phi^{(0)}\) and

\[
\Gamma_{aix,bjy; a'x',b'y'} = \delta^2 \Gamma / \delta \phi^{(I)}_{ab}(x, y) \delta \phi_{a'b'}^{(I)}(x', y').
\]

Since \(\phi^{(0)}\) satisfies Eq. (69), \(\Delta \phi_{\alpha}(x, y)\) satisfies Eq. (69) also. Equation (74) is our relation between \(J\) and \(\Delta \phi\) for small \(J\). We see that the right-hand side is independent of the indices \(ab\).

With these preliminaries, we now state our equation of motion and the stability criterion. The equation determining \(\phi^{(I)}_{ab}(x, y)\) is

\[
\delta \Gamma / \delta \phi^{(I)}_{ab}(x, y) |_\ast = 0,
\]

(75)

where \(\cdots |_\ast\) signifies that we use the constraint (69) after the differentiation by \(\phi^{(I)}_{ab}(x, y)\). For any indices \(ab\), Eq. (75) gives the same equation and it corresponds to Eq. (56). The stability of the solution \(\phi^{(I)}_{ab}(x, y)\) to Eq. (75) is determined by the eigenvalue equation

\[
\int d^4x' \int d^4y' (\Gamma_{aix,bjy; a'x',b'y'})_0 \Delta \phi_{a'b'}^{(I)}(x', y') = 0.
\]

(76)

This is the same equation for any indices \(ab\) since it is a special case \(J=0\) of Eq. (74). Equation (76) corresponds to Eq. (57). In the space of the indices \(ab\), Eq. (76) appears to involve a \(4 \times 4\) matrix but actually it can be reduced to \(1 \times 1\) by using Eq. (69) for \(\Delta \phi_{ab}^{(I)}(x, y)\). In fact all \(\Delta \phi_{ab}^{(I)}(x, y)\) is linearly expressed by a single \(\Delta \phi_{11}^{(I)}(x, y)\) for instance.

Let us see the relation of our zero eigenvalue equation (76) to the Green's functions. We note the following identity,

\[
\begin{align*}
- \delta_{aa'} \delta_{bb'} \delta_{ii'} \delta_{jj'} \delta^4(x-x') \delta^4(y-y') &= \int d^4x'' \int d^4y'' W_{aix,bjy; c'ix',d'jy'} \\
\times \Gamma_{c'ix',d'jy'; a'x',b'y'}.
\end{align*}
\]

(77)

\[
J_{\alpha}(x, y) = \left( -\right)^{a+b+1} \sum_{a, b=1}^{2} \int d^4x' \int d^4y' \Gamma_{aix,bjy; a'x',b'y'} \Delta \phi_{a'b'}^{(I)}(x', y'),
\]

(74)

where \((\cdots)_0\) means the value at \(\phi = \phi^{(0)}\) and

\[
\Gamma_{aix,bjy; a'x',b'y'} = \delta^2 \Gamma / \delta \phi^{(I)}_{ab}(x, y) \delta \phi_{a'b'}^{(I)}(x', y').
\]

Since \(\phi^{(0)}\) satisfies Eq. (69), \(\Delta \phi_{\alpha}(x, y)\) satisfies Eq. (69) also. Equation (74) is our relation between \(J\) and \(\Delta \phi\) for small \(J\). We see that the right-hand side is independent of the indices \(ab\).

With these preliminaries, we now state our equation of motion and the stability criterion. The equation determining \(\phi^{(I)}_{ab}(x, y)\) is

\[
\delta \Gamma / \delta \phi^{(I)}_{ab}(x, y) |_\ast = 0,
\]

(75)

where \(\cdots |_\ast\) signifies that we use the constraint (69) after the differentiation by \(\phi^{(I)}_{ab}(x, y)\). For any indices \(ab\), Eq. (75) gives the same equation and it corresponds to Eq. (56). The stability of the solution \(\phi^{(I)}_{ab}(x, y)\) to Eq. (75) is determined by the eigenvalue equation

\[
\int d^4x' \int d^4y' (\Gamma_{aix,bjy; a'x',b'y'})_0 \Delta \phi_{a'b'}^{(I)}(x', y') = 0.
\]

(76)

This is the same equation for any indices \(ab\) since it is a special case \(J=0\) of Eq. (74). Equation (76) corresponds to Eq. (57). In the space of the indices \(ab\), Eq. (76) appears to involve a \(4 \times 4\) matrix but actually it can be reduced to \(1 \times 1\) by using Eq. (69) for \(\Delta \phi^{(I)}_{ab}(x, y)\). In fact all \(\Delta \phi^{(I)}_{ab}(x, y)\) is linearly expressed by a single \(\Delta \phi^{(I)}_{11}(x, y)\) for instance.

Let us see the relation of our zero eigenvalue equation (76) to the Green's functions. We note the following identity,
This relation is used at $J=0$, or at $\phi = \phi^0$. Multiply Eq. (77) from the right by $\Delta \phi_{\alpha \beta}^{\gamma \delta}(x, y)$, then we get

$$-\Delta \phi_{\alpha \beta}^{\gamma \delta}(x, y) = \int d^4 x' \int d^4 y' \Gamma_{\alpha \beta \gamma \delta} \phi_{\alpha \beta}^{\gamma \delta}(x', y').$$

But from Eq. (74) $\Gamma \Delta \phi$ part of the right-hand side of Eq. (78) becomes independent of $cd$ so that by using the notation (71), Eq. (78) reduces to

$$-\Delta \phi_{\alpha \beta}^{\gamma \delta}(x, y) = \int d^4 x' \int d^4 y' \tilde{W}_{\alpha \beta}^{\gamma \delta}(k_x, k_y) \Gamma_{\alpha \beta \gamma \delta} \phi_{\alpha \beta}^{\gamma \delta}(x', y').$$

Now the $\Delta \phi_{\alpha \beta}^{\gamma \delta}(x, y)$'s are expressed by a single $\Delta \phi_{a_0b_0}$ as

$$\Delta \phi_{\alpha \beta}^{\gamma \delta}(x, y) = (C_{a_0b_0}) \phi_{\alpha \beta}^{\gamma \delta}(x, y),$$

where $C_{a_0b_0}$ includes the operation of the transpose $(ix) \leftrightarrow (jy)$ otherwise it is a numerical constant, as is seen from Eq. (69). In Eq. (80) there is no summation on $a_0b_0$. For the index $ab = a_0b_0$ we get, with no summation over $a_0b_0$, the relation,

$$\int d^4 x' \int d^4 y' \tilde{W}_{\alpha \beta}^{\gamma \delta}(k_x, k_y) \Gamma_{\alpha \beta \gamma \delta} \phi_{\alpha \beta}^{\gamma \delta}(x', y') \tilde{C}_{a_0b_0} = - \delta_{ii'} \delta_{jj'} \delta^4(x-x') \delta^4(y-y'),$$

where $\tilde{C}$ means that it acts on the left. This holds independently of the indices $a_0b_0$ and $cd$. Equation (81) is the analogue of Eq. (61).

We see that solving the zero eigenvalue equation (76) is equivalent to looking for the eigenvector of the pole of $\tilde{W}$. Remember that this is the pole in the two particle channel specified by the pair of indices $(ix)$ and $(jy)$ or $(kx')$ and $(ly')$ in Eq. (81). This pole therefore appears in the Fourier component conjugate to the difference of the center of mass coordinate $(x+y)/2 - (x'+y')/2$, see Eq. (37). By looking at Eq. (72), it is easily convinced that the position of the pole is independent of the indices $a_0b_0$.

In the following we show that $\tilde{W}$ has the property of the generalized retarded Green's function. Take for instance

$$\tilde{W}_{\alpha \beta \gamma \delta}^{\alpha' \beta' \gamma' \delta'} = \text{ave} \langle \Gamma_{\alpha \beta \gamma \delta} \phi_{\alpha \beta}^{\gamma \delta}(x, y) \phi_{\alpha' \beta'}^{\gamma' \delta'}(x', y') \rangle_x.$$
Stability Conditions in Quantum System

\[ \text{Im}\omega_1 > 0, \text{Im}\omega_2 > 0, \text{Im}\omega_3 > 0, \text{where } \text{Im}(\omega_1 + \omega_2 + \omega_3) > 0. \]

Therefore our stability criterion that \( \Delta \phi^{ab}_0(x, y) \) does not blow up for \( X_0 \equiv (x_0 + y_0)/2 \rightarrow +\infty \) is satisfied if \( \bar{W}^{11} \) does not have a pole for \( \text{Im}\omega > 0 \) where \( \hbar\omega = (\omega_1 + \omega_2 + \omega_3)\hbar \) is the total energy in the two particle channel. The above conclusion is also valid for \( \bar{W}^{12}; i=x', j', y', \bar{W}^{21}; \) and \( \bar{W}^{22}; \) They all have support for \( x_0', y_0' \) which are both smaller than \( \max(x_0, y_0). \)

The diagrammatic expansion of \( \Gamma[\phi] \) is readily obtained by the same expression as Eq. (38) where \( D \) is generalized to a \( 2 \times 2 \) matrix with \( D_{ij}^{0} = G_0(x, y), D_{ij}^{1} = G_1(x, y), D_{ij}^{2} = G_2(x, y) \) and \( D_{ij}^{3} = G_3(x, y) \) and \( \text{Tr} \) now involves also the trace over this matrix indices. The vertex rule is slightly modified to include minus sign if the vertex refers to “2” of the upper index of \( D_{ij}^{0}. \) We can use this expansion rule for the discussion of non-zero temperature case just like the zero-temperature case. The Schwinger-Dyson equation for Bosonic case is, using the notation \( \phi^{ab}_0(x, y) = D_{ij}^{0}(x, y), \)

\[
\delta \Gamma / \delta D^{ab}_0(x,y) = D_{ij}^{1-1ab}(x,y) - \bar{D}_{ij}^{-1ab}(x,y) - \Sigma^{ab}_0(x,y) = 0,
\]

\[
\Sigma^{ab}_0(x,y) = (2i/\hbar) \delta\gamma^{(a)}[D] / \delta D^{ab}_0(y,x), \tag{82}
\]

where \( \bar{D}_{ij}^{-1} = 1/\hbar D_{ij}^{-1} \) with \( D_{ij} \) the free propagator matrix. Denoting the solution to Eq. (82) by \( D^{(0)} \), we find the B-S equation

\[
\int d^4x' \int d^4y'(-D^{(0)-1ac}_h(x,x')D^{(0)-1db}_0(y',y') + T^{(4)}_{aix, biy; cx'i', dy'}')AD^{(0)}_{ab}(x', y') = 0,
\]

where

\[
T^{(4)}_{aix, biy; cx'i', dy'} = - (\delta \Sigma^{ab}_0(x,y) / \delta D^{ab}_0(x', y')).
\]

Finally the zero temperature limit is obtained by using the formula (50) for any Green’s function \( \langle \cdots \rangle_\gamma \) in the \( \gamma \)-representation (68). If the necessary conditions similar to the ones given in § 5 are satisfied, \( \bar{W}^{11} \) for example becomes identical to \( \langle 0| T(\phi_i(x) \phi_j(y) \phi_i(x') \phi_j(y')|0 \rangle \), which is the \( G^{(4)} \) appearing in § 4.

References
1) E. E. Salpeter and H. A. Bethe, Phys. Rev. 84 (1951), 1232.