The $S$ Matrix Method in Pion Reactions

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A new method which is convenient for the phenomenological analysis to explore the mutual correlations among various pion reactions is presented based on the $S$ matrix formalism.

We first study the general properties of the $S$ matrix as it is necessary for practical purposes.

Next, utilizing the results we discuss the intimate correlations between the photo-meson production and pion-nucleon scattering.

The results seem to be promising in that it can correctly give the ratio:

$$\sigma(\pi^+\rho) / \sigma(\pi^-\rho) = 3,$$

(at 130 Mev)

which can never be deduced from the lowest order perturbation theory.

Finally, we apply this method to the qualitative investigation of another process, i.e. the elastic nucleon-nucleon scattering at extremely high energies, from which we can conclude that the scattered nucleons will be concentrated in the forward and backward directions in the centre of mass system.

§ 1. Introduction and summary

From the recent studies on pions, it has already been known that the lowest order perturbation theory cannot even qualitatively predict the correct behaviour of the artificially produced pions. Hence it is a serious problem to consider which kind of method we should rely upon.

In quantum electrodynamics, we have achieved such a remarkable success that allows us consistently to describe the electromagnetic phenomena in terms of the Hamiltonian representing the interaction between electron-positron and electromagnetic fields.

On the other hand, a number of pionic phenomena can hardly be explained in terms of the field theory of mesons.

Although it is suspected that higher order corrections might overcome such a dishonourable situation, yet at present one cannot find a unique Hamiltonian from which one can calculate the matrix elements of all pionic processes with confidence.

Under such circumstances that the present Hamiltonian formalism is not so powerful once applied to pionic phenomena, it is quite probable that the correct theory of pions might be beyond the frame of the Hamiltonian formalism. Thus we shall bear on the $S$ matrix formalism\(^1\) rather than the questionable Hamiltonian formalism, since the former has wider applicability and is more convenient in describing pionic phenomena than the latter. But since we have as yet no concrete means to construct $S$ matrix, we may assume that all conditions satisfied by the $S$ matrix in quantum field theory independently of the special form of the Hamiltonian will hold also in the future theory of $S$ matrix.
One of such conditions is the unitarity of $S$:

$$S^+S = SS^+ = 1,$$

where $S^+$ is the Hermitian conjugate of $S$.

In section 2, we shall show the special importance of this condition in studying the mutual correlations among various pion reactions. Other conditions such as the invariances of $S$ under Lorentz and gauge transformations are also important. The essence of the content of section 3 consists in these invariance conditions.

In the unitarity condition of $S$, both $S$ and $S^+$ appear and the relation between $S$ and $S^+$ comes into our question. Hence, in section 4, the procedure to derive $S^+$ from $S$ is investigated in detail.

With these preliminaries, we directly apply the $S$-matrix formalism to the investigation of the mutual correlations among various pion reactions. As an example, a simultaneous integral equation connecting the most important two pionic processes, i.e. the photo-meson production and pion-nucleon scattering, is derived in section 5. From this equation we can calculate the matrix element for the latter provided that those for the former processes are already known. For this purpose, the following properties of pions are necessary to know.

1. **The transformation properties of pions**
   It has already been established from the recent experiments on artificial pions that pions are pseudoscalar.

2. **The charge dependence of the pion-nucleon interaction**
   We assume the most promising and simplest symmetrical theory for the time being. It has long been known that the charge independence of nuclear forces which seems plausible at low energies can be achieved both on the basis of the symmetrical theory and the pure neutral theory and their combinations. However, the symmetrical theory seems superior to other theories such as the pure neutral theory in various processes, though there is yet no definite proof in favour of or against the symmetrical theory.

   Perhaps the most promising clue will be gained by the investigation of the process $\gamma + d \rightarrow d + \pi^0$, since the result predicted from the symmetrical theory shows a marked contrast, to the pure neutral theory, and the recent experimental data seem to show the evidences in favour of the symmetrical theory.

3. **The possibility of the partial wave expansion**
   It must be emphasized that the forces between a pion and a nucleon are of the short range type. This fact guarantees the possibility of treating low energy pionic phenomena by means of the partial wave expansion or synonymously the expansion in energies.

   Accordingly we can make use of "the shape independent approximation in meson theory" similar to the phenomenological theory of nuclear forces. As for this point, we shall discuss in section 6.

Regrettably the $S$ matrix method is still impossible to determine the matrix element...
of a single process, although it is quite powerful in exploring the mutual correlations among pionic phenomena. Because of this inevitable reason, we must more or less depend on the Hamiltonian formalism, and we shall employ the matrix elements of the photo-meson production derived from a rather phenomenological Hamiltonian, about which we shall discuss in section 7.

In section 8, adopting the matrix elements for the photo-meson production mentioned above to the integral equations, we get those for the pion-nucleon scattering. The result shows a good agreement with the experimental data, especially we obtained the ratio:

\[ \sigma(\pi^+ p)/\sigma(\pi^- p) = 3, \text{ (at 130 Mev)} \]

which can never be deduced from the lowest order perturbation theory. Thus we may say that the phenomenological approach in the field theory is fairly promising at the present stage.

Finally in section 9, we apply this method to nucleon-nucleon collisions at extremely high energies. If the total cross-section of the nucleon-nucleon collisions at extremely high energies is nearly equal to the geometrical cross-section, the angular distribution of the elastic nucleon-nucleon scattering can be deduced, namely the scattered nucleons will be concentrated in the forward and backward directions in the centre of mass system.

§ 2. Unitarity of S matrix

One of the most important properties of S matrix is its unitarity which guarantees the conservation of probability.

Defining the transition matrix \( R \) with

\[ S = 1 + R, \]

the unitarity of S matrix is expressed by

\[ R + R^+ + R^+ R = R + R^+ + RR^+ = 0. \]

Now we shall consider the matrix elements of (2) corresponding to the process:

\[ \gamma + N \rightarrow N + \pi \quad (N: \text{nucleon}). \]

Omitting the symbol \( N \) for simplicity, we have

\[ (\pi | R + R^+ | \gamma) + (\pi | R | \pi) (\pi | R^+ | \gamma) + (\pi | R | \gamma) (\gamma | R^+ | \gamma) \]

\[ + \ldots \ldots \text{(higher order terms in } e^2/\hbar c) = 0. \]

We cannot apply the perturbation method to the meson-nucleon interaction, but can utilize the power series expansion in terms of the fine structure constant \( e^2/4\pi\hbar c = 1/137 \).

Physically speaking, the \( \gamma \)-ray can be regarded as a test body to investigate the nature of the pion-nucleon interaction, since the photon has so weak interactions with other fields that the perturbation theory holds and its disturbances to other fields can safely be discarded.

Thus picking up only the lowest order terms in the fine structure constant we have from (4)
\[
(\pi|R + R^+|\gamma) + (\pi|R|\pi)(\pi|R^+|\gamma) = 0. \tag{2.5}
\]
This integral equation enables us to derive the matrix elements for pion-nucleon scattering, provided that those for the photo-meson production are already known.

In completely the same way as (4) or (5), we can derive various relations among various pion reactions, but we shall give only one more example. The unitarity relation, applied to nucleon-nucleon collisions at energies a little above the threshold of meson production gives

\[
(2N|R + R^+|2N) + (2N|R|2N)(2N|R^+|2N) + (2N|R|2N, \pi)(\pi,2N|R^+|2N) = 0. \tag{2.6}
\]

From this relation, it is easily understood that the total cross-section of pion production can be derived if the matrix elements of (elastic) nucleon-nucleon scattering are known (Cf. § 8,9).

In the present phenomenological theory of nuclear forces, however, we are obliged to make use of real potentials even at high energies, so that the unitarity relation is satisfied by the matrix elements of the elastic scattering only.

\[
(2N|R + R^+|2N) + (2N|R|2N)(2N|R^+|2N) = 0. \tag{phenomenological 2.7}
\]

Namely, the contribution from the real meson production is inevitably neglected.

At 340 Mev. we know experimental cross-sections:

<table>
<thead>
<tr>
<th>reaction</th>
<th>total cross-section</th>
</tr>
</thead>
<tbody>
<tr>
<td>(p+p \to p+p)</td>
<td>(\sim 5 \times 10^{-26} \text{ cm}^2)</td>
</tr>
<tr>
<td>(p+p \to n+p+\pi^+)</td>
<td>(\sim 2 \times 10^{-28} \text{ cm}^2)</td>
</tr>
</tbody>
</table>

It is true that the cross-section of the inelastic collision, i.e. the real meson production is very small compared to the elastic one, but the effect of virtual meson production would convert the nature of the elastic scattering as damping. (See § 9) One can see from the above examples that the \(S\) matrix method would be an important clue to explore the correlations among pion reactions.

3. General forms of the transition matrices

In this section, we shall derive such possible forms of matrix elements of \(R\) on the basis of the Lorentz and gauge invariances that are convenient for the phenomenological analyses of pions.

The Dirac equation is given in the momentum representation as

\[
(i\gamma \cdot p + M)\psi(p) = \bar{\psi}(p)(i\gamma \cdot p + M)\psi(p) = 0, \tag{3.1}
\]

where current notations are employed.

The following algebraic relations for the \(\gamma\) matrices are often used throughout this work.
For any four vectors $a$ and $b$,

$$(\gamma a)(\gamma b) = 2(ab) - (\gamma b)(\gamma a),$$

and

$$\gamma\mu (\gamma a) = 2\alpha\mu - (\gamma a)\gamma\mu,$$

where $\mu = 1, 2, 3, 4$.

Making use of (2), the matrix elements of any pion reaction can be reduced to a linear combination of some convenient fundamental forms with simple $\gamma$-dependences.\(^7\)

We shall illustrate the fundamental forms by two examples, the photo-meson production and pion-nucleon scattering.

(a) photo-meson production

From the requirements of Lorentz and gauge invariances, we have, as the only possible form of the matrix elements of the photo-meson production,

$$(\pi | R | \gamma) \sim \tilde{\psi}(F) P_{\mu\nu}(k, I, F, \gamma) \psi(I) A_\mu(k) R, \phi(q),$$

where $P_{\mu\nu}$ is an antisymmetric tensor and can be reduced to the linear combination of the following four fundamental forms:

$$\gamma_5 (I \mu F_\nu - I \nu F_\mu), \quad \gamma_5 (\gamma_\mu I_\nu - \gamma_\nu I_\mu),$$

$$\gamma_5 (I \mu F_\nu - I \nu F_\mu), \quad \gamma_5 (\gamma_\nu I_\mu - \gamma_\nu I_\mu).$$

We can obtain $P_{\mu\nu}$ by summing up these expressions multiplied by certain scalar c-numbers. It must be noticed that terms appearing in (4) are deduced only for the pseudoscalar theory.

(b) pion-nucleon scattering

Assuming the symmetrical theory for pions, the charge dependence of the matrix elements of pion-nucleon scattering is written as

$$\tilde{\phi}(F) \left( Q \tau_{LM} + T \delta_{LM} \right) \psi(I) \phi_L(q_f) \phi_M(q_i),$$

where $\tau_{LM} = [\tau_L, \tau_M]/2i$, and the charge dummy indices $L$ and $M$ run from 1 to 3. Next the $\gamma$-dependences of $Q$ and $T$ are reduced to

$$Q = Q'(I, F; q_f) + Q''(I, F; q_f) (q_f),$$

$$(q = q_i \text{ or } q_f)^*$$

$$T = T'(I, F; q_f) + T''(I, F; q_f) (q_f).$$

just as in the case of photo-meson production.

Inserting the above reduced forms into the equation (2.5), we obtain a simultaneous integral equation for $Q$ and $T$.

* Notice the relation: $\phi(F) (q_f \gamma) \psi(I) = \tilde{\phi}(F) (q_f \gamma) \psi(I)$. 
§ 4. Derivation of the conjugate matrix $R^+$ from $R$

Since the expression of the unitarity condition of $S$ involves both $R$ and $R^+$, it is necessary to seek for the procedure to derive $R^+$ from $R$. This problem itself is interesting in connection with the principle of detailed balancing.

As readily be seen from the expression

$$ (a | R^+ | b) = (b | R | a)^* $$  \hspace{1cm} (*: complex conjugate)  \hspace{1cm} (4.1) 

we have to derive matrix element of the inverse process $a \rightarrow b$ from that of $b \rightarrow a$. Hence we shall investigate the procedure to derive $(b | R | a)$ from $(a | R | b)$. Later on, we shall see that the pseudoscalar character of the pion plays an important role in this problem.

First we shall study how to derive the matrix element of the inverse process $\bar{A}$ from the matrix element of a process $A$ based on the Feynman theory.

Let the diagrams corresponding to the process $A$ be $G_1, G_2, \ldots$, then the sum of contributions from their inverse diagrams $G_1^*, G_2^*, \ldots$ constitute the matrix element of the inverse process $\bar{A}$. Namely, the diagrams for the processes $A$ and $\bar{A}$ have one-to-one correspondence.

Thus our problem is reduced to a more simplified one how to get a general procedure (common to all diagrams) in order to derive $c(G)$ from $c(G)$, where $c(G)$ means the contribution from a diagram $G$ to $S$ matrix. The diagram $G^*$ is obtained from $G$ by reversing the sense of future and past (See fig. 3 and 4).

The contribution of the diagram $G$ to $S$ matrix is given by

$$ c(G) = A \int (dx_1) \cdots \int (dx_n) e^{-\bar{\psi} \gamma^\mu \psi (p_1) Q_1 S_F (x_1-x_2) Q_2 \cdots Q_n \psi (p_1) e^{\bar{\psi} \gamma^\mu \psi}} $$

where the first line in the integrand corresponds to an open Fermion polygon, and it also stands for other open Fermion polygons if they exist. On the other hand, we need not know the explicit representation of the Boson lines as they are undirected and common to both $c(G)$ and $c(G^*)$. $A$ is a numerical factor.

In the meson part, Hermitian $\phi_1, \phi_2$ and $\tau_1, \tau_2$ should be used instead of $\phi, \phi^*$ and $\tau, \tau^*$ and summation with regard to the charge dummy indices should be performed. As for the closed Fermion lines, they can be treated as if undirected, for only the sum of contributions from both directions comes into our question.

The contribution of the diagram $G^*$ is given by

$$ c(G^*) = A \int (dx_1) \cdots \int (dx_n) e^{-\bar{\psi} \gamma^\mu \psi (p_1) Q_1 S_F (x_2-x_1) Q_2 \cdots Q_n \psi (p_1) e^{\bar{\psi} \gamma^\mu \psi}} $$

(4.2)
where the dotted line represents the undirected part common to both $c(G)$ and $c(\tilde{G})$. We shall decompose the coupling matrix $Q$ into the direct product of a Dirac matrix $O$ and an isotopic matrix $T$:

$$Q = O \cdot T.$$  \hspace{1cm} (4.4)

$T$ is an Hermitian isotopic matrix such as $1$, $\tau_1$, $\tau_2$, $\tau_3$, $\tau_\rho$, as mentioned before.

The problem is how to get (3) from (2).

First we shall apply the following two transformations:

$$(1^*) \quad \phi(p_i) \rightarrow (\bar{\phi}(p_i) C)^T, \quad \bar{\phi}(p_f) \rightarrow (C^{-1} \bar{\phi}(p_f))^T. \quad \text{(charge conjugation)} \hspace{1cm} (4.5)$$

where $(+)$, $(-)$ signs correspond to the positive and negative frequency parts respectively and have been omitted so far in $c(G)$ and $c(\tilde{G})$.

$$(II) \quad p_i \rightarrow -p_i, \quad p_f \rightarrow -p_f, \quad q_i \rightarrow -q_i, \quad q_f \rightarrow -q_f. \hspace{1cm} (4.6)$$

Now let $c(G)$ be changed to $c'(G)$ by these transformations, then making use of the formula

$$c'[G,S_p(x_1-x_2)Q_2\cdots Q_n]^T C^{-1} = \tilde{O}_n \cdots \tilde{O}_2 S_p(x_2-x_1) \tilde{O}_1 T_n^r \cdots T_2^r T_1^r, \hspace{1cm} (4.7)$$

we see

$$c'(G) = A \{ (dx_1) \cdots (dx_n) e^{-ipq_{x_{1}}^{\mu}} \bar{\phi}(p_i) \tilde{O}_n T_n^r \cdots \tilde{O}_2 T_2^r S_p(x_2-x_1) \tilde{O}_1 T_1^r \phi(p_f) e^{ipq_{x_{1}}} \times e^{-iq_{x_{1}}^\mu} \phi(q_i) e^{-iq_{x_{1}}^\mu} \bar{\phi}(q_f) e^{ipq_{x_{1}}},$$

$$\hspace{1cm} (4.8)$$

The Boson part in $c'(G)$ agrees completely with that in $c(\tilde{G})$.

The only difference is that $Q_n = O_n T_n^r$ (in $c(G)$) is replaced by $\tilde{O}_n T_n^r$ (in $c'(G)$), where $\tilde{O} = CO^r C^{-1}$ and is given in the following table.$^8$

We shall compare $c'(G)$ with $c(\tilde{G})$ for some special but important cases.

(a) quantum electrodynamics

In this case, we can discard the isotopic matrix $T$, and the only difference is

Table of $\tilde{O}$

<table>
<thead>
<tr>
<th></th>
<th>$S$</th>
<th>$P_3$</th>
<th>$P_\nu$</th>
<th>$V$</th>
<th>$T$</th>
<th>$P_T(T)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$O$</td>
<td>1</td>
<td>$\tau_5$</td>
<td>$\tau_5 \gamma^\mu$</td>
<td>$\tau_5$</td>
<td>$\sigma_{\mu \nu}$</td>
<td>$\gamma_5 \sigma_{\mu \nu}$</td>
</tr>
<tr>
<td>$\tilde{O}$</td>
<td>1</td>
<td>$\tau_5$</td>
<td>$\tau_5 \gamma^\mu$</td>
<td>$-\tau_5$</td>
<td>$-\sigma_{\mu \nu}$</td>
<td>$-\gamma_5 \sigma_{\mu \nu}$</td>
</tr>
</tbody>
</table>

| Parity | even | odd |

$$O \rightarrow \tilde{O}. \hspace{1cm} (4.9)$$

$^*$ $\phi$ and $\bar{\phi}$ are treated as c-numbers, i.e. commutative but not anticommutative with each other. $C$ is Schwinger's charge conjugation matrix.
In quantum electrodynamics, $O$ is $\gamma_\mu$ (odd), so we have

$$c'(G) = (-1)^N c(\tilde{G}) \quad (N: \text{the order of the process})$$

The factor $(-1)^N$ is related only to the parity of the process, i.e. even or odd, and this property is common to all diagrams. Namely

$$c'(G) = c(\tilde{G}) \quad \text{(even order process)} \quad (4.10)$$

$$c'(G) = -c(\tilde{G}) \quad \text{(odd order process)}$$

These relations completely settle the solution of our problem in the case of quantum electrodynamics.

(b) symmetrical meson theory

In this case, the situation is not so simple as in the former case because of the appearance of isotopic matrices. Yet there is some interesting feature. Namely, quite analogous to the charge conjugation matrix $C$ for the Dirac matrices:

$$c^{-1} r^L c = - r^L \quad (L=1,2,3)$$

we can define an isotopic matrix $c$ by

$$c^{-1} \tau_L c = - \tau_L \quad (L=1,2,3) \quad (4.12)$$

For the conventional representation of the isotopic matrices, i.e.,

$$\tau_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \tau_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \tau_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

we may choose $c$ as

$$c = \tau_2 \quad (1.13)$$

Thus applying the charge conjugation $c$ in the isotopic space just like the transformation (1) in the ordinary space, we have

$$\tilde{O} T c \rightarrow - \tilde{O} T \quad (T = \tau_1, \tau_2, \tau_3) \quad (1.14)$$

Here we have only to apply $c(\ ) c^{-1}$ to the isotopic parts lying between $\tilde{\eta}(\rho_1)$ and $\eta(\rho_3)$.

Let $c'(G)$ be thus changed to $c''(G)$ by the transformation (III), then we have

$$c''(G) = (-1)^N c(\tilde{G}) \quad \text{if } O \text{ is even}, \quad (4.15)$$

$$c''(G) = c(\tilde{G}) \quad \text{if } O \text{ is odd}.$$

where $N$ (mod. 2), the order of the process, is common to all diagrams as before. (We do not consider the non-linear interactions between nucleon and meson fields.)

In the case of the pseudoscalar meson theory, we can obtain $c(G)$ by a unified procedure since both pseudoscalar and pseudovector couplings are even, while in the scalar meson theory, we cannot treat both scalar and vector couplings at the same time since they are of different parities.

In view of this difference, we may say that the pseudoscalar theory is a specially convenient case.
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(c) photo-meson production

What is necessary for our later purpose is the matrix element of the inverse process of the photo-meson production.

We shall give the procedure for only linear terms in the coupling constant $\epsilon$, but the result can also be applied to higher order terms.

The discussion is slightly complicated in this case, because now we have three coexisting fields, nucleonic, mesonic and electromagnetic.

The essential relations in this case are

$$
\tau_1^T = \tau_1, \quad \tau_2^T = -\tau_2, \quad \tau_3^T = \tau_3. \quad (4 \cdot 16)
$$

Adopting these relations properly, we are able to get $\epsilon(G)$. For simplicity, we shall assume the symmetrical theory. (This assumption is not necessarily required and the procedure to be shown is also valid for the pure neutral theory.)

The interaction Lagrangian of pseudoscalar meson, nucleon, and radiation fields is given by

$$
L_{\text{int}} = -if\bar{\psi}_f \gamma_\mu \psi_f \gamma_\mu + eA_\mu \left( \frac{\partial \phi_1}{\partial x_\mu} - \frac{\partial \phi_2}{\partial x_\mu} \right) - \frac{1}{2} \epsilon^2 A_\mu \left( \phi_1^2 + \phi_2^2 \right) \{\rho\} \text{ only} \\
- i \frac{x^2}{x} \bar{\psi}_f \tau_\mu \psi_f \frac{\partial \phi_1}{\partial x_\mu} + \frac{x}{x}\bar{\psi}_f \tau_\mu \left[ \gamma_\mu \tau_\lambda \gamma_\mu \right] \psi_f A_\mu \phi_2 \{\rho\} \text{ added.} \quad (4.17)
$$

(Natural units $\hbar = c = 1$ are employed.)

To this we may add the following Pauli term if necessary:

$$
\bar{\psi}_\mu \gamma_\nu (\sigma_\mu \tau_\nu + \sigma_\nu \tau_\mu) \psi \cdot F_{\mu\nu}.
$$

First we shall study the case of pseudoscalar coupling only, and then prove that the procedure obtained is also valid for the case of pseudovector coupling. In the former case, the photon is absorbed only by the meson or nucleon current, while in the latter case it can also be absorbed by another mechanism through a contact interaction among nucleon, meson, and radiation fields.

The difference of transformation properties in charge space between meson and nucleon fields brings about a positive to negative ratio not equal to unity in the photo-meson production and $\mu_p/\mu_n = -1$ in the anomalous magnetic moments of the proton and neutron.

(i) absorption of a photon by the nucleon current

In $\epsilon(G)$, there appears $O \cdot T^\nu$ corresponding to $O \cdot T$ in $\epsilon(G)$.

When a photon is absorbed by the nucleon current, there is only one vector coupling, i.e. odd $\gamma_\mu$ as $O$, and in other parts there are only pseudoscalar coupling, i.e. even $\gamma_\nu$.

Thus on the whole, the factor $(-1)^1$ arises from the Dirac part since there is only one odd coupling.
On the other hand, we see that \( \tau' \)'s in \( T \)'s changes sign in \( c'(G) \) in a manner indicated in (16). Hence, we see

\[
c'(G_{\text{nucleon}}) = (-1)^{N_2} c(G),
\]

(4.18)

where \( N_2 \) is the number of \( \tau' \)'s in \( c(G) \).

If we call the pions corresponding to the wave functions \( \phi_1, \phi_2 \) and \( \phi_3 \) as \( \phi_1 \), \( \phi_2 \), and \( \phi_3 \) pions respectively, virtual \( \phi_2 \) pions are always accompanied with \( \tau' \)'s and contracted in pairs, so that \( N_2 \) may be interpreted as the number of \( \phi_2 \) pions produced.

The discussion developed above about the interaction \( \bar{\psi} \gamma_\mu \psi \cdot A_\mu \) can also be applied to the Pauli term \( \bar{\psi} \sigma_{\mu\nu} (\alpha_\mu \tau_\nu + \alpha_\nu \tau_\mu) \psi \cdot F_{\mu\nu} \) for \( \sigma_{\mu\nu} \) as well as \( \gamma_\mu \) is odd.

(ii) absorption of a photon by the meson current

The expression of the meson current is given by

\[
\frac{\partial \phi_i}{\partial x_\mu} - \frac{\partial \phi_j}{\partial x_\mu} \phi_\mu.
\]

In this case, all \( \phi_i \)'s are even since no odd matrix exists, so that the factor \((-1)\) does not arise from this Dirac part, but from the isotopic part. For when a photon is absorbed by the meson current, there is always a difference of 1 between the numbers of \( \tau' \)'s and \( \phi_2 \)'s, so if \( N_2 \) stands for the number of produced \( \phi_2 \) mesons as in the case (i), \( c'(G) \) due to the meson current can be written as

\[
c'(G_{\text{meson}}) = (-1)^{N_2} c(G).
\]

(4.19)

Since (18) and (19) are of the same form, the general expression for the case of pseudoscalar coupling is given by

\[
c'(G) = (-1)^{N_2} c(G),
\]

(4.20)

where \( N_2 \) is the number of produced \( \phi_2 \) mesons.

Next we shall apply this rule to the case with both pseudoscalar and pseudovector couplings.

The discussions about nucleon and meson currents are completely the same with the pseudoscalar coupling case since both pseudoscalar and pseudovector couplings are even. So we shall only be concerned with the contact interaction. The coupling matrix \( \gamma_\mu (P_\nu) \) is even and there is no sign change in the Dirac part, while in the isotopic part, the numbers of \( \tau' \)'s and \( \phi_2 \)'s differ by 1, because

\[
[\tau_2 \gamma_\mu] \phi_2 = i (\tau_2 \phi_2 - \tau_3 \phi_2).
\]

(4.21)

We thus see the same situation as in the case of the meson current, and the formula (20) is still valid.

That the pseudoscalar and pseudovector couplings have the same parity is very fortunate. This proof can also be applied to other cases such as the pure neutral theory.

It must be noted that we have to transform \( \phi_1 \) and \( \phi_2 \) into \( \phi \) and \( \phi^* \) for practical purposes, which we shall write down for the sake of completeness.
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\[ \tau^* = (1/2)(\tau_1 - i\tau_2), \quad \tau = (1/2)(\tau_1 + i\tau_2), \]
\[ \phi^* = (1/\sqrt{2})(\phi_1 - i\phi_2), \quad \phi = (1/\sqrt{2})(\phi_1 + i\phi_2), \quad (4.22) \]
\[ \phi_1 = (1/\sqrt{2})(\phi + \phi^*), \quad \phi_2 = (1/\sqrt{2}i)(\phi - \phi^*). \]

Comparing (20) and (22), we see that the factor \((-1)^s\) is obtained by changing the sign of \(\phi_2\), leaving \(\phi_1\) and \(\phi_8\) unchanged. In other words, the factor is achieved by the following transformation:

\[(\text{III}) \quad \phi \rightarrow \phi^*, \quad \phi^* \rightarrow \phi. \quad (\text{charge conjugation}) \quad (4.23)\]

If we denote the expression obtained from \(c(G)\) by applying (I), (II) and (III) as \(c''(G)\), we have in the case of the photo-meson production

\[ c''(G) = -c(G). \quad (4.24) \]

With the above procedure, we shall calculate \(R^+\) from \(R\) for the case of photo-meson production.

The submatrix corresponding to the photo-meson production can be written as

\[ (\pi | R | \gamma) \sim \sum_{s=1}^{4} \delta_s \bar{\psi}(F) P_{\mu_s} \phi(I) A_{\mu}(k) \phi(q), \quad (4.25) \]

where the coefficients \(b_s\)'s are scalar c-numbers, being functions of scalar products of \(I,F,k\) and \(q\), and consequently invariant under the transformation (II). \(P_s\)'s are given by (3.4).

To apply the transformations (I) and (II) to (25), we have only to replace the \(P_s\) as

\[ P^* \rightarrow \epsilon_s P^*, \quad (4.26) \]

where

\[ \epsilon_s = \begin{cases} 1, & (s=1) \\ -1, & (s=2,3,4) \end{cases} \quad (4.27) \]

Hence

\[ (\gamma | R^+ | \pi) \sim \sum_{s=1}^{4} \delta_s \bar{\psi}(F) P_{\mu_s} \phi(I) A_{\mu}(k) \phi(q), \quad (4.28) \]

in which the role of creation and destruction operators should be interpreted in the opposite manner to (25).

Next, from the relation

\[ (\pi | R^* | \gamma) = (\gamma | R | \pi)^*, \]

we finally obtain

\[ (\pi | R^+ | \gamma) \sim \sum_{s=1}^{4} \epsilon_s \eta_s \delta_s \bar{\psi}(F) P_{\mu_s} \phi(I) A_{\mu}(k) \phi(q), \quad (4.29) \]

where

\[ \eta_s = \begin{cases} 1, & (s=4) \\ -1, & (s=1,2,3) \end{cases} \quad (4.30) \]
§ 5. Integral equations

In this section, we shall discuss the integral equations connecting the matrix element of the photo-meson production and that of pion-nucleon scattering.

The elementary photo-meson processes which are experimentally observed are

\[ \gamma + p \rightarrow p + \pi^0, \]
\[ \gamma + p \rightarrow n + \pi^+. \]

Corresponding to the above two processes, we have

\[ (\pi^0|\vec{R} + R^+|\vec{p} \gamma) + (\pi^0|\vec{R}|\rho \pi^n) (\pi^0|\vec{R}^+|\vec{p} \gamma) \]
\[ + (\pi^0|\vec{R}|n \pi^+)(\pi^+ n|\vec{R}^+|\vec{p} \gamma) = 0, \]  \hspace{1cm} (5.1)
\[ (\pi^+ n|\vec{R} + R^+|\vec{p} \gamma) + (\pi^+ n|\vec{R}|\rho \pi^n) (\pi^0|\vec{R}^+|\vec{p} \gamma) \]
\[ + (\pi^+ n|\vec{R}|n \pi^+)(\pi^+ n|\vec{R}^+|\vec{p} \gamma) = 0. \]  \hspace{1cm} (5.2)

which are the concrete expressions of (2.5).

According to the symmetrical theory, the amplitudes of pion-nucleon scattering can be expressed as the linear combinations of two independent quantities \( Q \) and \( T \), so that the above two equations are necessary and sufficient to determine the matrix elements of all modes of pion-nucleon scattering.

We shall rewrite \( \phi_1 \) and \( \phi_2 \) by \( \phi \) and \( \phi^* \), namely
\[ (Q \tau_{LM} + T \delta_{LM}) \phi_3 \phi_4 \]
\[ = Q [i \tau_3 (\phi \phi^* - \phi^* \phi) + i \sqrt{2} (\tau \phi^* \phi - \tau \phi \phi^*) \phi_3 - i \sqrt{2} \phi_3 (\tau \phi^* - \tau \phi \phi^*)] \]
\[ + T [\phi \phi^* + \phi^* \phi + \phi \phi^*]. \]  \hspace{1cm} (5.3)

Then we have only to insert this relation into (1) and (2).

Next, we employ the following normalizations of the wave functions,

\[ \phi^{(\pm)}(x) = \frac{1}{\sqrt{2(2\pi)^3}} \int dq \phi^{(\pm)}(q) e^{\pm iqx}, \]
\[ \phi^{(+)}(x) = \frac{1}{\sqrt{2(2\pi)^3}} \int dp \phi^{(+)}(p) e^{ipx}, \]  \hspace{1cm} (5.4)
\[ \phi^{(-)}(x) = \frac{1}{\sqrt{2(2\pi)^3}} \int dp \phi^{(-)}(p) e^{-ipx}. \]

which are convenient for covariant calculations.

Then we have

\[ \langle \phi^{(+)}(q) \phi^{(-)}(q') \rangle = g_0 \delta(q - q'), \]
\[ \langle \psi(\rho) \bar{\psi}(\rho') \rangle = (-i \gamma \rho + M) \rho_0 \delta(\rho - \rho'). \]  \hspace{1cm} (5.5)

We shall write the transition matrices of the photo-meson production as follows:
The S-Matrix Method in Pion Reactions

\[
\langle \pi^0 p | R | \eta \rangle \equiv \int \frac{dq}{q_0} \frac{dF}{F_0} \frac{dI}{I_0} \frac{dk}{k_0} (q + F - I - k) \\
\times \left[ \sum_{\alpha} \phi_\alpha(q) \bar{\phi}(F) P_\alpha \zeta \phi(I) A_\alpha(k) k_\alpha \right],
\]
(5.6)

Similarly, the transition matrices of the pion-nucleon scattering are

\[
\langle \pi^+ n | R | \eta n \rangle \equiv \int \frac{dq}{q_0} \frac{dF}{F_0} \frac{dI}{I_0} \frac{dk}{k_0} (q + F - I - q) \\
\times \bar{\phi}(q) \phi_\phi(F)[T'' + T''(q\eta)] \bar{\tau} \phi(I) \phi_\phi(q),
\]
(5.7)

Inserting the above expressions into (1) and (2), and remembering (5), we reach finally to the integral equations in question.

Next, we must eliminate wave functions such as \( \phi, \bar{\phi} \) and \( A \). But notice that the relation

\[
\bar{\phi} O \phi = 0
\]
does not necessarily require

\[
O = 0.
\]

Instead, we have

\[
A(F) O A(I) = 0
\]
(5.9)

from

\[
\phi(F) O \phi(I) = 0,
\]

where

\[
A(p) = -i p_T + M.
\]

Similarly we cannot conclude

\[
O_{\mu\nu} = 0
\]
from

\[
O_{\mu\nu} A_\mu(k) k_\nu = 0.
\]

(5.11)
In this case, we utilize the result that the $S$ matrix integrated in a covariant manner is equivalent to that integrated after eliminating the longitudinal waves.\textsuperscript{13} Namely, we express the electromagnetic potential by the operators which destruct photons with two independent polarizations, and put each coefficient equal to zero.

Then, choosing the direction of $k$ as $z$-axis, we can rewrite (11) as follows:

$$(O_x)_z + (O_y)_y = 0, \quad (O_x)_y - (O_y)_x = 0,$$  \hspace{1cm} (5.12)

where

$$O_x = (O_{25} O_{31} O_{12}), \quad O_y = (O_{30} O_{20} O_{30}).$$

The integrals arising in the matrix multiplications can be reduced to solid angle integrations.

Taking the energy-momentum four vectors in the matrix elements as in fig. 8, we have

$$\int \frac{d\Omega'}{q_0'} \frac{dQ'}{F_0'} \delta(q' + F' - F - q) \cdots = \frac{|q|}{q_0 + F_0} \int d\Omega(q') \cdots $$  \hspace{1cm} (5.13)

![Diagram 8]

We obtain as the final integral equations, after taking traces to eliminate Dirac matrices

$$\sum \left[ \tilde{b}_s (q,k) + \varepsilon_s b_s (q,k) \right] \mathfrak{M}_{\mu\nu}$$

$$+ \frac{|q|}{q_0 + F_0} \int d\Omega' \left[ \sum \varepsilon_s \delta^s (q',k) \{ T' (q,q') \mathfrak{B}_{\mu\nu} + T'' (q,q') \mathfrak{C}_{\mu\nu} \} ight. $$

$$\left. + \left( i \sqrt{2} \right) \sum \varepsilon_s \eta_s a_s^* (q',k) \{ Q' (q,q') \mathfrak{B}_{\mu\nu} + Q'' (q,q') \mathfrak{C}_{\mu\nu} \} \right] = 0,$$

$$\sum \left[ a_s (q,k) + \varepsilon_s b_s^* (q,k) \right] \mathfrak{N}_{\mu\nu}$$

$$+ \frac{|q|}{q_0 + F_0} \int d\Omega' \left[ \sum \varepsilon_s \delta_s (q',k) \{ Q' (q,q') \mathfrak{B}_{\mu\nu} + Q'' (q,q') \mathfrak{C}_{\mu\nu} \} ight. $$

$$\left. + \left( T' (q,q') \mathfrak{B}_{\mu\nu} + T'' (q,q') \mathfrak{C}_{\mu\nu} \right) \right] = 0,$$  \hspace{1cm} (5.14)

where the equality sign ($\equiv$) means that the equality holds if $\mathfrak{M}_{\mu\nu}$, $\mathfrak{B}_{\mu\nu}$ and $\mathfrak{C}_{\mu\nu}$ in (14) are substituted for in the same way as (12) was obtained from (11). The coefficients $a$ and $b$ are the functions of angles, and $\mathfrak{M}_{\mu\nu}$, $\mathfrak{B}_{\mu\nu}$ and $\mathfrak{C}_{\mu\nu}$ are given by
The SMatrix Method in Pion Reactions

\[ \mathfrak{S}_{\mu\nu} = S\mathcal{A}(F) P_{\mu\nu}(F, I) A(I) \],
\[ \mathfrak{B}_{\mu\nu} = S\mathcal{A}(F) A(F') P_{\mu\nu}(F', I) A(I) \],
\[ \mathfrak{C}_{\mu\nu} = S\mathcal{A}(F) (\gamma q) A(F) P_{\mu\nu}(F', I) A(I) \], \quad (5.15)

where \( I \) is an arbitrary Dirac matrix.

The equations (14) are simultaneous Fredholm's integral equations of the first kind. We shall separate these equations.

In the symmetrical theory there are following relations\(^{12}\)

\[ \sqrt{2} \left( \pi^0 p | R \rho n \right) + \left( \pi^+ n | R \rho p \right) = \sqrt{2} \left( \pi^+ p | R \rho p \right), \]
\[ \sqrt{2} \left( \pi^0 p | R n \rho p \right) + \left( \pi^+ n | R n \rho n \right) = \left( \pi^+ p | R n \rho n \right), \quad (5.16) \]
apart from isotopic operators.

Combining (1), (2), and (16), we get

\[ \sqrt{2} \left( \pi^0 p | R + R^+ | \rho n \right) + \left( \pi^+ n | R + R^+ | \rho n \right) \]
\[ + \left( \pi^+ p | R n \rho p \right) = 0. \quad (5.17) \]

This is an integral equation for \( \left( \pi^+ p | R \rho n \right) \) only, so we can first solve this separated equation (17).

It is interesting that the matrix element of pion-nucleon scattering first obtained from those of photo-meson production is relevant to the process \( \pi^+ + p \rightarrow \rho + \pi^+ \), i.e. the state with charge 2.

We have studied the case of symmetrical theory so far, but we had better discuss here the case of pure neutral theory.

In this case there appear more than two independent amplitudes, and therefore we need informations about

\[ \gamma + n \rightarrow n + \pi^0, \quad \gamma + n \rightarrow \rho + \pi^+, \]
in addition to the processes

\[ \gamma + p \rightarrow \rho + \pi^0, \quad \gamma + p \rightarrow n + \pi^+. \]

The complete integral equations are

\[ \left( \pi^0 p | R + R^+ | \rho n \right) + \left( \pi^0 p | R n \rho \right) \left( \pi^0 p | R^+ | \rho n \right) = 0, \]
\[ \left( \pi^+ n | R + R^+ | \rho n \right) + \left( \pi^+ n | R n \rho \right) \left( \pi^+ n | R^+ | \rho n \right) = 0, \]
\[ \left( \pi^0 n | R + R^+ | \eta \right) + \left( \pi^0 n | R n \rho \right) \left( \pi^0 n | R^+ | \eta \right) = 0, \]
\[ \left( \pi^+ p | R + R^+ | \eta \right) + \left( \pi^+ p | R n \rho \right) \left( \pi^+ p | R^+ | \eta \right) = 0, \]
\[ \left( \pi^- p | R + R^+ | \eta \right) + \left( \pi^- p | R n \rho \right) \left( \pi^- p | R^+ | \eta \right) = 0. \]

\[ + \left( \pi^- p | R | \rho \right) \left( \pi^- p | R^+ | \eta \right) = 0. \]
All the integral equations obtained from (2.5) are given above. Since the matrix elements of inverse processes can be derived from those of original processes by the procedure given in section 4, only independent ones are

(a) \( \pi^0 + p \rightarrow p + \pi^0 \)
(b) \( \pi^0 + n \rightarrow n + \pi^0 \)
(c) \( \pi^+ + n \rightarrow n + \pi^+ \)
(d) \( \pi^- + p \rightarrow p + \pi^- \)
(e) \( \pi^0 + p \rightarrow n + \pi^- \)
(f) \( \pi^0 + n \rightarrow p + \pi^- \)

The charge dependence of the interaction in the pure neutral theory is given by

\[ H \sim f(\tau \phi^* + \tau^* \phi) + f_0 (1 \cdot \phi_0). \]

This interaction is invariant under the substitutions

\[ \phi \rightarrow \phi^*, \quad \tau \rightarrow \tau^*. \]

The transformation \( \tau \rightarrow \tau^* \) is achieved by

\[ \tau_1^{-1} \tau \tau_1 = \tau^*, \quad \tau_1^{-1} \tau_0 \tau_1 = \tau. \]

Since \( R \) is invariant under this transformation

\[ R_a = R_{ba}, \quad R_c = R_{ab}, \quad R_e = R_f. \]

Namely only three of the six transition matrices of (a), \( \ldots \), (f) are independent, and the four integral equations given above are enough. In this case, however, we can obtain no information on the processes \( \pi^+ + p \) and \( \pi^- + n \).

§ 6. Partial wave expansion

In order to solve integral equations, it is convenient to expand \( T \) and \( Q \) in Legendre polynomials,

\[ T(\cos \theta) = \sum_{l=0}^{\infty} T_l P_l(\cos \theta). \quad (6 \cdot 1) \]

The values of the maximum \( l \)'s come into question for the purpose of numerical integration, so that we shall roughly evaluate them.

(a) pion-nucleon scattering

The interaction between a pion and a nucleon is of short range, if the Coulomb forces are discarded.

Let the impact parameter in pion-nucleon collision be \( b \), then the condition that the collision takes place is given by
The value of $r_f$ is strongly energy dependent for the pseudoscalar pion, but we may regard it as constant in the order of magnitude theory. As the most probable value of $r_f$, we choose the Compton wave length of pion, i.e.

$$r_f \sim \frac{\hbar}{\mu c}, \quad (6.3)$$

then the conditions for collision is

$$b \lesssim \frac{\hbar}{\mu c},$$

with the corresponding angular momentum

$$l \hbar = pb \lesssim \frac{p(h/\mu c)}{1/\mu c}.$$ (6.4)

Thus the angular momenta which contribute to the scattering are restricted to

$$l \lesssim \frac{p}{\mu c}.$$ (6.4)

The correspondence between $E$ and pion angular momentum is tabulated in the following table. ($E$: meson kinetic energy)

<table>
<thead>
<tr>
<th>$E$ (in Mev)</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>60</td>
<td>170</td>
<td>300</td>
<td>440</td>
<td></td>
</tr>
</tbody>
</table>

(b) photo-meson production

In the pion-nucleon scattering, even a low energy incident pion can be scattered, while in the photo-meson production only a photon with energy higher than the threshold can create a pion. So our discussion must be somewhat altered.

The condition that an incident photon with momentum $k$ can create a pion is given by

$$kc \gtrsim \mu c^2.$$ (6.5)

Assuming the same force range with the former case, we have

$$l \hbar \gtrsim \frac{k}{\mu c} = \frac{kc}{\mu c^2} \lesssim \hbar.$$ (6.6)

Thus $P$ waves as well as $S$ waves can appear at low energies slightly above the threshold. This fact is surely guaranteed by experiments on $\gamma - \pi^0$ process.

In the above consideration, a photon is treated as though a scalar particle, but it will be enough for such a rough discussion.

As can readily be seen from the above examples, the partial waves that contribute to the process in question are strongly dependent upon the special features of the process.

In pion-nucleon and nucleon-nucleon scatterings, both the initial and final state can be in low energy region, so that only $S$ waves contribute mainly at low energies.
§ 7. Remarks on the photo-meson production

The photo-meson production will be one of the most fundamental reactions among various pionic elementary processes.

In this section, we shall discuss this process from the field theoretical viewpoint.

In the pseudoscalar meson theory, it is well known that the convergence of the perturbation will not be good, but in practice, it is prohibitively difficult to compute higher order contributions. So some device to take account of the higher order effects has to be performed in a more of less phenomenological manner.

Recently Minami calculated the matrix elements of $\gamma - \pi^0$ up to the order $e\mu$, and then introduced a Pauli-type interaction phenomenologically in order to estimate the effects of orders higher than $e\mu$.

It has already been emphasized that in order to account for the large cross-section of $\gamma - \pi^0$ process the anomalous magnetic moment of a nucleon will play an important rôle, but since the lowest order ($e\mu^2$) calculation of anomalous magnetic moment is unsatisfactory we are compelled to introduce this effect in a very complicated way as Minami did.

The Pauli-type interaction can express the effects of the static magnetic moment, but not the non-static effects. Accordingly he calculated the radiative corrections to the Pauli-type interaction.

Thus the orders of the whole expression are

$$ef + ef^3 + e\mu^2 \sigma f + e\mu' f^3,$$

where $e\mu' f$ is the static anomalous magnetic moment expressing the static effects, and $e\mu' f^3$ is expressing the non-static effects.

The results show a fairly good agreement with experimental data, so that we shall employ his matrix elements in the integral equations (5.14).

In practical calculations, the following point should be noticed: in the equation

$$\langle \pi | R + R^+ | \gamma \rangle + \langle \pi | R | \pi \rangle \langle \pi | R^+ | \gamma \rangle = 0,$$

the lowest order in the second term is

$$(f^2) \times (ef) = ef^3,$$

therefore the orders that survive in the first term are only

$$ef^3 + e\mu f^3.$$ (7.2)

If there were only terms of order $ef^3$, our approximation in the calculation of $\langle \pi | R | \pi \rangle$ would no more be better than the calculation in the lowest order, but since we are assuming that the $e\mu f^3$ term expresses phenomenologically the effects of orders higher than $ef^3$, our approximation will presumably be better than the lowest order calculation.

§ 8. Pion-nucleon scattering

Assuming the matrix elements of $\gamma - \pi$ processes discussed in the previous section, we shall solve the integral equations for the pion-nucleon scattering. Since the matrix elements
of \(r-\pi\) processes are given numerically, we must perform numerical integration. For this purpose, the method of partial wave expansion is useful.

As the integral equations concerned are very complicated, we shall illustrate the method by a much simplified example.

The equations in question are Fredholm's integral equations of the first kind and the integration variables are the solid angles.

The simplest equation with the above property is

\[
\int f(\cos \theta) F(\cos \phi) dQ = g(\cos \phi),
\]

(8.1)

where \(f\) and \(g\) are known functions, and \(F\) an unknown function. In order to solve the above equation, expansions in Legendre polynomials are convenient.

\[
f(\cos \theta) = \sum f_i P_i(\cos \theta),
\]

\[
g(\cos \theta) = \sum g_i P_i(\cos \theta),
\]

\[
F(\cos \theta) = \sum F_i P_i(\cos \theta).
\]

(8.2)

Choosing the direction of \(c\) as the z-axis, we write the polar coordinates of \(a\) and \(b\) as

\[
a: \theta, 0; \quad b: \theta, \phi
\]

(8.3)

Then we have

\[
\cos \angle ab = \cos \theta \cos \phi + \sin \theta \sin \phi \cos \phi,
\]

\[
\cos \angle bc = \cos \theta,
\]

\[
\cos \angle ac = \cos \theta_1.
\]

(8.4)

Next from the addition theorem of Legendre polynomials, we have

\[
P_i(\cos \angle ab) = P_i(\cos \theta) P_i(\cos \phi) + 2 \sum_{m=1}^{i} \frac{(i-m)!}{(i+m)!} P_i^{m}(\cos \theta) P_i^{m}(\cos \phi) \cos m \phi,
\]

(8.5)

then substituting this relation into (1), we get from the orthogonality of spherical harmonics

\[
F_i = \frac{2l+1}{4\pi} \frac{g_i}{f_i}
\]

or

\[
F(\cos \theta) = \sum_i \frac{(2l+1)}{4\pi} \frac{g_i}{f_i} P_i(\cos \theta).
\]

(8.6)

The most serious problem is the convergence of this series, but we assume that it is guaranteed by physical requirements such as the possibility of partial wave expansions discussed in section 6. The equations (5.14) are much more complicated, but the method is essentially the same with the above simplified example.

Next we shall combine the results with Möller's formula for the collision cross-sections.\(^{10}\)
Suppose that the transition matrix for the pion-nucleon scattering is given by
\[ \langle \pi N | R | Nn \rangle = \frac{d\mathbf{q}_1}{(q_1)_{0}} \cdot \frac{d\mathbf{q}_0}{(q_0)} \cdot \frac{d\mathbf{I}}{I_0} \cdot \frac{d\mathbf{F}}{F_0} \delta^4(q_1 + F - q_0) \phi(F) O \phi(I) \phi(q_1) \phi(q_0), \] (8.7)
then referring to the normalizations in section 5, the total cross-section is given by
\[ \sigma = \frac{4\pi^2}{\mathcal{B}} \int \frac{d\mathbf{q}_1}{(q_1)_{0}} \cdot \frac{d\mathbf{F}}{F_0} \delta^4(q_1 + F - q_0) |u(F) O u(I)|^2, \] (8.8)
with the replacement
\[ |u(F) O u(I)|^2 \rightarrow \frac{1}{2} S_p(A(I) O^+ A(F) O). \] (8.9)

Let the differential solid angle representing the direction of \( q_f \) be \( d\Omega \), then the differential cross-section becomes as
\[ \frac{d\sigma}{d\Omega} = \frac{4\pi^2}{\mathcal{B}} \frac{|q|}{q_0 + F_0} \frac{1}{2} S_p(A(I) O^+ A(F) O), \] (8.10)
in the centre of mass system (8.11)
where
\[ \mathcal{B} = \sqrt{|q_0^I - q_0|^2 - |q_i \times I|^2}. \]

So the formula (10) is also rewritten as
\[ \frac{d\sigma}{d\Omega} = \frac{4\pi^2}{(F_0 + q_0)^2} \frac{1}{2} S_p(A(I) O^+ A(F) O). \] (8.12)

The matrix \( O \) is tabulated in the following.

<table>
<thead>
<tr>
<th>Reaction mode</th>
<th>( O )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \pi^+ + p \rightarrow \pi^+ + n )</td>
<td>( T + iq )</td>
</tr>
<tr>
<td>( \pi^- + p \rightarrow \pi^- + n )</td>
<td>( T - iq )</td>
</tr>
<tr>
<td>( \pi^- + n \rightarrow \pi^- + p )</td>
<td>( -i\sqrt{2} O )</td>
</tr>
</tbody>
</table>

where
\[ T = T' + T''(q_f), Q = Q' + Q''(q_f), \]
\[ (q_f)^+ = -(q_f). \]
The total cross-sections can be computed more easily.

Combining Möller’s formula (8) with the following unitarity conditions:
\[(\pi^+ p | R + R^+ | \rho n^+) + (\pi^+ p | R | \rho n^+) (\pi^+ p | R^+ | \rho n^+) = 0, \]
\[(\pi^- p | R + R^+ | \rho n^-) + (\pi^- p | R | \rho n^-) (\pi^- p | R^+ | \rho n^-) + (\pi^- p | R^+ n^{0}) (\pi^0 n | R^+ | \rho n^-) = 0, \] (8.13)
we have for the total cross-section
\[ \sigma(\pi N) = -\frac{4\pi^2}{\mathcal{B}} \frac{1}{2} S_p[A(I) (O + O^*)_0 = 0], \] (8.14)
where \( \theta = 0 \) means to put the vector \( q_f \) equal to \( q_i \). Inserting the results obtained by solving the integral equations, we have the total cross-sections of the pion-nucleon scattering, i.e.
The S Matrix Method in Pion Reactions

\[ \sigma(\pi^+p) = 105 \text{ mb} \]
\[ \sigma(\pi^-p) = 33 \text{ mb} \]  
(at about 130 Mev (lab.)) \hspace{1cm} (8.15)

To our regret numerical errors in \( T \) and \( Q \) are as large as about 10\(^\text{th} \)\( -20 \text{\%} \), so that we cannot calculate differential cross-sections from (8), since (8) is quadratic in \( T \) and \( Q \). (But it must be emphasized that the relations (13) are satisfied by our solution within numerical error.) The results, however, bear the following important feature. Its order of magnitude is in good agreement with experiments, especially the ratio

\[ \sigma(\pi^+p)/\sigma(\pi^-p) \approx 3 \] \hspace{1cm} (8.16)

is consistent with the recent experimental data\(^{50} \) even if the numerical errors are taken into account. Remember that this ratio cannot be deduced from the lowest order perturbation theoretical calculations based on the pseudoscalar meson theory with both pseudoscalar and pseudovector couplings.

From this result we may conclude:

Although the effects of higher orders can hardly be estimated in the standard manner, we can still take account of these effects consistently for various pion reactions provided that suitable interactions such as the Pauli term are phenomenologically added.

Another method to describe various pion reactions in a unified manner may be to assume the existence of nucleon isobars and study its effects in pion reactions\(^{50} \).

§ 9. Elastic nucleon-nucleon collisions at extremely high energies

In previous sections, our interests are limited only to pion reactions at low energies. In contrast to previous sections, we shall now briefly touch upon the problem of nucleon-nucleon collisions at extremely high energies where partial wave analysis is not available.

In this section we shall discuss the correlation between excitation functions and angular distributions of elastic nucleon-nucleon collisions based on the unitary character of \( S \) matrix.

Our assumptions in this section are the followings.

(I) There exists a divergence-free \( S \) matrix.

It is not necessary that this \( S \) matrix is constructed from the renormalizable field theory.

(II) The asymptotic wave functions of particles can be expanded in plane waves.

The latter assumption is valid only when forces in question are of the short range character, so that Coulomb forces cannot be included.

The fundamental formulas are

\[ R + R^* + RR^* = 0, \quad \text{(unitarity)} \] \hspace{1cm} (9.1)

and \( M^{\text{\text{\kern-1em}r}} \text{ller's formula.} \)

In nucleon-nucleon collisions at extremely high energies, the multiple production of pions can take place, and possibly also the pair creation of nucleons. Thus, at extremely high energies we get from (1)

\[ -(2N|R + R^*|2N) = (2N|R|2N)(2N|R^*|2N) \]
\[ + (2N|R|2N, \pi)(\pi, 2N|R^*|2N) + \cdots \cdots \] \hspace{1cm} (9.2)
From now on we consider the problem in the centre of mass system. Then the total energy of this system is twice the energy of the incident nucleon. At extremely high energies,

\[ p_0 = \sqrt{p^2 + M^2} \approx \rho \left( |p| \right), \]

so we can measure the order of magnitudes of all energies, momenta by the unit \( \rho \).

Now let the transition matrix for the elastic nucleon-nucleon collision be

\[ (2N|R|2N) = \int \frac{dp_f}{(p_f)} \frac{dp_i}{(p_i)} \delta(p_i + p_f - p_i - p_f') \psi(p_i) \psi(p_f') \]

\[ T(p_i, p_f', p_i, p_{f'}) \psi(p_i) \psi(p_f'). \]  

Then total cross-section of the elastic collision is given by

\[ \sigma_{el} = \frac{4 \pi^2}{B} \int \frac{dp_i}{(p_i)} \frac{dp_f'}{(p_f')} \delta(p_i + p_f' - p_i - p_f') \frac{1}{4} (S(p) \cdot S(p') A(p_i) A(p_f') \cdot (T + T^*) \bigg|_{q=0}), \]  

for neutron-proton system. \( (9.4) \)

\[ B = 2 \rho \cdot \rho_0. \]

On the other hand the total cross-section including both elastic and inelastic ones is given by

\[ \sigma_{total} = \frac{4 \pi^2}{B} \frac{1}{4} (S(p) \cdot S(p') A(p_i) A(p_f') \cdot (T + T^*) \bigg|_{q=0}), \]  

just as in the previous section.

If there are following relations between \( Q(p) \) and \( f(p) \),

\[ \lim_{p \to \infty} \frac{Q(p)}{f(p)} = \text{finite} \]  

\[ \lim_{p \to \infty} \frac{Q(p)}{f(p)} = \text{finite or zero} \]  

then we symbolically write as

\[ Q(p) = O(f(p)) \]  

\[ Q(p) \leq O(f(p)). \]

And we have

\[ B = O(p^2), A = O(p). \]  

\[ (9.8) \]

The total cross-section is the sum of elastic and inelastic ones:

\[ \sigma_{total} = \sigma_{el} + \sigma_{inel}, \]  

so that it is clearly seen that

\[ \sigma_{total} \geq \sigma_{el}. \]  

\[ (9.10) \]

It is very important that both \( \sigma_{total} \) and \( \sigma_{el} \) in \( (10) \) can be expressed in terms of \( T \) and \( T^* \) without referring to the precise form of the matrix elements of inelastic collisions. \( (Cf. \ (4) \ and \ (5)) \)
In the formula (4), $T$ is the function of energy and angle, and the integration is reduced to the one with respect to the solid angle.

Now we assume the energy dependence of $T$ as follows:

$$T = O(f(p)),$$  \hspace{1cm} (9·11)

except for the domain of measure zero with regard to the solid angle integration. If we assume that the direction $\theta = 0$ does not belong to this exceptional domain, we see from the formulas (4), (5) and (11)

$$\sigma_d = O(p^2 f^2(p)), \quad \sigma_{\text{total}} = O(f(p)).$$  \hspace{1cm} (9·12)

Combining (12) with the inequality (10), we get

$$O(f(p)) \leq O(p^{-2}),$$  \hspace{1cm} (9·13)

which amounts to

$$\sigma_{\text{total}} \leq O(p^{-2}).$$  \hspace{1cm} (9·14)

On the other hand we know from the cosmic ray data that the total cross-section of nucleon-nucleon collision is nearly equal to the geometrical cross-section, i.e.

$$\sigma_{\text{total}} = \sigma_{\text{geo}} = O(1).$$  \hspace{1cm} (9·15)

Accordingly (14) contradicts the experimental data, and we infer that the angle $\theta = 0$ indeed belongs to the exceptional domain. Thus

$$T_{\theta=0} = O(f(p)) \quad (9·16a), \quad T_{\theta=0} < O(f(p)). \quad (9·16b).$$

(16b) is also excluded by the same reason with the case

$$T_{\theta=0} = O(f(p)).$$

Therefore we can deduce

$$T_{\theta=0} = O(f(p)).$$

In proton-proton collisions, $\theta = \pi$ also belongs to the exceptional domain. The exceptional domain will perhaps consist of only two angles $\theta = 0$ and $\theta = \pi$, since we cannot consider any special angle other than these two.

Thus we can predict the following relation:

$$\lim_{p \to \infty} \left( \frac{d\sigma_{d}}{d\Omega} \right)_{\theta} \left| \frac{d\sigma_{d}}{d\Omega} \right|_{\theta=0} = 0, \quad (0 < \theta < \pi)$$  \hspace{1cm} (9·17)

if the effect of Coulomb scattering can be eliminated.

Similar discussions can also be applied to pion-nucleon collisions. In conclusion, the authors express their hearty thanks to Prof. Y. Nambu, Prof. S. Hayakawa and Mr. Y. Yamaguchi for their kind interests taken in this work, and to Mr. S. Minami for his helpful information on the photo-meson production.
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