Characterization of Statistically Homogeneous Fluctuations

— Fluctuation Spectrum and Spatial Correlation —

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A new characterization of statistically homogeneous spatial fluctuations is developed by observing how a scale-dependent local average approaches the ensemble average as the scale is extended. This enables us to study the global fluctuation characteristics together with the spatial correlations. The present approach informs various statistical aspects which cannot be described by two-point correlation functions.

Recently we have developed a new statistical-physical approach to one-variable sequential fluctuations generated by stochastic or chaotic dynamics. This is done by observing how a local average of fluctuations taken over large but finite scale approaches the ensemble average as the scale is increased. We have shown that such procedure produces the fluctuation spectrum concept, which plays an important role for the global characterization of fluctuations. Furthermore, especially when we are concerned with a time sequence, statistical characteristics which cannot be described by the global analysis are studied with the aid of the generalized time correlation function. We have also presented a systematic, practical approximation method for the fluctuation spectrum and the generalized time correlation function in the frame of the continued fraction expansion.

Many physical systems, on the other hand, are often defined as the field quantities. Famous physical systems are, e.g., magnetic systems in the thermal equilibrium, fluid dynamics, especially locally homogeneous and isotropic turbulence, and the dynamics of the oscillator community with large degrees of freedom, etc.

Let \( \{ u(\mathbf{r}, t) \} \) be a scalar fluctuation variable at the position \( \mathbf{r} \) in a \( d \)-dimensional system at the time \( t \), contained in the system volume \( \Omega \). In the Ising spin system \( u(\mathbf{r}, t) \) is a local spin density. We assume that \( u(\mathbf{r}, t) \) is statistically homogeneous in the sense that its statistical properties do not depend on \( \mathbf{r} \), and that without loss of generality \( u(\mathbf{r}, t) \) is bounded as \( u_{\text{min}} < u(\mathbf{r}, t) < u_{\text{max}} \). Furthermore the statistical properties of \( \{ u(\mathbf{r}, t) \} \) are assumed to be invariant in the course of time sufficiently after the initial time. Therefore we shall discuss the statistical properties at a certain time. We hereafter write \( u(\mathbf{r}, t) \) as \( u(\mathbf{r}) \).

A conventional statistical analysis of spatial fluctuations is carried out with the two-point correlation function

\[
C_2(\mathbf{r}_1 - \mathbf{r}_2) \equiv \langle \delta u(\mathbf{r}_1) \delta u(\mathbf{r}_2) \rangle,
\]

(\( \delta u = u - \langle u \rangle \)), where \( \langle \cdots \rangle \) is the ensemble average. The homogeneity assumption

\[
\text{such that} \quad C_2(\mathbf{r}_1 - \mathbf{r}_2) = C_2(\mathbf{r}_2 - \mathbf{r}_1)
\]

is valid for the fluctuation spectrum. In the Ising spin system it is shown that such assumption holds, whereas in other systems it may not hold. The fluctuation spectrum is defined by

\[
F(\mathbf{k}) = \frac{1}{\Omega} \int_{\Omega} \delta u(\mathbf{r}) e^{i \mathbf{k} \cdot \mathbf{r}} d\mathbf{r},
\]

where \( \mathbf{k} \) is the wave vector. The fluctuation spectrum concept allows us to study the global fluctuation characteristics together with the spatial correlations.
Characterization of Statistically Homogeneous Fluctuations

Fig. 1. A schematic pattern of spatial fluctuations in a 2-dimensional system, where $u$ can take two values denoted by black point and no mark. Depending on positions of regions $\{R_r(r_0)\}$ indicated by circles, the local averages $\langle a_r(r_0) \rangle$ take different values.

Let $R_r(r_0)$ be the interior region of a $d$-dimensional sphere with the radius $r$, whose center is located at $r_0$. The local space average over $R_r(r_0)$ is given by

$$a_r(r_0) = \frac{1}{V_r} \int_{|r'| < r} u(r' + r_0) dr',$$

(2)

where

$$V_r = \frac{\pi^{d/2}}{\Gamma(1 + d/2)} r^d,$$

(3)

$\Gamma(z)$ being the gamma function, is the volume of $R_r$. In Fig. 1, regions $R_r(r_0)$ are drawn for several different centers, being denoted by circles. In each region, the local space averages $a_r(r_0)$ have different numerical values.

We introduce two kinds of averages. One is the space average,

$$\langle A \rangle_a = \frac{1}{\Omega} \int_\Omega A(r_0) dr_0 = \int_\infty^\infty a p_a(a) da$$

(4)

with

$$p_a(a) = \frac{1}{\Omega} \int_\Omega \delta(A(r_0) - a) dr_0$$

(5)

for an arbitrary variable $A(r_0)$ which is statistically homogeneous with respect to $r_0$. The symbol $\int$ means the integration over the whole system with the volume $\Omega$. The other is the ensemble average $\langle A \rangle = \int_{\Omega = 0}^\infty a p(a) da$ with the "invariant" probability density $p(a) = \langle \delta(A - a) \rangle$. Postulating that the space average for $\Omega \to \infty$ can be
replaced by the ensemble average, we get \( \langle A \rangle = \langle A \rangle_\infty \), i.e., \( p(a) = p_\infty(a) \). Hereafter we consider an infinitely large system, \( (\Omega \to \infty) \). Then we find

\[
A_\infty = \langle u \rangle,
\]

which is free from \( r_0 \) because of the homogeneity condition.

In order to study the fluctuation characteristics of \( a_r (= a_r(r_0)) \), we consider the set of moments

\[
M(q; r) = \langle \exp(q V_r a_r) \rangle = \int_{-\infty}^{\infty} \rho(a'; r) \exp(q V_r a') da', \quad (-\infty < q < \infty)
\]

where \( q \) is a parameter with the inverse dimension of the product of \( u \) and \( r^d \). The \( \rho(a'; r) \) is the probability density that \( a_r \) takes values between \( a' \) and \( a' + da' \). We assume that \( \rho(a'; r) \) is asymptotically given by

\[
\rho(a'; r) \sim \exp[-\sigma(a') V_r]
\]

for a large \( V_r \), where \( \sigma(a') \) is defined by

\[
\sigma(a') = -\lim_{r \to \infty} \frac{1}{V_r} \ln \rho(a'; r).
\]

The limit \( \lim_{r \to \infty} \) should be read as \( \lim_{r \to \infty} \lim_{Q \to \infty} \). On the other hand, the extensivity of \( V_r a_r(r_0) \) suggests the asymptotic form

\[
M(q; r) \sim \exp(q \lambda_q V_r)
\]

for a large \( V_r \), where the order-\( q \) characteristic function \( \lambda_q \) has been defined by

\[
\lambda_q = \frac{1}{q} \lim_{r \to \infty} \frac{1}{V_r} \ln M(q; r). \quad \left( \frac{d\lambda_q}{dq} \geq 0 \right)
\]

We note that \( \lambda_q \) has the same dimension as \( u(r) \).

Inserting (8) into (7) and applying the saddle-point technique, we obtain the relation between \( \lambda_q \) and \( \sigma(a) \) as

\[
\lambda_q = -\frac{1}{q} \min_a [\sigma(a) - qa].
\]

This leads to the thermodynamics formalism among \( q, \lambda_q, \alpha(=d(q\lambda_q)/dq) \) and \( \sigma(a) (=q^2 d\lambda_q/dq = q(a - \lambda_q)) \). One should note that the inequalities \( da/dq \geq 0 \) and \( d^2 \sigma(a)/da^2 \geq 0 \) hold. We call \( \sigma(a) \) the fluctuation spectrum. It is an important function for the global characterization of \( \{u(r)\} \), evaluating the fluctuation range \( (-\infty \leq a \leq \infty) \) and the realization probability of \( a_r \) (Eq. (8)).

The variance of \( a_r \) obeys the asymptotic law

\[
\langle (a_r - a_\infty)^2 \rangle \approx \frac{2D}{V_r}
\]

for a large \( V_r \), where \( D \) is a positive constant, provided that the correlation range of the two-point correlation function of \( u(r) \) is finite. Furthermore the prefactor in
Characterization of Statistically Homogeneous Fluctuations

Eq. (8) turns out to be proportional to $\sqrt{V_r}$ for a large $r$.

Equation (10) is more precisely written as

$$M(q; r) = \exp[q\Lambda_q(r)V_r],$$

where

$$\Lambda_q(r) = \lambda_q - q\phi_q(r).$$

The $\lambda_q$ is the limiting value of $\Lambda_q(r)$, and $\phi_q(r)$ indicates how $\Lambda_q(r)$ approaches $\lambda_q$ as $r \to \infty$ and satisfies

$$\lim_{r \to \infty} \phi_q(r) = 0.$$

For a moment, assume that $\lambda_q$ and $\phi_q(r)$ can be expanded in the form of the cumulant expansions as

$$\lambda_q = \lambda_0 + \sum_{n=2}^{\infty} \frac{\lambda_0}{n!} q^{n-1},$$

$$\phi_q(r) = \sum_{n=2}^{\infty} \frac{\phi_0(r)}{n!} q^{n-2}$$

with

$$f_n = \int_C C_n(x_1, x_2, ..., x_{n-1}),$$

$$g_n(r) = \frac{1}{V_r} \int_{|y| < r} \int_{|y+x| < r} \int_{|y+x|^2} \int_{|y+x|^2} C_n(x_1, x_2, ..., x_{n-1}),$$

where we have used the abbreviations $f_j \cdots \equiv \int dx_j$ and $f_j \cdots \equiv \int_{|y+x| < r} \cdots dx_j$. Here $\lambda_0 = \langle u \rangle$ and

$$C_n(x_1, x_2, ..., x_{n-1}) = \langle \prod_{j=0}^{n-1} u(x_0 + x_j) \rangle_C,$$

$(x_0 = 0)$, is the $n$-th order cumulant, being independent of $r_0$ because of the homogeneity of fluctuations.

As was seen above, the characteristic function $\lambda_q$ is relevant to the global characterization of fluctuations. The function $\phi_q(r)$, on the other hand, gives information about the fluctuation characteristics which cannot be described by $\lambda_q$. It should be noted that $\phi_q(r)$ vanishes if fluctuations at different positions are independent of each other, i.e., if

$$C_n(x_1, x_2, ..., x_{n-1}) \propto \delta(x_1) \delta(x_2) \cdots \delta(x_{n-1}).$$

On the contrary, if they are correlated, it remains finite. Therefore $\phi_q(r)$ estimates the spatial correlation effects of $\{u(r)\}$, and will be hereafter called the order-$q$ correlation function.

Let us turn to the evaluation of the asymptotic $r$ dependence of $\phi_q(r)$ for a large $r$. Assume that the correlation ranges of cumulants $\{C_n\}$ are finite. Let $b$ be the maximum value of them. Relative vectors $x_1, x_2, \text{etc.},$ satisfying $|x_j| < b,$ $(j = 1, 2, \cdots)$. 

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contribute to $\phi_q(r)$, (Eq. (19b)). One configuration of position vectors $y, x_1, x_2, \text{etc.}$, relevant to $\phi_q(r)$ is illustrated in Fig. 2. Since $b$ is finite, the $y$ region which contributes to $\phi_q(r)$ is restricted near the surface of $R_r$ for $r \gg b$, and the integration in (19b) is $O(dV_r/dr)$. This yields

$$g_n(r) \propto \frac{1}{V_r} \frac{dV_r}{dr} \sim r^{-1}$$

(22)

for all $n$. Therefore, we get

$$\phi_q(r) \sim \eta q r^{-1}$$

(23)

for a large $r(\gg b)$. The quantity $\eta_q$ defined by

$$\eta_q = \lim_{r \to \infty} r \phi_q(r)$$

(24)

remains finite, provided that the correlation ranges of cumulants are finite.

When fluctuations have no spatial correlation, we get $\eta_q = 0$. Therefore it is natural to regard $\eta_q$ as the quantity evaluating the magnitude of spatial correlations of $\{u(r)\}$. Depending on the parameter $q$, the strengths of correlations may be different. This implies that $\eta_q$ has a dispersion with respect to $q$.

The cumulant expansions (17) and (18) are valid only when $|q|$ is less than their radii of convergence $\kappa_3$ for (17) and $\kappa_\phi$ for (18). By postulating $\kappa_3 \sim \kappa_\phi (\equiv \kappa)$, the expansions (17) and (18) are valid only for $|q| < \kappa$. The finiteness of $\kappa$ is namely due to the non-perturbative non-gaussian characteristics of fluctuations $\{u(r)\}$. Especially, for $|q| \ll \kappa$, (17) is approximated as

$$\lambda_q = \lambda_0 + Dq$$

(25)

where the quantity $D$ is the same as in (13), and is given by

$$D = \frac{1}{2} \int C_2(x) dx$$

(26)

If $\{u(r)\}$ is gaussian, then $\kappa = \infty$ and Eq. (25) holds for any $q$. The relevant variables corresponding to $|q| \ll \kappa$ are

$$a = \lambda_0 + 2Dq$$

(27a)

$$\sigma(a) = \frac{(a - \lambda_0)^2}{4D}$$

(27b)

and

$$\phi_q(r) \approx \frac{1}{2V_r} \int_{|y| < r} dy \int_{|y + x| > r} dx \ C_2(x) \equiv \phi_0(r)$$

(28)
Accordingly \( \eta_2 \approx \eta_0 \). The parabolic form (27b) in the vicinity of \( \alpha = \lambda_0 \) agrees with the central limit theorem result. This law is valid for \( \alpha \) satisfying \( |\alpha - \lambda_0| \ll |\alpha^* - \lambda_0| \), where \( \alpha^* = \alpha(q = \kappa) \). We note that Eq. (28) gives a direct interrelation between the two-point correlation function and the present statistics for \( q \to 0 \). For a 1-dimensional system \( (d = 1) \), Eq. (28) is written as a more explicit relation between \( \phi_0(r) \) and \( C_3(|x|) = C_3(x) \) as

\[
\phi_0\left(\frac{r}{2}\right) = \int_0^r C_3(x) \, dx + \frac{1}{r} \int_0^r x C_3(x) \, dx ,
\]

or equivalently

\[
C_3(r) = -\frac{d^2}{dr^2} \left[ r \phi_0\left(\frac{r}{2}\right) \right].
\]

As \( |q| \) is gradually increased, we must take into account higher cumulants \( C_3, C_4, \ldots \). The expansions (17) and (18) enable us to perturbationally get deviations from the asymptotic forms (25) and (28), provided that the expansions converge, i.e., \( |q| < \kappa \). However, when \( |q| \) exceeds the convergence radius \( \kappa \), the cumulant expansions do no longer converge, and we go into a completely different world.

So for \( |q| \gg \kappa \), the cumulant expansions (17) and (18) have no meanings. Especially instead of (17), we postulate the asymptotic expansion:

\[
\lambda_q \approx \lambda_{\infty} - \frac{1}{q} \left[ \frac{1}{\Omega_e} - c_e \cdot \exp(-\gamma_e|q|) \right],
\]

for \( eq \gg \kappa \), \( (e = \pm) \), where \( \Omega_e, \gamma_e \) and \( c_e \) are positive constants. Equation (30) immediately yields

\[
\alpha - \lambda_{\infty} \approx -e c_e \gamma_e \exp(-\gamma_e|q|) ,
\]

\[
\sigma(\alpha) - \frac{1}{\Omega_e} \approx -\frac{1}{\gamma_e} |\alpha - \lambda_{\infty}| \ln \left[ \frac{a_e}{|\alpha - \lambda_{\infty}|} \right] \text{,}
\]

where \( a_e = ec_e \gamma_e(>0) \). The derivative \( d\sigma(\alpha)/d\alpha \) logarithmically diverges as \( \alpha \to \lambda_{\pm \infty} \).

The implications of \( \lambda_{\pm \infty} \) and \( \Omega_e \) are given as follows. For a large \( r \), depending on positions \( r_0 \), the local averages \( \alpha_r(r_0) \) take various values, (Fig. 1). The \( \lambda_{\pm \infty} \) is the maximum value among them. Let \( S_+ \) be a sub-region giving \( \alpha_r(r_0) = \lambda_{\pm \infty} \). In the sense that fluctuations \( \{\alpha(r)\} \) have large values in \( S_+ \), \( S_+ \) is a coherent region and is called a \( u \)-rich region. In other words, \( \lambda_{\pm \infty} \) is the average of \( u(r) \) over a \( u \)-rich region. As \( r \) is increased, such coherent region terminates. This is due to the existence of a "non-coherent" region which does not give the largest value of \( \alpha_r \). \( \Omega_e \) has the dimension of \( V_r \), and estimates the characteristic volume of one \( u \)-rich region. This is understood by the asymptotic probability density

\[
\rho(\alpha; r) \sim \exp\left( -\frac{V_r}{\Omega_e} \right),
\]

near \( \alpha = \lambda_{\pm \infty} \), where we have neglected the logarithmic term in (31b). On the contrary, \( \lambda_{- \infty} \) is the minimum value of \( \alpha_r(r_0) \), i.e., the average over a \( u \)-poor region \( S_- \), and \( \Omega_\cdot \) measures the extent of the volume of this region. Therefore, we obtain \( \rho(\alpha; r) \)
\( \sim \exp(-V_r/Q_v) \) in the vicinity of \( \alpha = \lambda \). The key point is that in contrast to that the characteristics for \( \vert q \vert \ll \kappa \) are completely determined with the two-point correlation function, those for \( \vert q \vert \gg \kappa \) cannot be explained with any kind of correlation functions for \( \{u(r)\} \) of finite order.

In this paper we have presented a new characterization of statistically homogeneous spatial fluctuations. Their global aspect can be studied in the frame of the thermodynamics formalism constructed by \( q, \lambda, \alpha \) and \( \sigma(\alpha) \). To study the global fluctuation characteristics is the new, remarkable tide in the statistical physics, and much work in this direction has been recently carried out. Others which cannot be described by the global analysis were shown to be relevant mainly to spatial correlations. These can be evaluated with the function \( \phi(q) \). It was shown that our approach is suitable for the description of both characteristics. The fluctuations \( \{u(r)\} \) generally have many different statistical aspects. The parameter \( q \) plays the role to selectively single out the statistical aspect characteristic of it, and is called the filtering parameter.\(^6\) The existence of the convergence radius \( \kappa \) clearly separates three kinds (\( q \ll -\kappa, \vert q \vert \ll \kappa \) and \( \kappa \ll q \)) of typical statistical behaviors. The region \( \vert q \vert \ll \kappa \) is specified with the two-point correlation function. On the other hand, the characteristics in the regions \( \vert q \vert \gg \kappa \) cannot be perturbationally connected to that in \( \vert q \vert \ll \kappa \), and represent highly coherent structures of \( \{u(r)\} \).

Even if the fluctuations \( \{u(r)\} \) are produced by a numerical simulation or an experimental observation, the present approach can be immediately applied, since we have imposed no restriction on the generation law of the fluctuations. We believe that by applying the present method to various systems, new and deep insights into the homogeneous spatial fluctuations will be obtained.

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