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Discussion

Discussion: "Normal Indentation of Elastic Half-Space With a Rigid Frictionless Axisymmetric Punch" (Fu, G., and Chandra, A., 2002, ASME J. Appl. Mech., 69, pp. 142–147)

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The authors presented an interesting consideration of an axisymmetric frictionless contact problem with the aid of mathematical software MATHEMATICA. Evidently, the use of modern analytical software gives a possibility to obtain new results, check known solutions, and correct possible misprints. However, some papers in the field should be added to their reference list.

In 1939 an analytical solution for a punch described by a monomial function of r of a positive even degree α was obtained by Shtaerman [1]. It is worth mentioning that after A. E. H. Love had obtained his solution, the problem for conical punch was also solved by Lur'e [2] in 1941. The problem for a punch described by a monomial function of r of an arbitrary real degree α was solved by Galin (see Chap. 2, paragraph 5 in Ref. [3]). Then this problem was also analyzed by Sneddon [4]. In 1957 the problem was analyzed by Segedin [5] for a punch whose shape is represented by a series (a polynomial function of r) with integer degrees α. For a punch described by a fractional power series of r, the problem was analyzed in Ref. [6]. The analysis in Ref. [6] was based on the Galin's solution ([3]). It was shown that the solution can be also used in the case when the punch is a transversally isotropic solid and the half space has homogeneous initial stresses. In particular, a formula similar to formula (17) was obtained.

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Closure to "Discussion of 'Normal Indentation of Elastic Half-Space With a Rigid Frictionless Axisymmetric Punch' " (2003, ASME J. Appl. Mech., 70, pp. 783)

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Our work ([1]) was based on Green's solution ([2]). We thank the discussors for pointing out a different approach taken by Borodich [3] following the work of Galin [4]. At the time of publication, we were unaware of the work by Borodich. The usage of our derived solution is straightforward. With modern mathematical software, hypergeometric function will be like a regular elementary function and the final result is easy to be obtain. It can also be used to check analytical expressions for possible misprints.

As it is pointed out in the paper, the power of the "polynomial" can be any non-negative number, such as 0, 2, 1/12, e, π. With this solution, one can use multiple terms to define the punch shape instead of a monomial function of the punch radius.

We appreciate the fact that there exist numerous contributions to this field in the Russian literature, and our understanding of this work is mainly based on the books by Gladwell [5] and Sneddon [6]. Johnson [7] also mentioned the solutions by Shtaerman and Galin in his book.

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Discussion: "Dynamic Condensation and Synthesis of Unsymmetric Structural Systems" (Rao, G. V., 2002, ASME J. Appl. Mech., 69, pp. 610–616)

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The dynamic condensation method ([1]) was successfully extended by Rao [2] to handle the unsymmetric systems with damping. This method is very interesting and useful in the finite element modeling, vibration control, etc. However, one misunderstanding occurred when this approach was utilized in substructure synthesis.

As stated by the author in Sec. 4, the reduced order matrices $[M_R]$ and $[K_R]$ of each substructure in Eqs. (16) and (17) have the form

$$[M_R]=\begin{bmatrix}[0] & [M_{mmR}]\\ -[M_{mmR}] & -[C_{mmR}]\end{bmatrix},\quad [K_R]=\begin{bmatrix}[M_{mmR}] & [0]\\ [0] & [K_{mmR}]\end{bmatrix} \tag{1}$$

in which $[M_{mmR}]$, $[C_{mmR}]$, and $[K_{mmR}]$ are the reduced order mass, damping, and stiffness matrices of order $m\times m$.

Actually, if the reduced order matrices $[M_R]$ and $[K_R]$ are computed from Eqs. (16) and (17) as indicated by the author, these two matrices are generally fully populated and do not have the forms shown in Eq. (1). This will be explained in detail later. Hence one cannot simply convert these two matrices into the displacement space with the explicit forms of the reduced order matrices $[M_{mmR}]$, $[C_{mmR}]$ and $[K_{mmR}]$. If the matrices on the right-hand sides of Eq. (1) are known and those on the left-hand sides are unknowns, the relations shown in this equation are right. However, the problem is how we get the reduced matrices $[M_{mmR}]$, $[C_{mmR}]$, and $[K_{mmR}]$ before we have $[M_R]$ and $[K_R]$.

To simplify the discussion, consider a symmetric problem. After the simplification, the full order matrices $[\bar{M}]$ and $[\bar{K}]$ in Ref. [2] become

$$[\bar{M}]=-\begin{bmatrix}[0] & [M]\\ [M] & [C]\end{bmatrix},\quad [\bar{K}]=\begin{bmatrix}-[M] & [0]\\ [0] & [K]\end{bmatrix}, \tag{2}$$

if the eigenproblem in Sec. 3 rather than the dynamic equations of equilibrium in Sec. 2 is considered. The transformation matrices $[R]$ and $[S]$ are the same and indicated by $[R]$. The corresponding governing equation for the transformation matrix is given by

$$[R]=[\bar{K}_{ss}]^{-1}[([\bar{M}_{sm}]+[\bar{M}_{ss}][R])[M_R]^{-1}[K_R]-[\bar{K}_{sm}]] \tag{3}$$

and the initial approximation is

$$[R]^{(0)}=-[\bar{K}_{ss}]^{-1}[\bar{K}_{sm}]. \tag{4}$$

A very simple numerical example is given to show the form of reduced order matrices $[M_R]$ and $[K_R]$. In this example, the mass, damping, and stiffness matrices are

$$[M]=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix},\quad [C]=\begin{bmatrix}1&0&0\\0&0&0\\0&0&0\end{bmatrix},\quad [K]=\begin{bmatrix}2&-1&0\\-1&2&-1\\0&-1&1\end{bmatrix}. \tag{5}$$

Two cases that the first and the third degrees of freedom are, respectively, selected as the master degrees of freedom are considered. The resulted reduced order matrices $[M_R]$ and $[K_R]$ from the initial approximation and the first three iterations are listed in Table 1. The results show that reduced order matrices $[M_R]$ and $[K_R]$ obtained from the initial approximation, that is, Guyan condensation, have the forms given in Eq. (2). This conclusion can be proven simply. After two iterations, both reduced order matrices are fully populated. The further discussion on the dynamic condensation of viscously damped, symmetric models may be found in Refs. [3–6].

Table 1 Reduced order matrices $[M_R]$ and $[K_R]$ during iteration

	Case 1				Case 2			
Iteration	$[M_R]$		$[K_R]$		$[M_R]$		$[K_R]$	
0	0	−1	−1	0	0	−1	−1	0
	−1	−1	0	1	−1	−0.1111	0	0.3333
1	0	−3	−3	0	0.2469	−1.4815	−1.4815	0
	−3	−1	0	1	−1.4815	−0.1111	0	0.3333
2	−3.3333	−4.6667	−2.4444	0.5556	0.1963	−1.5716	−1.5042	−0.0311
	−4.6667	−1	0.5556	1.5556	−1.5716	−0.1962	−0.0311	0.3512
3	−3.0183	−5.1174	−3.0159	0.6003	0.2750	−1.6296	−1.6196	−0.04670
	−5.1174	−0.6281	0.6003	1.6537	−1.6296	−0.2148	−0.04670	0.3507

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Closure to "Discussion on 'Dynamic Condensation and Synthesis of Unsymmetrical Systems'" (2003, ASME J. Appl. Mech., 70, p. 784)

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I appreciate Zu-Qing Qu for the interest shown and for making useful comments on the contents of my paper.

The reduced order matrices M_R and K_R retain the same form as given in Eq. (1) as the iterations progress. The computation results presented by Zu-Qing Qu seem to be incorrect.

With the help of the procedure in Sec. 3 of my paper, the computations for the first three iterations are carried out on the numerical example cited by Zu-Qing Qu. The results of the first three iterations are given in Table 1 hereunder.

In addition to the above, the eigenvalues are also extracted for the full system and the two cases of master selection. The converged eigenvalues are shown in Table 2 below.

Further to the above, I wish to add here that since the formulation in my paper finally falls into the category of unsymmetric matrices—particularly so in the case of the mass matrix in Eq. (2)—the assumption by Zu-Qing Qu that the transformation matrices [R] and [S] are the same even for a symmetric structure is not valid.

Table 1 M_R and K_R for three iterations

	Case 1				Case 2			
Iteration	M_R		K_R		M_R		K_R	
0	0	1	1	0	0	1	1	0
	−1	−1	0	1	−1	0.1111	0	0.333
1	0	1	1	0	0	1	1	0
	−3.0	−1	0	1	−1.481	−0.111	0	0.333
2	0	1	1	0	0	1	1	0
	−3.0	−0.5165	0	1	−1.50	−0.167	0	0.333
3	0	1	1	0	0	1	1	0
	−4.1111	−0.4444	0	1	−1.572	−0.161	0	0.333
10	0	1	1	0	0	1	1	0
	−4.765	−0.516	0	1	−1.588	−0.172	0	0.333

Table 2 Eigenvalues

Mode no.	Full system	Case 1	Case 2
1	−0.0542+J 0.4549	−0.0507+J 0.4553	−0.05419+J0.4549
2	−0.0542−J 0.4549	−0.0507−J 0.4553	−0.05419−j 0.4549
3	−0.3336+J 1.2374		
4	−0.3336−J 1.2374		
5	−0.11225+J 1.6996		
6	−0.11225−J 1.6996		

