

Discussion: “Normal Indentation of Elastic Half-Space With a Rigid Frictionless Axisymmetric Punch” (Fu, G., and Chandra, A., 2002, ASME J. Appl. Mech., 69, pp. 142–147)

**F. M. Borodich
L. M. Keer**

Northwestern University, Evanston, IL 60208-3109

The authors presented an interesting consideration of an axisymmetric frictionless contact problem with the aid of mathematical software MATHEMATICA. Evidently, the use of modern analytical software gives a possibility to obtain new results, check known solutions, and correct possible misprints. However, some papers in the field should be added to their reference list.

In 1939 an analytical solution for a punch described by a monomial function of r of a positive even degree α was obtained by Shtaerman [1]. It is worth mentioning that after A. E. H. Love had obtained his solution, the problem for conical punch was also solved by Lur'e [2] in 1941. The problem for a punch described by a monomial function of r of an arbitrary real degree α was solved by Galin (see Chap. 2, paragraph 5 in Ref. [3]). Then this problem was also analyzed by Sneddon [4]. In 1957 the problem was analyzed by Segedin [5] for a punch whose shape is represented by a series (a polynomial function of r) with integer degrees α . For a punch described by a fractional power series of r , the problem was analyzed in Ref. [6]. The analysis in Ref. [6] was based on the Galin's solution ([3]). It was shown that the solution can be also used in the case when the punch is a transversally isotropic solid and the half space has homogeneous initial stresses. In particular, a formula similar to formula (17) was obtained.

References

- [1] Shtaerman, I. Ya., 1939, "On the Hertz Theory of Local Deformations Resulting From the Pressure of Elastic Solids," *Dokl. Akad. Nauk SSSR*, **25**, pp. 360–362 (in Russian).
- [2] Lur'e, A. I., 1941, "Some Contact Problems of the Theory of Elasticity," *Prikl. Mat. Mekh.*, **5**, pp. 383–408 (in Russian); Lur'e, A. I., 1964, *Three-Dimensional Problems of the Theory of Elasticity*, Interscience Publishers, New York.
- [3] Galin, L. A., 1953, *Contact Problems in the Theory of Elasticity*, Gostekhizdat, Moscow-Leningrad. (English translation by H. Moss, edited by I. N. Sneddon, North Carolina State College, Departments of Mathematics and Engineering Research, NSF Grant No. G16447, 1961).
- [4] Sneddon, I. N., 1965, "The Relation Between Load and Penetration in the Axisymmetric Boussinesq Problem for a Punch of Arbitrary Profile," *Int. J. Eng. Sci.*, **3**, pp. 47–57.
- [5] Segedin, C. M., 1957, "The Relation Between Load and Penetration for a Spherical Punch," *Mathematika*, **4**, pp. 156–161.
- [6] Borodich, F. M., 1990, "Hertz Contact Problems for Elastic Anisotropic Half-Space With Initial Stress," *Soviet Appl. Mechanics*, **26**, pp. 126–132.

Closure to “Discussion of ‘Normal Indentation of Elastic Half-Space With a Rigid Frictionless Axisymmetric Punch’ ” (2003, ASME J. Appl. Mech., 70, pp. 783)

**A. Chandra
G. Fu**

Mechanical Engineering Department, Iowa State University, Ames, IA 50011

Our work ([1]) was based on Green's solution ([2]). We thank the discussors for pointing out a different approach taken by Borodich [3] following the work of Galin [4]. At the time of publication, we were unaware of the work by Borodich. The usage of our derived solution is straightforward. With modern mathematical software, hypergeometric function will be like a regular elementary function and the final result is easy to be obtain. It can also be used to check analytical expressions for possible misprints.

As it is pointed out in the paper, the power of the “polynomial” can be any non-negative number, such as 0, 2, 1/12, e , π . With this solution, one can use multiple terms to define the punch shape instead of a monomial function of the punch radius.

We appreciate the fact that there exist numerous contributions to this field in the Russian literature, and our understanding of this work is mainly based on the books by Gladwell [5] and Sneddon [6]. Johnson [7] also mentioned the solutions by Shtaerman and Galin in his book.

References

- [1] Fu, G., and Chandra, A., 2002, "Normal Indentation of Elastic Half-Space With a Rigid Frictionless Axisymmetric Punch," *ASME J. Appl. Mech.*, **69**, pp. 142–147.
- [2] Green, A. E., and Zerna, W., 1954, *Theoretical Elasticity*, Oxford University Press, London, Great Britain.
- [3] Borodich, F. M., 1990, "Hertz Contact Problems for Elastic Anisotropic Half-Space With Initial Stress," *Soviet Appl. Mechanics*, **26**, pp. 126–132.
- [4] Galin, L. A., 1953, *Contact Problems in the Theory of Elasticity*, Gostekhizdat, Moscow-Leningrad (English translation by H. Moss, edited by I. N. Sneddon, North Carolina State College, Departments of Mathematics and Engineering Research, NSF Grant No. G16447, 1961).
- [5] Gladwell, G. M. L., 1980, *Contact Problems in the Classical Theory of Elasticity*, Sijthoff & Noordhoff, Alphen aan den Rijn, the Netherlands.
- [6] Sneddon, I. N., 1966, *Mixed Boundary Value Problems in Potential Theory*, North-Holland Publishing Company, Amsterdam, Holland.
- [7] Johnson, K. L., 1985, *Contact Mechanics*, Cambridge University Press, Cambridge, UK.

Discussion: “Dynamic Condensation and Synthesis of Unsymmetric Structural Systems” (Rao, G. V., 2002, ASME J. Appl. Mech., 69, pp. 610–616)

Z.-Q. Qu¹

Department of Civil Engineering, University of Arkansas, Fayetteville, AR 72701

The dynamic condensation method ([1]) was successfully extended by Rao [2] to handle the unsymmetric systems with damping. This method is very interesting and useful in the finite element modeling, vibration control, etc. However, one misunderstanding occurred when this approach was utilized in substructure synthesis.

As stated by the author in Sec. 4, the reduced order matrices $[M_R]$ and $[K_R]$ of each substructure in Eqs. (16) and (17) have the form

$$[M_R] = \begin{bmatrix} [0] & [M_{mmR}] \\ -[M_{mmR}] & -[C_{mmR}] \end{bmatrix}, \quad [K_R] = \begin{bmatrix} [M_{mmR}] & [0] \\ [0] & [K_{mmR}] \end{bmatrix} \quad (1)$$

in which $[M_{mmR}]$, $[C_{mmR}]$, and $[K_{mmR}]$ are the reduced order mass, damping, and stiffness matrices of order $m \times m$.

Actually, if the reduced order matrices $[M_R]$ and $[K_R]$ are computed from Eqs. (16) and (17) as indicated by the author, these two matrices are generally fully populated and do not have the forms shown in Eq. (1). This will be explained in detail later. Hence one cannot simply convert these two matrices into the displacement space with the explicit forms of the reduced order matrices $[M_{mmR}]$, $[C_{mmR}]$ and $[K_{mmR}]$. If the matrices on the right-hand sides of Eq. (1) are known and those on the left-hand sides are unknowns, the relations shown in this equation are right. However, the problem is how we get the reduced matrices $[M_{mmR}]$, $[C_{mmR}]$, and $[K_{mmR}]$ before we have $[M_R]$ and $[K_R]$.

To simplify the discussion, consider a symmetric problem. Af-

ter the simplification, the full order matrices $[\bar{M}]$ and $[\bar{K}]$ in Ref. [2] become

$$[\bar{M}] = - \begin{bmatrix} [0] & [M] \\ [M] & [C] \end{bmatrix}, \quad [\bar{K}] = \begin{bmatrix} -[M] & [0] \\ [0] & [K] \end{bmatrix}, \quad (2)$$

if the eigenproblem in Sec. 3 rather than the dynamic equations of equilibrium in Sec. 2 is considered. The transformation matrices $[\bar{R}]$ and $[\bar{S}]$ are the same and indicated by $[R]$. The corresponding governing equation for the transformation matrix is given by

$$[R] = [\bar{K}_{ss}]^{-1}([\bar{M}_{sm}] + [\bar{M}_{ss}][R])[M_R]^{-1}[K_R] - [\bar{K}_{sm}] \quad (3)$$

and the initial approximation is

$$[R]^{(0)} = -[\bar{K}_{ss}]^{-1}[\bar{K}_{sm}]. \quad (4)$$

A very simple numerical example is given to show the form of reduced order matrices $[M_R]$ and $[K_R]$. In this example, the mass, damping, and stiffness matrices are

$$[M] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad [C] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad (5)$$

$$[K] = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}.$$

Two cases that the first and the third degrees of freedom are, respectively, selected as the master degrees of freedom are considered. The resulted reduced order matrices $[M_R]$ and $[K_R]$ from the initial approximation and the first three iterations are listed in Table 1. The results show that reduced order matrices $[M_R]$ and $[K_R]$ obtained from the initial approximation, that is, Guyan condensation, have the forms given in Eq. (2). This conclusion can be proven simply. After two iterations, both reduced order matrices are fully populated. The further discussion on the dynamic condensation of viscously damped, symmetric models may be found in Refs. [3–6].

Table 1 Reduced order matrices $[M_R]$ and $[K_R]$ during iteration

Iteration	Case 1				Case 2			
	$[M_R]$		$[K_R]$		$[M_R]$		$[K_R]$	
0	0	-1	-1	0	0	-1	-1	0
	-1	-1	0	1	-1	-0.1111	0	0.3333
1	0	-3	-3	0	0.2469	-1.4815	-1.4815	0
	-3	-1	0	1	-1.4815	-0.1111	0	0.3333
2	-3.3333	-4.6667	-2.4444	0.5556	0.1963	-1.5716	-1.5042	-0.0311
	-4.6667	-1	0.5556	1.5556	-1.5716	-0.1962	-0.0311	0.3512
3	-3.0183	-5.1174	-3.0159	0.6003	0.2750	-1.6296	-1.6196	-0.04670
	-5.1174	-0.6281	0.6003	1.6537	-1.6296	-0.2148	-0.04670	0.3507

References

- [1] Qu, Z.-Q., and Fu, Z.-F., 1998, “New Structural Dynamic Condensation Method for Finite Element Models,” AIAA J., **36**, pp. 1320–1324.
- [2] Rao, G. V., 2002, “Dynamic Condensation and Synthesis of Unsymmetric Structural Systems,” ASME J. Appl. Mech., **69**, pp. 610–616.
- [3] Rivera, M. A., Singh, M. P., and Suarez, L. E., 1999, “Dynamic Condensation Approach for Nonclassically Damped Structures,” AIAA J., **37**, pp. 564–571.
- [4] Qu, Z.-Q., and Selvam, R. P., 2000, “Dynamic Condensation Methods for Viscously Damped Models,” *Proceedings of the 18th International Modal Analysis Conference & Exhibit* (San Antonio, Texas), Society for Experimental Mechanics, CT, USA, pp. 1752–1757.
- [5] Qu, Z.-Q., and Chang, W., 2000, “Dynamic Condensation Method for Viscously Damped Vibration Systems in Engineering,” Eng. Struct., **22**, pp. 1426–1432.
- [6] Qu, Z.-Q., and Selvam, R. P., 2002, “Efficient Method for Dynamic Condensation of Nonclassically Damped Vibration Systems,” AIAA J., **40**, pp. 368–375.

¹Current address: 2409 Wynncrest Circle, 6205, Arlington, TX 76006.

Closure to “Discussion on ‘Dynamic Condensation and Synthesis of Unsymmetrical Systems’” (2003, ASME J. Appl. Mech., 70, p. 784)

G. V. Rao

EMRC 607/900 Barton Centre, M. G. Road, Bangalore, Karnataka 560001, India

I appreciate Zu-Qing Qu for the interest shown and for making useful comments on the contents of my paper.

The reduced order matrices M_R and K_R retain the same form as given in Eq. (1) as the iterations progress. The computation results presented by Zu-Qing Qu seem to be incorrect.

With the help of the procedure in Sec. 3 of my paper, the computations for the first three iterations are carried out on the numerical example cited by Zu-Qing Qu. The results of the first three iterations are given in Table 1 hereunder.

In addition to the above, the eigenvalues are also extracted for the full system and the two cases of master selection. The converged eigenvalues are shown in Table 2 below.

Further to the above, I wish to add here that since the formulation in my paper finally falls into the category of unsymmetric matrices—particularly so in the case of the mass matrix in Eq. (2)—the assumption by Zu-Qing Qu that the transformation matrices $[R]$ and $[S]$ are the same even for a symmetric structure is not valid.

Table 1 M_R and K_R for three iterations

Iteration	Case 1				Case 2			
	M_R		K_R		M_R		K_R	
0	0 -1	1 -1	1 0	0 1	0 -1	1 0.1111	1 0	0 0.333
1	0 -3.0	1 -1	1 0	0 1	0 -1.481	1 -0.111	1 0	0 0.333
2	0 -3.0	1 -0.5165	1 0	0 1	0 -1.50	1 -0.167	1 0	0 0.333
3	0 -4.1111	1 -0.4444	1 0	0 1	0 -1.572	1 -0.161	1 0	0 0.333
10	0 -4.765	1 -0.516	1 0	0 1	0 -1.588	1 -0.172	1 0	0 0.333

Table 2 Eigenvalues

Mode no.	Full system	Case 1	Case 2
1	-0.0542+J 0.4549	-0.0507+J 0.4553	-0.05419+J0.4549
2	-0.0542-J 0.4549	-0.0507-J 0.4553	-0.05419-j 0.4549
3	-0.3336+J 1.2374		
4	-0.3336-J 1.2374		
5	-0.11225+J 1.6996		
6	-0.11225-J 1.6996		