Reducing the Huge Symmetry Occurring in the Compactification of a String

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For the problem of the huge symmetry which occurs after the compactification, we propose a mechanism of reducing the symmetry. The realistic symmetry is obtained by introducing the para-quantization, whose number and order are determined by the masslessness of graviton. We also discuss the interpretation of our formulation.

In the present stage no one doubts that the superstring theory is a promising candidate for a unified theory including gravity. Of course there are many problems in the superstring theory. For example, we have no reliable method of extracting non-perturbative information. Owing to this drawback we cannot decide the unique and stable vacuum on which the physical space of the string theory should be built.

Ordinarily the theory is formulated in the critical dimension, \( d = 26 \) for the bosonic string, \( d = 10 \) for the fermionic string. Then if we attempt to contact with the real world, it is necessary to compactify the extra dimensions. Recently many interesting ideas are proposed, Calabi-Yau compactification, Narain type compactification, orbifold compactification, etc. Furthermore there is a drastic progress which permits the reinterpretation of the role played by the internal systems. This theory, say, the 4-d string theory is just being developed. So we have many possible classical solutions. We shall concentrate on this problem, the so-called compactification problem, in this paper.

Whichever way you may choose to 4-dimensions, you will encounter the huge symmetry problem. For example, the heterotic superstring may have \( O(44) \) symmetry in 4-dimensions. Generally speaking, we have a symmetry much larger than 4, which is the rank of \( SU(3) \otimes SU(2) \otimes U(1) \), the symmetry group of the standard model. The Type II superstring theory is the most economical one among the closed string theories. It is, however, not possible to extract the standard model from Type II superstring theory. The method of proving this no go theorem is so powerful that one can apply it to other phenomenological problems.

Of course, there remains a problem for the open string. However, we will not enter here into the question of how to compactify the extra dimensions in the open string theory. Now the question is as follows:

Can we evade the huge symmetry problem in the closed string theory?

Certainly we can say that the answer is yes. The method we will propose to avoid this difficulty is tricky in a sense, but it is attractive enough and very suggestive. The reason why we say tricky is using unconventional quantization, the so-called para-quantization. In this point we are inspired by the para-string theory.
which is indeed formulated in 4-dimensions. There is, however, an important difference in the treatment of the 4-d Minkowski space. Ours is the ordinary 4-d Minkowski space, although the internal space includes the para-space. On the other hand, the original para-string theory is formulated in the para-Minkowski space, the physical interpretation of which is rather vague and difficult. The reason why we say attractive is the existence of a kind of confinement mechanism. And the reason why we say suggestive is its freedom which leads to more general construction. Until now only level 1 representations of Kac-Moody algebra had been utilized for model construction, but with this mechanism we can use arbitrary level representations of Kac-Moody algebra as a symmetry. This direction is related with recently developed topics, the so-called new conformal field theory.\textsuperscript{13)}

Let us first summarize the para-statistics.\textsuperscript{10} Within the conventional formalism of quantum mechanics the relation combining frequency and energy is a direct consequence of the Heisenberg equation of motion:

\[ i\hbar \frac{dA}{dt} = [A, H], \]  

where \( A \) stands for an arbitrary observable and \( H \) is the Hamiltonian. It is nothing but the mathematical manifestation of the duality of matter and wave. Therefore, the Heisenberg equation is regarded as the most fundamental equation of quantum theory. On the other hand, we need to retain Bohr's correspondence principle. The consistency between the Heisenberg and classical equations of motion imposes certain restrictions on the commutation relations for the basic variables. Here we do not assume the canonical commutation relations. As a result the dynamical variables in para-field theories satisfy not bilinear but trilinear relations,

\[ ([a_j^+, a_k]_z, a_l] = -2\delta_{jl} a_k, \quad ([a_j, a_k]_z, a_l] = 0, \]  

in which, contrary to the Bose or Fermi case, the requirement of the (anti)commutation relations is relaxed. These relations are solved by the following ansatz:

\[ a_k = \sum_{\sigma=1}^{Q} a_k^{(\sigma)}, \]  

where \( Q \) is the order of the para-field, \( \sigma \) is the Green index,\textsuperscript{14} and \( a_k^{(\sigma)} \) are Green components. Here,

\[ [a_k^{(\sigma)}, a_l^{(\rho)}] = 2\delta^{\sigma\rho}, \quad [a_k^{(\sigma)}, a_l^{(\rho)}] = 0; \quad \sigma \neq \rho. \]  

The Hilbert space of the para-field is defined by a Fock-type irreducible representation with a unique vacuum,

\[ a_k|0\rangle = 0, \quad a_k a_l|0\rangle = Q\delta_{kl}|0\rangle. \]  

This representation can be most conveniently characterized by specifying the order of para-fields through the Green's ansatz. A number of theorems concerning the correspondence of a single para-field of order \( Q \) with an ordinary field with \( Q \) degrees of freedom had been established.\textsuperscript{10} Especially we need the information about the locality here. A theorem given by Ohnuki and Kamefuchi\textsuperscript{15} is based on the require-
ment of strong locality

\[ [F(V), \phi(X)] = 0, \]

in which \( \phi(X) \) and \( \phi^*(X) \) are the para-fields with order \( Q \) and \( F(V) \) is a Hermitian functional of the para-fields representing an observable. Here \( V \) represents a certain spatial domain and \( X \) separates from \( V \) space-like. They then show that \( F(V) \) depends on the parafields only through the bilinears

\[ [\phi(X), \phi(Y)], [\phi(X), \phi^*(Y)], [\phi^*(X), \phi^*(Y)], X, Y \in V. \]

These expressions are \( SO(Q) \) invariants.

Now let us consider a free two-dimensional Majorana para-fermi theory of order \( Q \) with a chiral \( SO(N) \) symmetry. From here we use an alternative to the Green ansatz suggested by Greenberg and Macrae, \( \text{(16)} \)

\[
\phi(x) = \sum_{a=1}^{Q} e^a \phi^a(x); a = 1, \ldots, Q,
\]

where \( \phi^a(x) \) are ordinary fermions, and \( e^a \) are elements of the real Clifford algebra:

\[
\{e^a, e^b\} = 2 \delta^{ab}.
\]

Moreover, \( [e^a, \phi^b] = 0 \). The Majorana parafermions satisfy the trilinear relations,

\[
[[\phi(x), \phi(y)] - \langle [\phi(x), \phi(y)] \rangle_0, \phi(z)] = -4 \delta^a(x-z) \phi(y),
\]

where the symbol \( \langle \rangle_0 \) stands for the vacuum expectation value. The action can be written as

\[
S = \frac{i}{4} \int dz dz^\prime [\phi^i(z), \phi^j(z)] + [\phi^i(z), \partial \phi^j(z)],
\]

\[
= \frac{i}{2} \int dz dz^\prime [\phi^i(z) \overline{\phi}^j(z) + \phi^i(z) \overline{\partial \phi}^j(z)].
\]

The \( SO(N) \) currents, \( J^{ij} \), of the theory are given by

\[
J^{ij}(z) = \frac{1}{2} : [\phi^i(z), \phi^j(z)] :, \quad (7)
\]

where the normal ordering is defined with respect to the para-fields \( \phi^i \). Using Green components, we get

\[
J^{ij}(z) = : \phi^i(z) \phi^j(z) : . \quad (8)
\]

This expression has the \( SO(N) \otimes SO(Q) \) Kac-Moody symmetry. This symmetry, which manifests itself in the Fock space spanned by Green components, is fictitious. Because we now consider the para-field theory, the Fock space is the para Fock space Eq. (5) and the true symmetry is \( SO(N) \) Kac-Moody algebra with level \( Q \). Reversing this process is our idea. Starting from a large Hilbert space we will go to a small Hilbert space. We can interpret this process as a kind of confinement or projection. A similar situation occurs in a fermionic construction (we follow the notation of Antoniadis et al.\( ^{17} \)) in which the only requirement is the modular invariance of the
partition function. For example, reducing the symmetry from $SO(5)$ to $SO(3) \otimes SO(2)$ can be achieved by a projection. The partition function of the $SO(3) \otimes SO(2)$ system is given as follows:

\[
Z = \left[ Tr_{N_S} \frac{1+(-)^{F_1}}{2} q^{L_1} \bar{q}^{\bar{L}_1} + Tr_{N_F} \frac{1+(-)^{F_1}}{2} q^{L_1} \bar{q}^{\bar{L}_1} \right] \times \left[ Tr_{N_S} \frac{1+(-)^{F_2}}{2} q^{L_2} \bar{q}^{\bar{L}_2} + Tr_{N_F} \frac{1+(-)^{F_2}}{2} q^{L_2} \bar{q}^{\bar{L}_2} \right] \\
= Tr_{F_1} \frac{1+(-)^{F_1}}{2} q^{L_1} \bar{q}^{\bar{L}_1} + Tr_{F_2} \frac{1+(-)^{F_2}}{2} q^{L_2} \bar{q}^{\bar{L}_2} \\
+ Tr_{F_3} \frac{1+(-)^{F_1}}{2} q^{L_1} \bar{q}^{\bar{L}_1} + Tr_{F_4} \frac{1+(-)^{F_2}}{2} q^{L_2} \bar{q}^{\bar{L}_2} \\
= \sum_{a \in \mathbb{Z}} \sum_{\beta \in \mathbb{Z}} \frac{1+(-)^{a}}{2} q^{L_a} \bar{q}^{\bar{L}_a}, \\
\tag{9}
\]

where $F_1, F_2$ are the fermion sets which make the symmetry $SO(3)$ and $SO(2)$ respectively and $E = \{ \phi, F_1, F_2, F = F_1 F_2 \}$. The original $SO(5)$ symmetric theory corresponds to choosing $\phi$ and $F$ only. This formulation can be easily generalized to asymmetric cases. The above projection satisfying the modular invariance is essentially equivalent to the following constraints:

\[
J^{Aa} := \phi^A \phi^a : \approx 0, \\
\tag{10}
\]

where $A, a$ is the index of $F_1, F_2$ respectively. One can go one more step further, i.e., the following constraint, which gives rise to a modular invariant partition function, can also be imposed

\[
J^{a\beta} := \phi^a \phi^\beta : \approx 0. \\
\tag{11}
\]

As a result we obtain the symmetry $SO(3)$ from $SO(3) \otimes SO(2)$. This is the case of the para-fermion fields. In this case a guiding principle is the modular invariance and the para-statistics. This strategy is supported by para-quantization and stringy arguments. For the case of string theory the fundamental principle is the conformal invariance of the world sheet physics. Therefore, any conformal fields, even para-fields, can be allowed as the internal space. Our formulation will be a modest first step to the program treating the general conformal fields as the internal space.

Now we shall concentrate on the closed bosonic string theory for simplicity, although extension to the superstring theory is trivial. The central charge of the closed bosonic string must be 26 to cancel the ghost central charge. If we keep the 4-d Minkowski space and consider the fact that the order $Q$ para-field has the central charge $Q$ we guess the relation $\Sigma_i d_i Q_i = 22$, where the subscript $i$ means species of the para-fields and $d_i$ is its number.

Then we derive this relation from the condition that the graviton should be massless. Let $X^\mu(r, \sigma)$ be the 4-d Minkowski coordinates and $y^a(r, \sigma)$ be the internal para-bose fields with order $Q$, where $a = 1, \cdots, d$. In the light cone gauge, letting $A$ be a transverse index we get
\[ X^A(\tau, \sigma) = x^A + 2\alpha^A p^A \tau \]
\[ + \sqrt{2\alpha} \sum_{\sigma=1}^2 \sum_{n=1}^{\infty} \frac{1}{2\sqrt{n}} [a_n^A \exp(-in(\tau-\sigma)) + \bar{a}_{n}^{A*} \exp(-in(\tau+\sigma))] , \]
\[ y^a(\tau, \sigma) = \sum_{a=1}^{d} e^a y^{aa}(\tau, \sigma) , \]
\[ y^{aa}(\tau, \sigma) = x^{aa} + 2\alpha^a p^{aa} \tau \]
\[ + \sqrt{2\alpha} \sum_{\sigma=1}^2 \sum_{n=1}^{\infty} \frac{1}{2\sqrt{n}} [a_n^{aa} \exp(-in(\tau-\sigma)) + \bar{a}_{n}^{aa*} \exp(-in(\tau+\sigma))] . \] (12)

So, the (mass)\(^2\) operator can be written as
\[
M^2 = \frac{1}{\alpha} \sum_{a=1}^{d} \sum_{n=1}^{\infty} n[a_n^{aa*} a_n^{aa} + \bar{a}_{n}^{aa*} \bar{a}_{n}^{aa}] 
+ \frac{1}{\alpha} \sum_{A=1}^{2} \sum_{n=1}^{\infty} n[a_n^{A*} a_n^{A} + \bar{a}_{n}^{A*} \bar{a}_{n}^{A}] + M_0^2 , \] (13)

where
\[ \alpha' M_0^2 = Qd \sum_{\sigma=1}^2 n + 2 \sum_{\sigma}^2 n = (Qd + 2) \sum_{\sigma}^2 n^{-1} = -(Qd + 2) \zeta(1) = -\frac{Qd + 2}{12} . \] (14)

Thus, the (mass)\(^2\) of the ground state is fixed at
\[ \alpha' M^2|0\rangle = -\frac{Qd + 2}{12} |0\rangle . \] (15)

The graviton state is
\[ |\text{graviton}\rangle = \frac{1}{2} (a_1^{A*} a_1^{A} + \bar{a}_{1}^{A*} a_1^{A})|0\rangle , \] (16)

and the mass eigenvalue of this state is
\[ \alpha' M^2|\text{graviton}\rangle = \left[ -\frac{Qd + 2}{12} + 2 \right]|\text{graviton}\rangle . \] (17)

In order to get a massless graviton, we must choose \( Qd = 22 \). In general we obviously get \( \sum_i d_i Q_i = 22 \).

From this consideration we can guess the case of fermions, \( \sum_i d_i Q_i = 44 \) for the Majorana fermions, \( \sum_i d_i Q_i = 22 \) for the Dirac fermions. In the original Hilbert space we have 44 Majorana fermions as usual. Therefore, it seems that the \( SO(44) \) works in the original theory, however, we can get smaller symmetry in a projected Hilbert space. For example, we will get \( SU(3) \times SU(2) \times U(1) \) from 3 Dirac para-fermions with order 5, 2 Dirac para-fermions with order 3 and 1 Dirac fermion. It is nothing but the easy equation, \( 3Q_1 + 2Q_2 + Q_3 = 22 \). As we keep the 4-d Minkowski space, the kinematical structures of the theory do not change from the ordinary one. The only difference is the levels of the representation of the Kac-Moody algebra. The phenomenological significance of this fact is yet to be studied.

To conclude this paper, we must mention the implication of our idea. Let us see
the explicit form of the action,

\[ S = -\frac{1}{4\pi\alpha'} \int d^2z \left[ \partial X^\alpha \overline{\partial} X_\alpha + \frac{1}{2} [\psi^i, \overline{\psi}^j] \right] \]

\[ = -\frac{1}{4\pi\alpha'} \int d^2z \left[ \partial X^\alpha \overline{\partial} X_\alpha + \psi^{ia} \overline{\psi}^{ia} \right]. \quad (18) \]

This action manifestly has the conformal invariance. Using the bosonization technique and some arguments, we can regain the original Nambu-Goto action from the second line in Eq. (18). Nambu-Goto action has the elegant interpretation that the string world sheet must be a minimal surface.\(^{(17)}\) In our case this interpretation, however, may be misleading. As a starting point we take a conformal principle which assumes the existence of the 4-d Minkowski space and the 2-d conformal invariance. As a result the geometrical picture in the first line of Eq. (18) is abandoned. Nevertheless, there exists a suggestion\(^{(18)}\) that the free para-fermion theory with order \(k\) is equivalent to the WZW model with level \(k\). Therefore, we have to investigate this equivalence and understand it deeply. If this turns out to be correct, the simple cases of our model would be equivalent to the WZW model with level \(k\). Of course, combining our method and the spin-structure construction we will have more complicated models. So we can say that we arrive at the arbitrary level fermionic construction. It is interesting to study the constraints of modular invariance of this formulation.

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