On the \( \beta \)-Ray Angular Correlations

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The angular distribution functions of \( \beta \)-rays that are used to obtain the angular correlation functions are calculated taking into account the effect of the nuclear charge. The calculations are performed up to the second forbidden transitions for each \( \beta \)-decay interaction type of the Fermi theory separately but including the mixture of the nuclear matrix elements. To obtain the angular correlation functions easily in the case of the mixed nuclear matrix elements they are given in considerably simple forms in two interesting cases. Also in the appendix, transformation coefficients \( \lb j_1, 3 \ell, m_\ell \rb | \lb j_2, 3 j, m \rb \) are given.

§ 1. Introduction

When two particles are emitted successively from a nucleus, there is angular correlation between them.\(^1\) This evidence is now well known. Studies of the angular correlation is useful for understanding spins and parities of nuclei. Besides them, when \( \beta \)-decays are concerned, it can be used to decide the \( \beta \)-decay interaction type.

Many theories have been issued concerning the angular correlation. These theories are divided into two, the one is that which is common to all kinds of radiations and the other is the study of special cases. The first was investigated by Falkoff and Uhlenbeck,\(^2\) Racah,\(^3\) and Lloyd.\(^4\) As to the latter, there are studies by Hamilton,\(^5\) Goertzel,\(^6\) Ling Jr. and Falkoff,\(^7\) Lloyd\(^8\) and others for \( \gamma - \gamma \), and for the cases internal conversions are included, there are detailed calculations by Rose \( et \ al.\)\(^9\).

For the cases \( \beta \)-rays are included, Falkoff and Uhlenbeck\(^10\) and Fuchs and Lennox\(^11\) studied, but they either omitted the effect of the nuclear charge\(^10\) or took it into account only in the approximation of \( aZ \ll 1.\)\(^11\) Therefore, in this paper we calculate the angular distribution functions* of \( \beta \)-rays taking into account the effect of the nuclear charge perfectly.\(^1\) \(1\)

§ 2. General theory

In this section we introduce the forms of the angular correlation function \( \Theta(\theta) \)

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* As to the meaning of angular distribution function see Section 2.
† But the effect of the finite nuclear size is not considered explicitly.
\( \dagger \) Recently the authors knew that Fuchs and Al-Ghita investigated this problem; M. Fuchs, Thesis, (University of Michigan, 1951); M. K. Al-Ghita, Thesis, (University of Michigan, 1951). But the authors have not seen them yet.
which are convenient in the cases of the mixed nuclear matrix elements. Mainly we follow
the procedures of Falkoff and Uhlenbeck.\(^{2}\)

We consider only the directions of particles, and do not observe the polarizations of
them. Then the angular correlation function \(\mathcal{O}(\theta)\), the relative probability that two
particles are emitted with the angle \(\theta\), is\(^{1}\)

\[
\mathcal{O}(\theta) = \sum_{m} \left| \sum_{l} \langle B_{m} | H_{l}(\mathbf{k}_1) | A_{l} \rangle \right|^2 \sum_{n} \left| \langle C_{n} | H_{s}(\theta) | B_{m} \rangle \right|^2 .
\]  

(1)

The notation is as follows: the nucleus changes \(A_{l} \rightarrow B_{m} \rightarrow C_{n}\), \(l\), \(m\) and \(n\) are the \(z\)-
components of the total angular momenta, \(\mathbf{k}_1\) and \(\mathbf{k}_2\) are the directions of the first and
the second particles, and \(S_{1}\) and \(S_{2}\) indicate the sums over all the directional arguments
(spin etc.) except for \(\mathbf{k}_1\) and \(\mathbf{k}_2\). Generally, in the right-hand side of Eq. (1) there
are interferences between different \(m\)'s; but when we select the \(z\)-axis as \(\mathbf{k}_1\), the interferences
vanish\(^{13}\) and Eq. (1) becomes

\[
\mathcal{O}(\theta) = \sum_{m} \left| \sum_{l} \langle B_{m} | H_{l}(\theta) | A_{l} \rangle \right|^2 \sum_{n} \left| \langle C_{n} | H_{s}(\theta) | B_{m} \rangle \right|^2 .
\]  

(2)

We put
\[
S_{l} \left( \langle B_{m} | H_{l}(\theta) | A_{l} \rangle \right)^2 \equiv \rho_{lm}(\theta), \quad S_{l} \left( \langle C_{n} | H_{s}(\theta) | B_{m} \rangle \right)^2 \equiv P_{mn}(\theta).
\]  

(3)

\(\rho_{lm}(\theta)\) is the relative probability that the nucleus changes from \(A_{l}\) to \(B_{m}\) and the first
particle is emitted into the direction \(\theta\), and \(P_{mn}(\theta)\) is that of the second particle. Then
Eq. (2) becomes

\[
\mathcal{O}(\theta) = \sum_{l, m, n} \rho_{lm}(0) P_{mn}(\theta).
\]  

(4)

Let \(T_{l_{1},\ldots,l_{L}}\) be the irreducible tensors of the \(L\)-th rank for the rotations of 3-dimen-
sional space. The interaction Hamiltonian is invariant under the space rotations, so that
it can be represented as follows:

\[
H = \sum_{T, \mathbf{X}_i} a \left[ T(\mathbf{X}_i) \right] \sum_{l_{1},\ldots,l_{L}} T_{l_{1},\ldots,l_{L}}(\mathbf{X}_i) T_{l_{1},\ldots,l_{L}}(\mathbf{A}_i),
\]  

(5)

where \(\mathbf{X}_i\) are vector operators for the nucleus, \(\mathbf{A}_i\) are argument vectors for the emitted
particle, \(a \left[ T(\mathbf{X}_i) \right]\) are numerical coefficients and \(\sum_{T, \mathbf{X}_i}\) denotes the sum over various irredu-
cible tensors (including various \(L\)'s). In the theory of the angular correlation, the magnetic
quantum numbers play important parts, so that it is convenient to use the polarized solid harmonics\(^{2}\) \(\mathcal{O}_{L,j,m}\) normalized as follows:

\[
\sum_{M=-L}^{L} \mathcal{O}_{L,M}(\mathbf{X}_i) \mathcal{O}_{L,M}(\mathbf{A}_i) = \sum_{l_{1},\ldots,l_{L}} T_{l_{1},\ldots,l_{L}}(\mathbf{X}_i) T_{l_{1},\ldots,l_{L}}(\mathbf{A}_i).
\]  

(6)

These polarized solid harmonics have the following properties. Let \(j, m\) and \(a\) be the
total angular momentum, its \(z\)-component and other quantum numbers of the initial nucleus
respectively, and \(j', m'\) and \(a'\) be those of the final nucleus, then

\[
(\alpha' j' m' | \mathcal{O}_{L,j,m}(\mathbf{X}_i) | \alpha j m) = \mathcal{M}(\alpha, \alpha', j, j', \mathbf{X}_i) (jLMjM | jLj' m'),
\]  

(7)

where \(\mathcal{M}(\alpha, \alpha', j, j', \mathbf{X}_i)\) is a complex number independent of \(m\) and \(m'\), and is called
a reduced nuclear matrix element. \((JLM | JJ' m')\) are transformation coefficients for the vector addition, and real numbers. They vanish unless \(m + M = m'\), and this shows the selection rule for the magnetic quantum number.

From Eqs. (5) and (6)

\[
H = \sum_{X_i} a \left[ T(X_i) \right] \sum_{M} \mathcal{O}_{LM}(X_i) \mathcal{O}_{L'M'}(A_i).
\]

By Eqs. (7) and (8), Eq. (3) becomes

\[
\rho_{mm'}(\theta) = \sum_{X_i} a \left[ T(X_i) \right] \mathcal{M}(X_i) \left( JLM | JJ' m' \right) \mathcal{O}_{L'M'}(A_i) \]

with \(m + M = m'\).

In the right-hand side of Eq. (9) the \(\theta\)-dependent parts are \(\mathcal{O}_{L'M'}(A_i)\). Let \(\sum_{T(X_i)}\) denote the sum over various irreducible tensors retaining the \(L\) constant, then Eq. (9) becomes

\[
\rho_{mm'}(\theta) = \sum_{T(X_i)} a \left[ T(X_i) \right] \mathcal{M}(X_i) \left( JLM | JJ' m' \right) \mathcal{O}_{L'M'}(A_i) \]

with \(m + M = m'\).

We put

\[
F_{LL'}(\theta) = \sum_{T(X_i)} \mathcal{O}_{L'M'}(A_i) \mathcal{O}_{L'M'}(A_j) + c.c.,
\]

then Eq. (10) becomes

\[
\rho_{mm'}(\theta) = \sum_{L_1 \leq L_2} \left( JLM | JL' m' \right) \left( JL_1 m_1 | JL_2 m_2 \right) F_{LL'}(\theta)
\]

with \(m + M = m'\).

\(F_{LL'}(\theta)\) are called angular distribution functions. We obtain the angular correlation function by inserting Eq. (12) into Eq. (4).

In the formula (12) the transformation coefficients are determined entirely by the group-theoretical method, and the angular distribution functions \(F_{LL'}(\theta)\) are to be determined in each case. \(M\) takes the values from \(-L_1\) to \(L_1\). However, these \(2L_1 + 1\) functions are not independent of each other, and if we know a suitable one among them, the others can be known mathematically. The reason is as follows. As \(S \mathcal{O}_{L_1 \rightarrow 0} \mathcal{O}_{L_2 \rightarrow 0}\)

is the only part that is connected with \(\theta\), we observe that part only.

\[
S \mathcal{O}_{L_1 \to 0} \mathcal{O}_{L_2 \to 0} = S (-1)^M \mathcal{O}_{L_1 \to 0} \mathcal{O}_{L_2 \to 0} \mathcal{O}_{L_3 \to -M}
\]

\[
= S \sum_{j} (-1)^M (L_3 L_1 - MM) \mathcal{O}_{L_1 \to 0} \mathcal{O}_{L_2 \to 0}.
\]
There are following relations:

\[ S_{J_0}^{2L \pm M} = u_J Y_{J \pm 0}, \quad J: \text{even}, \]
\[ = 0, \quad J: \text{odd}, \]  

(14)

where \( u_J \) are independent of \( M \), and \( Y_{J0} \) are the spherical harmonics. We put \( J = 2n \), \( (n: \text{integer}) \) then

\[ S_{J_0}^{2L \pm M} = \sum_n (-1)^n (L_2 L_1 - M M | L_2 L_1 2n 0) u_{2n} Y_{2n 0}, \]

\[ L_2 - L_1 \leq 2n \leq L_2 + L_1, \]  

(15)

From Eqs. (11) and (15) we obtain the following parametrization:

\[ F_{L_1 L_2}^M (\theta) = \sum_n (-1)^n (L_2 L_1 - M M | L_2 L_1 2n 0) \delta_{L_1 L_2}^{(2n)} P_{2n} (\cos \theta), \]

\[ L_2 - L_1 \leq 2n \leq L_2 + L_1, \]  

(16)

where \( \delta_{L_1 L_2}^{(2n)} \) are parameters and \( P_{2n} (\cos \theta) \) are the Legendre polynomials:

\[ P_n (x) = 1/(2^n n!) \cdot (d/dx)^n \{ (x^2 - 1)^n \}. \]  

(17)

The first four of Eq. (17) (even \( n \)) are

\[ P_0 (\cos \theta) = 1, \]
\[ P_2 (\cos \theta) = (3/2) \cos^2 \theta - (1/2), \]
\[ P_4 (\cos \theta) = (35/8) \cos^4 \theta - (15/4) \cos^2 \theta + (3/8), \]
\[ P_6 (\cos \theta) = (231/16) \cos^6 \theta - (315/16) \cos^4 \theta + (105/16) \cos^2 \theta - (5/16). \]  

(18)

Therefore, the problem reduces to the determination of \( \delta_{L_1 L_2}^{(2n)} \).

We write Eq. (16) explicitly in several cases. We multiply \( \delta_{L_1 L_2}^{(2n)} \) by suitable numerical factors, and write \( a_{L_1 L_2}^{(2n)} \):

1) \( L_1 = 1, \ L_2 = 1 \)

\[ L_{11}^{\pm} (\theta) = a_{11}^{(0)} + a_{11}^{(T)} P_2 (\cos \theta), \]
\[ L_{11}^{0} (\theta) = a_{11}^{(0)} - 2 a_{11}^{(T)} P_2 (\cos \theta). \]  

(19)

2) \( L_1 = 1, \ L_2 = 2 \)

\[ L_{12}^{1} (\theta) = - L_{12}^{1} (\theta) = a_{12}^{(0)} P_3 (\cos \theta), \]
\[ L_{12}^{0} (\theta) = 0. \]  

(20)

3) \( L_1 = 2, \ L_2 = 2 \)

\[ L_{22}^{\pm} (\theta) = a_{22}^{(0)} + 2 a_{22}^{(T)} P_3 (\cos \theta) + a_{22}^{(S)} P_4 (\cos \theta), \]
\[ L_{22}^{\pm} (\theta) = a_{22}^{(0)} - 2 a_{22}^{(T)} P_3 (\cos \theta) - 4 a_{22}^{(S)} P_4 (\cos \theta), \]
\[ L_{22}^{0} (\theta) = a_{22}^{(0)} - 2 a_{22}^{(T)} P_3 (\cos \theta) + 6 a_{22}^{(S)} P_4 (\cos \theta). \]  

(21)

4) \( L_1 = 2, \ L_2 = 3 \)

\[ L_{23}^{0} (\theta) = - L_{23}^{0} (\theta) = 5 a_{23}^{(0)} P_4 (\cos \theta) + 2 a_{23}^{(S)} P_4 (\cos \theta), \]
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$$F_{13}^1(\theta) = - F_{23}^1(\theta) = \sqrt{10} a_{23}^{(0)} P_2(\cos \theta) - \sqrt{10} a_{23}^{(0)} P_4(\cos \theta),$$
$$F_{23}^0(\theta) = 0. \quad (22)$$

5) $L_1 = 3, L_2 = 3$

$$F_{33}^{\pm 1}(\theta) = a_{33}^{(0)} P_2(\cos \theta) + a_{33}^{(0)} P_4(\cos \theta),$$
$$F_{33}^{\pm 3}(\theta) = - 7 a_{33}^{(0)} P_4(\cos \theta) - 6 a_{33}^{(0)} P_6(\cos \theta),$$
$$F_{33}^{\pm 5}(\theta) = a_{33}^{(0)} - 3 a_{33}^{(0)} P_6(\cos \theta) + 15 a_{33}^{(0)} P_8(\cos \theta),$$
$$F_{33}^0(\theta) = a_{33}^{(0)} - 4 a_{33}^{(0)} P_6(\cos \theta) + 20 a_{33}^{(0)} P_8(\cos \theta). \quad (23)$$

The angular distribution functions of $\gamma$-rays are given in reference 7 for the dipole and the quadrupole radiations.

$$F_{1}^{(0)}(\theta) = 2|\alpha|^2\{1 - P_2(\cos \theta)\},$$
$$F_{2}^{(0)}(\theta) = 2|\beta|^2\{1 + (5/7) P_2(\cos \theta) - (12/7) P_4(\cos \theta)\},$$
$$F_{3}^{(0)}(\theta) = 2\sqrt{15} R_{12}(\alpha\beta^*) P_4(\cos \theta). \quad (24)$$

$\alpha$ and $\beta$ are the reduced nuclear matrix elements of the dipole and the quadrupole radiations respectively.

We obtain the angular correlation function $\mathcal{G}(\theta)$ from $F_{L1}M(\theta)$ above shown, and Eqs. (12) and (4). In this case the sums of the products of the transformation coefficients appear. There are several relations between these sums. Falkoff and Uhlenbeck$^2$ calculated these relations in some cases. By the same method we obtain them in other cases.

Let the spin changes of the nucleus be $j \rightarrow j \rightarrow j + dJ$. For simplicity we put

$$g_{il}^m = (j - dJ, i, m, 0) (j - dJ, i, j, m)^*,$$
$$d_{il}^m = (j - dJ, i, m - 1, 1) (j - dJ, i, j, m)^*,$$
$$e_{il}^m = (j - dJ, i, m - 2, 2) (j - dJ, i, j, m)^*,\quad f_{il}^m = (j - dJ, i, m - 3, 3) (j - dJ, i, j, m)^*,$$
$$d_{i+1}^{m+1} = (j - dJ, i, m - 1, 1) (j - dJ, i, j, m) (j - dJ, i, j, m) - (j - dJ, i, j, m)\times (j - dJ, i, j, m),$$
$$e_{i+1}^{m+1} = (j - dJ, i, m - 2, 2) (j - dJ, i, j, m) (j - dJ, i, j, m) - (j - dJ, i, j, m) (j - dJ, i, j, m) (j - dJ, i, j, m),$$
$$G_{il}^m = (j, i, m, 0) (j, i, j, m)^*,\quad D_{il}^m = (j, i, m, 1) (j, i, j, m)^*,\quad E_{il}^m = (j, i, m, 2) (j, i, j, m)^*,$$
$$D_{i+1}^{m+1} = (j, i, m, 1) (j, i, j, m)^*.\quad (25)$$
We attach \( ' \) to the second transition.

When the first transition involves \( L=1 \) and \( 2 \), and the second transition involves \( L=1 \) and \( 2 \), \( \mathcal{W}(\theta) \) is

\[
\mathcal{W}(\theta) = \sum_{m} \left[ g_{11}^{m} F_{11}^{(0)}(0) + d_{11}^{m} F_{11}^{(+1)}(0) + e_{11}^{m} F_{11}^{(-1)}(0) + d_{12}^{m} F_{12}^{(0)}(0) + e_{12}^{m} F_{12}^{(+1)}(0) \right] \\
\cdot \left[ g_{22}^{m} F_{22}^{(0)}(0) + d_{22}^{m} F_{22}^{(+1)}(0) + e_{22}^{m} F_{22}^{(-1)}(0) + d_{22}^{m} F_{22}^{(0)}(0) + e_{22}^{m} F_{22}^{(+1)}(0) \right] \\
+ \sum_{m} \left[ g_{11}^{m} F_{11}^{(0)}(0) + d_{11}^{m} F_{11}^{(+1)}(0) + e_{11}^{m} F_{11}^{(-1)}(0) + d_{12}^{m} F_{12}^{(0)}(0) + e_{12}^{m} F_{12}^{(+1)}(0) \right] \\
\cdot \left[ g_{22}^{m} F_{22}^{(0)}(0) + d_{22}^{m} F_{22}^{(+1)}(0) + e_{22}^{m} F_{22}^{(-1)}(0) + d_{22}^{m} F_{22}^{(0)}(0) + e_{22}^{m} F_{22}^{(+1)}(0) \right].
\]

(25)

When we take the \( z \)-axis as the direction of the second particle and the \( \theta \)-direction as that of the first particle, we obtain the same angular correlation function:

\[
\mathcal{W}(\theta) = \sum_{m} \left[ g_{11}^{m} F_{11}^{(0)}(\theta) + d_{11}^{m} F_{11}^{(+1)}(\theta) + e_{11}^{m} F_{11}^{(-1)}(\theta) + d_{12}^{m} F_{12}^{(0)}(\theta) + e_{12}^{m} F_{12}^{(+1)}(\theta) \right] \\
\cdot \left[ g_{22}^{m} F_{22}^{(0)}(\theta) + d_{22}^{m} F_{22}^{(+1)}(\theta) + e_{22}^{m} F_{22}^{(-1)}(\theta) + d_{22}^{m} F_{22}^{(0)}(\theta) + e_{22}^{m} F_{22}^{(+1)}(\theta) \right].
\]

(26)

The right-hand side of Eq. (26) is equal to that of Eq. (27) identically in regard to \( a_{12}^{(m)} \), \( a_{11}^{(m)} \), \( \theta \). From these conditions we obtain the following relations between sums. For simplicity we write \( g_{11}^{n} G_{11} \) etc. for \( \sum_{m} g_{11}^{m} G_{11}^{m} \) etc.

\[
\begin{align*}
g_{11}^{n} G_{11} & = (1/2) (d_{11}^{n} D_{11} - d_{11}^{n} G_{11}), \quad g_{11}^{n} D_{11} = d_{11}^{n} G_{11}, \\
g_{22}^{n} G_{11} & = (1/6) (5d_{22}^{n} G_{11} - d_{22}^{n} D_{11}), \quad g_{22}^{n} D_{11} = (1/3) (2d_{22}^{n} D_{11} - d_{22}^{n} G_{11}), \\
e_{22}^{n} G_{11} & = d_{22}^{n} D_{11} - d_{22}^{n} G_{11}, \quad e_{22}^{n} D_{11} = 2d_{22}^{n} G_{11}, \\
g_{11}^{n} G_{22} & = (1/6) (5g_{11}^{n} D_{22} - d_{11}^{n} G_{22}), \quad g_{11}^{n} E_{22} = d_{11}^{n} D_{22} - g_{11}^{n} D_{22}, \\
d_{11}^{n} G_{22} & = (1/3) (2d_{11}^{n} D_{22} - g_{11}^{n} D_{22}), \quad d_{11}^{n} E_{22} = 2g_{11}^{n} D_{22}, \\
g_{22}^{n} G_{22} & = (1/4) (2d_{22}^{n} D_{22} - d_{22}^{n} E_{22}), \quad g_{22}^{n} D_{22} = d_{22}^{n} E_{22}, \\
e_{22}^{n} G_{22} & = e_{22}^{n} G_{22} = (1/4) (-2d_{22}^{n} G_{22} + 3d_{22}^{n} E_{22}), \\
e_{22}^{n} E_{22} & = d_{22}^{n} E_{22}, \quad e_{22}^{n} E_{22} = (1/4) (6d_{22}^{n} G_{22} + 4d_{22}^{n} D_{22} - 3d_{22}^{n} E_{22}), \\
g_{11}^{n} D_{12} & = -d_{11}^{n} D_{12}, \\
d_{11}^{n} G_{11} & = -d_{11}^{n} D_{11}, \\
g_{22}^{n} D_{12} & = d_{22}^{n} D_{12} = -(1/2) e_{22}^{n} D_{12}, \\
d_{12}^{n} G_{22} & = d_{12}^{n} D_{22} = -(1/2) d_{12}^{n} E_{22}. 
\end{align*}
\]

(27)

(28)

Inserting Eqs. (19), (20), (21) and (28) into Eq. (26), we obtain the following comparatively simple form of the angular correlation function,

\[
\mathcal{W}(\theta) = \left\{ (3/2) a_{11}^{(0)} d_{11}^{(0)} \{ d_{11}^{n} G_{11} + d_{11}^{n} D_{11} \} + (5/2) a_{11}^{(0)} d_{11}^{(0)} \{ d_{12}^{n} G_{11} + d_{12}^{n} D_{11} \} \right\} \]

\[
\left\{ (5/2) a_{22}^{(0)} d_{22}^{(0)} \{ g_{11}^{n} D_{22} + d_{11}^{n} D_{22} \} + (5/2) a_{22}^{(0)} d_{22}^{(0)} \{ d_{22}^{n} G_{22} + d_{22}^{n} D_{22} + d_{22}^{n} E_{22} \} \right\}.
\]


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\[ W(\theta) = \left[ (5/2) a^{(0)}_{23} a^{(0)}_{11} \{ d_{23} G_{11} + d_{23} D_{11} \} + (7/2) a^{(0)}_{35} a^{(0)}_{11} \{ f_{35} G_{11} + f_{35} D_{11} \} \right. \]

\[ + (5/2) a^{(0)}_{23} a^{(0)}_{11} \{ d_{23} G_{32} + d_{23} D_{32} + d_{23} E_{32} \} \]

\[ + (7/2) a^{(0)}_{35} a^{(0)}_{11} \{ f_{35} G_{32} + f_{35} D_{32} + f_{35} E_{32} \} \]

\[ + [7 a^{(0)}_{35} a^{(0)}_{11} \{ 2 d_{23} G_{11} - d_{23} D_{11} \} - (42/5) a^{(0)}_{35} a^{(0)}_{11} \{ 2 f_{35} G_{11} - f_{35} D_{11} \} \]

\[ + 7 a^{(0)}_{35} a^{(0)}_{11} \{ 2 d_{23} G_{32} + d_{23} D_{32} - 2 d_{23} E_{32} \} \]

\[ - (42/5) a^{(0)}_{35} a^{(0)}_{11} \{ 2 f_{35} G_{32} + f_{35} D_{32} - 2 f_{35} E_{32} \} \]

When the first transition involves \( L=2 \) and \( 3 \), and the second transition involves \( L=1 \) and \( 2 \), the relations between sums are besides Eq. (28)

\[ g^{(35)}_{35} G_{11} = (1/10) \left( -f_{35} G_{11} + 3 f_{35} D_{11} \right), \quad g^{(35)}_{35} D_{11} = (1/5) \left( 3 f_{35} G_{11} + f_{35} D_{11} \right), \]

\[ d^{(35)}_{35} G_{11} = (1/15) \left( -f_{35} G_{11} + 8 f_{35} D_{11} \right), \quad d^{(35)}_{35} D_{11} = (1/15) \left( 16 f_{35} G_{11} + 7 f_{35} D_{11} \right), \]

\[ e^{(35)}_{35} G_{22} = (1/10) \left( 5 f_{35} G_{22} - 3 f_{35} D_{22} + 3 f_{35} E_{22} \right), \quad g^{(35)}_{35} G_{22} = (1/5) \left( 3 f_{35} G_{22} + 3 f_{35} D_{22} + 3 f_{35} E_{22} \right), \]

\[ d^{(35)}_{35} D_{22} = (1/15) \left( 7 f_{35} G_{22} + 8 f_{35} D_{22} \right), \quad e^{(35)}_{35} G_{22} = -f_{35} G_{22} + f_{35} D_{22} \]

\[ e^{(35)}_{35} D_{22} = (1/3) \left( 6 f_{35} G_{22} - 2 f_{35} D_{22} + 2 f_{35} E_{22} \right), \quad e^{(35)}_{35} E_{22} = (1/3) \left( 2 f_{35} D_{22} + f_{35} E_{22} \right), \]

\[ g^{(35)}_{35} E_{22} = (1/5) \left( 2 f_{35} G_{22} + 3 f_{35} D_{22} \right), \quad d^{(35)}_{35} D_{12} = - \frac{2}{5} f_{35} D_{12}, \quad d^{(35)}_{35} D_{12} = - \frac{3}{5} f_{35} D_{12}, \quad e^{(35)}_{35} D_{12} = 0, \]

\[ d^{(35)}_{35} E_{12} = - \frac{2}{5} f_{35} E_{12}, \quad e^{(35)}_{35} E_{12} = - \frac{2}{\sqrt{10}} \varepsilon^{(0)}_{35} D_{12}, \quad e^{(35)}_{35} G_{11} = - \varepsilon^{(0)}_{35} G_{11}, \]

\[ d^{(35)}_{35} G_{22} = 2 d^{(35)}_{35} E_{22} - \frac{5}{\sqrt{10}} \varepsilon^{(0)}_{35} E_{22}, \quad e^{(35)}_{35} G_{22} = (1/2) \left( - \sqrt{10} d^{(35)}_{35} E_{22} + \varepsilon^{(35)}_{35} E_{22} \right), \]

\[ d^{(35)}_{35} D_{22} = -3 d^{(35)}_{35} E_{22} + \frac{5}{\sqrt{10}} \varepsilon^{(0)}_{35} E_{22}, \quad e^{(35)}_{35} D_{22} = (1/2) \left( \sqrt{10} d^{(35)}_{35} E_{22} - 3 \varepsilon^{(35)}_{35} E_{22} \right), \]

\[ d^{(35)}_{35} D_{12} = - \frac{2}{\sqrt{10}} \varepsilon^{(0)}_{35} E_{12}. \]

The angular correlation function \( W(\theta) \) is

\( W(\theta) = \left[ (5/2) a^{(0)}_{23} a^{(0)}_{11} \{ d_{23} G_{11} + d_{23} D_{11} \} + (7/2) a^{(0)}_{35} a^{(0)}_{11} \{ f_{35} G_{11} + f_{35} D_{11} \} \right. \]

\[ + (5/2) a^{(0)}_{23} a^{(0)}_{11} \{ d_{23} G_{32} + d_{23} D_{32} + d_{23} E_{32} \} \]

\[ + (7/2) a^{(0)}_{35} a^{(0)}_{11} \{ f_{35} G_{32} + f_{35} D_{32} + f_{35} E_{32} \} \]

\[ + [7 a^{(0)}_{35} a^{(0)}_{11} \{ 2 d_{23} G_{11} - d_{23} D_{11} \} - (42/5) a^{(0)}_{35} a^{(0)}_{11} \{ 2 f_{35} G_{11} - f_{35} D_{11} \} \]

\[ + 7 a^{(0)}_{35} a^{(0)}_{11} \{ 2 d_{23} G_{32} + d_{23} D_{32} - 2 d_{23} E_{32} \} \]

\[ - (42/5) a^{(0)}_{35} a^{(0)}_{11} \{ 2 f_{35} G_{32} + f_{35} D_{32} - 2 f_{35} E_{32} \} \]
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\[+ (7/2) a_d^{(2)} a_d^{(3)} e_{m2} D_{12} + (42/5) a_{d3}^{(2)} a_d^{(3)} f_{m3} D_{12} + 21 a_{d3}^{(3)} e_{m3} D_{11}\]

\[+ (7/2) a_{d2}^{(3)} a_{d2}^{(3)} \{ \sqrt{10} d_{m2} E_{m2} + 5 e_{m2} E_{m2} \} + 7 a_{d2}^{(2)} a_{d2}^{(3)} e_{m2} D_{12} \] \[P_2(\cos \theta)\]

\[- (35/4) a_{d2}^{(4)} a_{d2}^{(4)} \{ 6 d_{m2} G_{m2} - 4 d_{m2} D_{m2} + d_{m2} E_{m2} \} \]

\[+ (77/3) a_{d3}^{(5)} a_{d3}^{(4)} \{ 6 f_{m3} G_{m3} - 4 f_{m3} D_{m3} + f_{m3} E_{m3} \} \]

\[- 35 a_{d3}^{(4)} a_{d3}^{(4)} \{ \sqrt{10} d_{m3} E_{m3} - 2 e_{m3} E_{m3} \} \] \[P_4(\cos \theta).\]

\[(31)\]

The independent sums that appear in Eqs. (29) and (31) can be calculated with the table of Condon and Shortley\textsuperscript{14} and the Appendix.

\section*{§ 3. Angular distribution functions of $\beta$-ray}

In this section we calculate the explicit forms of the angular distribution functions defined by Eq. (11) for the $\beta$-rays up to the second forbidden. We treat each five interaction type of the Fermi theory separately and do not consider the case of mixed interaction types.

The angular distribution functions are obtained as follows. At first we calculate the matrix elements of the interaction Hamiltonian corresponding to the definite change of the magnetic quantum number of the nucleus. We denote them with $H_{x,m}$, where $x$ and $m$ are the quantum numbers of the emitted electron. Neglecting the common factor, the asymptotic wave function of the emitted electron is\textsuperscript{18}

\[\psi = \sum_{x,m} H_{x,m} \psi_{x,m}.\]

\[(32)\]

$\psi_{x,m}$ are obtained by extracting the out-going waves, the parts containing $\exp(ikr')$, from the asymptotic forms of the eigenfunctions (standing waves) that contain the trigonometrical functions. We construct $\psi^* \psi$ from Eq. (32), sum over all the neutrino states and compare with Eq. (12), then the angular distribution functions are obtained.

We assume that the neutrino mass is zero. Then the Hamiltonian of a free neutrino commutes with $\gamma_5$. (We take $\gamma_5 = -iu_i u_j$ as Konopinski and Uhlenbeck.\textsuperscript{16}) From this fact the angular distribution for any nuclear matrix element is the same as that for the nuclear matrix element which is obtained from the former by exchanging $\gamma_5$ and $\alpha$ for $\sigma$ simultaneously. For example the angular distributions for $|\mathcal{M}(A_{12}^0)|^2$ and $|\mathcal{M}(B_{12}^0)|^2$ are the same. The angular distribution function for any nuclear matrix element containing $\beta$ (Dirac matrix) is obtained from the angular distribution function for that without $\beta$ by replacing $K$ (neutrino energy) by $-K$.

We use the following properties of the phases of the nuclear matrix elements. The reduced nuclear matrix elements for the $\beta$-decay defined by Eq. (7) are divided into two classes; the nuclear matrix elements belonging to the same class have the same phase (or phase difference $\pi$), and the phase differences between those belonging to the different classes are $\pi/2$ (or $3\pi/2$). Up to the second forbidden the two classes are as follows: the one consists of $\mathcal{M}(1)$, $\mathcal{M}(\beta)$, $\mathcal{M}(\gamma_5)$, $\mathcal{M}(\alpha)$, $\mathcal{M}(\sigma)$, $\mathcal{M}(\beta\sigma)$, $\mathcal{M}(\beta\alpha)$, $\mathcal{M}(\sigma \times r)$, $\mathcal{M}(\beta\sigma \times r)$, $\mathcal{M}(\beta\alpha \times r)$, $\mathcal{M}(\alpha \times r)$, $\mathcal{M}(R_{12})$, $\mathcal{M}(R_{12}^0)$, $\mathcal{M}(S_{1jk})$, $\mathcal{M}(S_{ijk})$ and the other...
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consists of \( M(\sigma \cdot r) \), \( M(\beta \sigma \cdot r) \), \( M(\beta \alpha \cdot r) \), \( M(\beta r) \), \( M(\beta r) \), \( M(A) \), \( M(A_t) \), \( M(T_d) \), \( M(T_\beta) \). For the tensors of the same rank the proof has already been given by Longmire and Messiah. For the tensors of the different ranks the proof is almost the same.

As an example, we consider \( A_{ij}^a \) and \( S_{ijkl}^a \). By the time reversal (\( \mathfrak{T} \) is the time reversal operator), the wave functions \( \Phi_j^M \) and \( \Phi_j'^{M'} \), \( (J \text{ and } J' \text{ are the total angular momenta, and } M \text{ and } M' \text{ are their } z \text{-components}) \), and the operators* \( A_{ij}^a \) and \( S_{ijkl}^a \) transform as follows:

\[
\mathfrak{T} \Phi_j^M = \epsilon^\varphi \Phi_j^{-M}, \quad \mathfrak{T} \Phi_j'^{M'} = \epsilon^\psi \Phi_j'^{-M'},
\]
\[
\mathfrak{T} A_{ij}^a = -A_{ij}^a, \quad \mathfrak{T} S_{ijkl}^a = -S_{ijkl}^a,
\]

where \( \varphi \) and \( \psi \) are arbitrary phases. The operator \( \mathfrak{T} \) can be represented as follows:

\[
\mathfrak{T} = U \mathfrak{T}_0.
\]

where \( \mathfrak{T}_0 \) transforms the wave function into its complex conjugate and \( U \) is a unitary operator. From Eqs. (33) and (34)

\[
A_{ij}^{M,M'} = \langle \Phi_j^{M'} | A_{ij}^a | \Phi_j^M \rangle = \langle \mathfrak{T} \Phi_j'^{M'} , \mathfrak{T} A_{ij}^a , \mathfrak{T} \Phi_j^M \rangle^* = -e^{i(\varphi - \psi)} \langle \Phi_j'^{-M'} | A_{ij}^a | \Phi_j^{-M} \rangle^* = -e^{i(\varphi - \psi)} A_{ij}^{-M,-M'}^*.
\]

Similarly

\[
S_{ijkl}^{M,M'} = -e^{i(\varphi - \psi)} S_{ijkl}^{-M,-M'}^*.
\]

On the other hand the transformation coefficients have the following relations:

\[
(JLMm | JLJ'm') = (-1)^{M - M' - L} (JL - m - M | JLJ' - m').
\]

Comparing Eqs. (7), (35), (36) and (37), we obtain

\[
\frac{M(A_{ij}^a)}{M(S_{ijkl}^a)} = -\frac{M^*(A_{ij}^a)}{M^*(S_{ijkl}^a)}.
\]

This indicates that the phases of \( M(A_{ij}^a) \) and \( M(S_{ijkl}^a) \) are different by \( \pi/2 \) (or \( 3\pi/2 \)) from each other.

Using the above results we obtain the angular distribution functions:

1) Allowed
   all isotropic.

2) 1st forbidden
   a) scalar

\[
F_1^\alpha(\theta) = |M(\beta r)|^2 \left[ \{ (1/3) K^3 L_0 + (2/3) KN_0 + 2L_1 + M_0 \} + \{ (4/3) KL_0 + 2L_1 + 4N_0 \} P_2(\cos \theta) \right].
\]

* Although Konopinski and Uhlenbeck have used this notation for matrix elements, we use it here for operators.
b) vector

\[ F_{10}^{2}(\theta) = |\mathfrak{M}(\alpha)|^2 L_0 + |\mathfrak{M}(r)|^2 \left\{ \left( \frac{1}{3} \right) K^2 L_0 - \frac{2}{3} K N_0 + 2 L_1 + M_6 \right\} \]

\[ + \left\{ - \left( \frac{4}{3} \right) K L_{12} + 2 L_1 + 4 N_{12} \right\} P_2(\cos \theta) \]

\[ - i \mathfrak{M}^* (\alpha) \mathfrak{M}(r) + c.c. \left\{ \left( \frac{1}{3} \right) K L_0 - N_0 - 2 L_{12} P_2(\cos \theta) \right\}. \] (V1)

c) tensor

\[ F_{00}^{2}(\theta) = |\mathfrak{M}(\beta \sigma \cdot r)|^2 \left\{ \left( \frac{1}{9} \right) K^2 L_0 + \frac{2}{3} K N_0 + \frac{1}{2} M_6 \right\}, \]

\[ F_{11}^{2}(\theta) = |\mathfrak{M}(\beta \alpha)|^2 L_0 + |\mathfrak{M}(\beta \sigma \times r)|^2 \left\{ \left( \frac{1}{6} \right) K^2 L_0 - \frac{1}{2} K N_0 + \frac{1}{2} L_1 + \frac{1}{2} M_6 \right\} \]

\[ + \left\{ - \left( \frac{2}{3} \right) K L_{12} + \frac{1}{2} L_1 - 2 N_{12} \right\} P_2(\cos \theta) \]

\[ - \left\{ \mathfrak{M}^* (\beta \alpha) \mathfrak{M}(\beta \sigma \times r) + c.c. \right\} \left\{ \left( \frac{1}{3} \right) K L_0 - N_0 + L_{12} P_2(\cos \theta) \right\}, \]

\[ F_{20}^{2}(\theta) = |\mathfrak{M}(\beta \alpha)|^2 \left\{ \left( \frac{1}{12} \right) K^2 L_0 + \frac{1}{2} L_1 \right\} + \frac{3}{4} L_1 P_2(\cos \theta) \],

\[ F_{22}^{2}(\theta) = |\mathfrak{M}(\beta \sigma \cdot r)\mathfrak{M}(B_{ij}) + c.c.| \left( \frac{1}{3} \right) K L_{12} + \frac{1}{3} N_{12} P_2(\cos \theta), \]

\[ F_{12}^{2}(\theta) = - \left\{ i \mathfrak{M}^* (\beta \sigma \times r) \mathfrak{M}(B_{ij}) + c.c. \right\} \frac{1}{6} L_{12} P_2(\cos \theta) \]

\[ - \left\{ i \mathfrak{M}^* (\beta \sigma \cdot r) \mathfrak{M}(B_{ij}) + c.c. \right\} \frac{1}{6} L_{12} P_2(\cos \theta). \]

\[ (T1) \]

d) pseudovector

\[ F_{00}^{2}(\theta) = |\mathfrak{M}(\gamma_5)|^2 L_0 + |\mathfrak{M}(\sigma \cdot r)|^2 \left\{ \left( \frac{1}{9} \right) K^2 L_0 - \frac{1}{3} K N_0 + \frac{1}{2} M_6 \right\} \]

\[ - i \mathfrak{M}^* (\gamma_5) \mathfrak{M}(\sigma \cdot r) + c.c. \left\{ \left( \frac{1}{3} \right) K L_0 - N_0 \right\}, \]

\[ F_{11}^{2}(\theta) = |\mathfrak{M}(\sigma \times r)|^2 \left\{ \left( \frac{1}{6} \right) K^2 L_0 + \frac{1}{2} K N_0 + \frac{1}{2} L_1 + \frac{1}{2} M_6 \right\} \]

\[ + \left\{ - \left( \frac{2}{3} \right) K L_{12} + \frac{1}{2} L_1 - 2 N_{12} \right\} P_2(\cos \theta), \]

\[ F_{22}^{2}(\theta) = |\mathfrak{M}(B_{ij})|^2 \left\{ \left( \frac{1}{12} \right) K^2 L_0 + \frac{1}{2} L_1 \right\} + \frac{3}{4} L_1 P_2(\cos \theta), \]

\[ F_{12}^{2}(\theta) = \left\{ i \mathfrak{M}^* (\gamma_5) \mathfrak{M}(B_{ij}) + c.c. \right\} \left( \frac{1}{3} \right) K L_{12} + \frac{1}{3} N_{12} P_2(\cos \theta), \]

\[ \left\{ i \mathfrak{M}^* (\sigma \cdot r) \mathfrak{M}(B_{ij}) + c.c. \right\} \left( \frac{1}{3} \right) K L_{12} + \frac{1}{3} N_{12} P_2(\cos \theta). \]

\[ (A1) \]
e) pseudoscalar (We define \( \mathfrak{M}(\beta \gamma_5) \) as the 1st forbidden).

\[ F_{00}^{2}(\theta) = |\mathfrak{M}(\beta \gamma_5)|^2 L_0. \]

\[ (P1) \]

3) 2nd forbidden

a) scalar

\[ F_{20}^{2}(\theta) = |\mathfrak{M}(R_{ij})|^2 \left\{ \left( \frac{1}{30} \right) K^4 L_0 + \frac{2}{15} \right\} K^2 N_0 + \frac{2}{3} K^2 L_1 + \frac{1}{3} K^2 M_6 \]

\[ + 2 K N_1 + \frac{9}{2} L_1 + 3 M_1 \right\} + \left\{ \frac{2}{3} \right\} K^2 L_{12} + \frac{1}{3} K^2 L_1 + \frac{2}{3} K^2 N_{12} \]

\[ + 2 K N_1 + \frac{9}{2} L_1 + 3 M_1 \right\} + \left\{ \frac{2}{3} \right\} K^2 L_{12} + \frac{1}{3} K^2 L_1 + \frac{2}{3} K^2 N_{12} \]
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\[
+ \frac{6}{7} KL_{23} + 2 KN_1 + \frac{36}{7} L_2 + 3 M_1 + \frac{18}{7} N_{23} \right] P_2(\cos \theta)
\]
\[+ \left[ \frac{36}{7} KL_{23} + \frac{27}{7} L_2 + \frac{108}{7} N_{23} \right] P_4(\cos \theta) \].

(S2)

b) vector

\[
F_{11}^a(\theta) = \mathcal{M}(\alpha \times \mathbf{r})^a \left[ \left\{ \frac{1}{6} K^2 L_0 + \frac{2}{3} K N_0 + \frac{1}{2} L_1 + M_0 \right\} + \left\{ -\frac{2}{3} KL_{12} + \frac{1}{2} L_1 - 2 N_{12} \right\} P_2(\cos \theta) \right],
\]
\[
F_{22}^a(\theta) = \mathcal{M}(A_{ij})^a \left[ \left\{ \frac{1}{12} K^2 L_0 + \frac{3}{4} L_1 + \frac{3}{4} L_1 P_2(\cos \theta) \right\} + \left\{ \frac{1}{30} K^2 N_0 + \frac{1}{2} K L_1 - \frac{3}{2} N_1 \right\} P_2(\cos \theta) \right]
\]
\[
- \frac{2}{3} KL_{12} + \frac{1}{2} K L_1 - K L_1 S - 3 N_1 \right] P_2(\cos \theta).
\]

(V2)

c) tensor

\[
F_{11}^b(\theta) = \mathcal{M}(\beta \alpha \cdot \mathbf{r})^b \left[ \left\{ \frac{1}{9} K^2 L_0 + \frac{2}{3} K N_0 + M_0 \right\} + \left\{ (1/2) K^2 L_0 + (3/4) L_1 + (3/4) L_1 P_2(\cos \theta) \right\} + \left\{ (1/15) K^2 N_0 + K^2 L_1 + K^2 M_0 \right\}
\]
\[
- 6 K N_1 + 6 L_2 + 9 M_1 \right] + \left\{ K^2 L_1 + 12/7) KL_{23} - 6 K N_1 + (48/7) L_2 \right] + \left\{ 9 M_1 - (36/7) N_{23} \right\} P_2(\cos \theta)
\]
\[
+ \left\{ (72/7) KL_{23} + (36/7) L_2 - (216/7) N_{23} \right\} P_4(\cos \theta) \right],
\]
\[
F_{22}^b(\theta) = \mathcal{M}(S_{ij})^b \left[ \left\{ \frac{1}{72} \right\} + \left\{ (1/15) K^2 L_0 + 2 K^2 L_1 + 15 L_2 \right\} + \left\{ (8/5) K^2 L_1 + (120/7) L_2 \right\} P_2(\cos \theta) + \left\{ (90/7) L_2 P_4(\cos \theta) \right\}
\]
\[
+ \left\{ (36/7) L_2 P_4(\cos \theta) \right\},
\]
\[
F_{33}^b(\theta) = \mathcal{M}(T_{ij})^b \left[ \left\{ \frac{1}{72} \right\} + \left\{ (1/15) K^2 L_0 + 2 K^2 L_1 + 15 L_2 \right\} + \left\{ (8/5) K^2 L_1 + (120/7) L_2 \right\} P_2(\cos \theta) + \left\{ (90/7) L_2 P_4(\cos \theta) \right\}
\]
\[
+ \left\{ (36/7) L_2 P_4(\cos \theta) \right\},
\]
\[
F_{44}^b(\theta) = \mathcal{M}(T_{ij})^b \left[ \left\{ \frac{1}{72} \right\} + \left\{ (1/15) K^2 L_0 + 2 K^2 L_1 + 15 L_2 \right\} + \left\{ (8/5) K^2 L_1 + (120/7) L_2 \right\} P_2(\cos \theta) + \left\{ (90/7) L_2 P_4(\cos \theta) \right\}
\]
\[
+ \left\{ (36/7) L_2 P_4(\cos \theta) \right\},
\]
\[
F_{21}^2(\theta) = -\{i\mathcal{M}^* (A_{ij}) \mathcal{M} (S_{ijk}) + \text{c.c.}\} \left( \sqrt{10} / \sqrt{3} \right) \left\{ \left( 1/60 \right) K^2 L_{12} + \left( 3/28 \right) L_{21} \right\} P_2(\cos \theta) + \left( 15/56 \right) L_{23} P_4(\cos \theta) \\
-\{i\mathcal{M}^* (T_{ij}) \mathcal{M} (S_{ijk}) + \text{c.c.}\} \left( \sqrt{10} / \sqrt{3} \right) \left\{ \left( 1/300 \right) ( - K^2 L_{12} + 5 K^2 N_{13} ) + \left( 1/28 \right) ( - K L_{23} + 2 L_2 + 3 N_{23} ) \right\} P_2(\cos \theta) \\
+ \left( 5/56 \right) \left\{ - K L_{23} + 2 L_2 + 3 N_{23} \right\} P_4(\cos \theta) .
\]

The notation is almost the same as that of Konopinski and Uhlenbeck,\(^6\) but we write \( \beta \) explicitly for the nuclear matrix elements including it. Moreover, we use the following new notation, which is necessary to describe the interference terms between different \( \phi_n \)'s.

\[
L_{12} = \left( \frac{\rho^2}{2\pi} F \right)^{-1} G_{-1} f_2 \cos ( \delta_{-1} - \delta_2 ) - f_2 G_{-2} \cos ( \delta_1 - \delta_2 ) \frac{4\pi \rho}{3 \sqrt{W}}, \\
L_{13} = \left( \frac{\rho^2}{2\pi} F \right)^{-1} G_{-1} G_{-3} \cos ( \delta_{-1} - \delta_3 ) + f_1 f_3 \cos ( \delta_1 - \delta_3 ) \frac{4\pi \rho^2}{15}, \\
L_{23} = \left( \frac{\rho^3}{2\pi} F \right)^{-1} G_{-3} f_3 \cos ( \delta_{-3} - \delta_3 ) - f_2 G_{-3} \cos ( \delta_2 - \delta_3 ) \frac{4\pi \rho^3}{45 \sqrt{W}}.
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\[ M_{12} = \left( \frac{\rho^2}{F} \right)^{-1} f_{-1} g_{-2} \cos(\delta_1 - \delta_2) - g_{-1} f_{-2} \cos(\delta_1 - \delta_2) \]
\[ \frac{4\pi \rho^3}{45 W} + \frac{\rho^3}{6 W} \frac{u Z^2}{4 \rho^3} + \frac{11 \rho^3}{90} \frac{u Z}{2 \rho} , \]

\[ N_{12} = \left( \frac{\rho^2}{2 \pi} \right)^{-1} f_{-1} g_{-2} \cos(\delta_1 - \delta_2) + g_{-1} f_{-2} \cos(\delta_1 - \delta_2) \]
\[ \frac{4\pi \rho^2}{21 \pi} + \frac{\rho^3}{3 \pi} \frac{u Z}{2 \rho} , \]

\[ N_{21} = \left( \frac{\rho^2}{2 \pi} \right)^{-1} g_{-1} g_{-2} \cos(\delta_1 - \delta_2) - f_{-1} f_{-2} \cos(\delta_1 - \delta_2) \]
\[ \frac{4\pi \rho^2}{21 \pi} + \frac{\rho^3}{3 \pi} \frac{u Z}{2 \rho} , \]

\[ N_{32} = \left( \frac{\rho^2}{2 \pi} \right)^{-1} f_{-2} f_{-3} \cos(\delta_1 - \delta_2) + g_{-2} g_{-3} \cos(\delta_1 - \delta_2) \]
\[ \frac{4\pi \rho^2}{21 \pi} + \frac{\rho^3}{90} \frac{u Z}{2 \rho} . \]

\[ (39) \]

In the above definition we mainly follow the notation of Konopinski and Uhlenbeck and Rose, but it should be noted that the suffixes of \( f \) and \( g \) are not those of Rose but quantum numbers \( x \). It is more natural and convenient to use the suffixes \( x \). The relation between the two suffixes is shown in Table I. \( \delta_x \) are the phases of the asymptotic wave functions. \( \rightarrow \) denotes the approximation which is valid for \( u Z W/\rho \ll 1 \). When this approximation can not be used, the calculation of Eq. (39) is considerably complicated, and moreover, for the heavy nuclei it is necessary to change \( f_x \) and \( g_x \) from the Coulomb wave functions and to take into account the effect of the finite nuclear size. As usual the values of \( f_x \) and \( g_x \) are to be taken at \( r = \rho \). (\( \rho \) is the nuclear radius.) From Eq. (39) it can be understood that \( M_{12} \) and \( N_{12} \) are strongly affected by the Coulomb field.

Finally we mention about the relation between the reduced nuclear matrix elements used here and the nuclear matrix elements used commonly in the theory of β-spectra. The latter ones have the forms of

\[ \sum_{i_1, \ldots, i_k} \sum_{j' m'} (j' m'| T_{i_1 \ldots i_k} | j m) \ast (j' m'| T_{i_1' \ldots i_k'} | j m) . \]
By Eq. (6) this equals to
\[ \sum_{m} \sum_{m'} (f' m' | Y_{LM} | m)^* (f' m' | Y_{LM} | m). \]
By Eq. (7) this equals to
\[ \mathbb{M}^*(T_{t_1 \ldots t_2}) \mathbb{M}(T'_{t_1 \ldots t_2}) \sum_{m'} (fLMm | JJ' m')^2 = (2J' + 1)/(2J + 1) \cdot \mathbb{M}^*(T_{t_1 \ldots t_2}) \mathbb{M}(T'_{t_1 \ldots t_2}). \]

For example,
\[ \sum A_{ij}^a T_{ij}^a = (2J' + 1)/(2J + 1) \cdot \mathbb{M}^*(A_{ij}^a) \mathbb{M}(T_{ij}^a). \]

(2J' + 1)/(2J + 1) is common to all the terms, so that this factor need not be considered in the angular correlation. Therefore, when we study the \( \bar{\beta} \)-spectra the factors such as \( \sum A_{ij}^a T_{ij}^a \) may be replaced by the factors such as \( \mathbb{M}^*(A_{ij}^a) \mathbb{M}(T_{ij}^a) \).

Comparison with the experiments will be reported in the subsequent paper.

The authors wish to express their sincere thanks to Professor S. Nakamura and Mr. M. Umezawa for their kind guidance, and to Professor T. Miyazima for the suggestion of this problem. They are also indebted to Mr. H. Taneichi for his valuable assistance.

**Appendix**

We calculate the transformation coefficients \( (j_1 j_2 m_1 m_2 | j_3 j_4 m_3 j_4 m_4) \), using the formula in reference 14 p. 75:
\[ (j_1 j_2 m_1 m_2 | j_3 j_4 m_3 j_4 m_4) = \delta(m_1 + m_2) \sqrt{\frac{(j_1 + j_2 - j_3)!}{(j_1 - j_3 - j_2)! (j_1 + j_2 + j_3)! (j_1 + m_1)! (j_2 + m_2)!}} \times \frac{(j_2 + j_4 - j_3)! (j_2 - j_3 - j_4)! (j_2 + m_2)! (j_2 + m_4)!}{(j_2 + j_3 + j_4)! (j_2 - j_3 - j_4)! (j_2 + m_2)! (j_2 + m_4)!} \]

The results are shown in the following table.

<table>
<thead>
<tr>
<th>( j = j_1 + 3 ),</th>
<th>( m_2 = 3, \sqrt{3} )</th>
<th>( \sqrt{6} )</th>
<th>( \sqrt{15} )</th>
<th>( \sqrt{20} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( (j_1 - m + 2)(j_1 + m - 1)(j_1 + m)(j_1 + m + 1)(j_1 + m + 2) )</td>
<td>( (j_1 - m + 2)(j_1 + m)(j_1 + m + 1)(j_1 + m + 2)(j_1 + m + 3) )</td>
<td>( (j_1 - m + 2)(j_1 + m)(j_1 + m + 1)(j_1 + m + 2)(j_1 + m + 3) )</td>
<td>( (j_1 - m + 2)(j_1 + m)(j_1 + m + 1)(j_1 + m + 2)(j_1 + m + 3) )</td>
<td></td>
</tr>
<tr>
<td>( (2j_1 + 1)(2j_1 + 2)(2j_1 + 3)(2j_1 + 4)(2j_1 + 5)(2j_1 + 6) )</td>
<td>( (2j_1 + 1)(2j_1 + 2)(2j_1 + 3)(2j_1 + 4)(2j_1 + 5)(2j_1 + 6) )</td>
<td>( (2j_1 + 1)(2j_1 + 2)(2j_1 + 3)(2j_1 + 4)(2j_1 + 5)(2j_1 + 6) )</td>
<td>( (2j_1 + 1)(2j_1 + 2)(2j_1 + 3)(2j_1 + 4)(2j_1 + 5)(2j_1 + 6) )</td>
<td></td>
</tr>
</tbody>
</table>
On the β-Ray Angular Correlations

\[ m^2 = -1, \sqrt{\frac{15}{N}} \left( \frac{j_i - m + 3}{2j_i + 1} \right) \left( \frac{j_i - m + 2}{2j_i + 2} \right) \left( \frac{j_i - m + 1}{2j_i + 3} \right) \left( \frac{j_i - m}{2j_i + 4} \right) \left( \frac{j_i + m + 3}{2j_i + 5} \right) \left( \frac{j_i + m + 2}{2j_i + 6} \right) \]

\[ m^2 = -2, \sqrt{\frac{6}{N}} \left( \frac{j_i - m + 3}{2j_i + 1} \right) \left( \frac{j_i - m + 2}{2j_i + 2} \right) \left( \frac{j_i - m + 1}{2j_i + 3} \right) \left( \frac{j_i - m}{2j_i + 4} \right) \left( \frac{j_i + m + 3}{2j_i + 5} \right) \left( \frac{j_i + m + 2}{2j_i + 6} \right) \]

\[ m^2 = -3, \sqrt{\frac{6}{N}} \left( \frac{j_i - m + 3}{2j_i + 1} \right) \left( \frac{j_i - m + 2}{2j_i + 2} \right) \left( \frac{j_i - m + 1}{2j_i + 3} \right) \left( \frac{j_i - m}{2j_i + 4} \right) \left( \frac{j_i + m + 2}{2j_i + 5} \right) \left( \frac{j_i + m + 1}{2j_i + 6} \right) \]

\[ j = j_i + 2 \]

\[ m^2 = -1, \sqrt{\frac{15}{N}} \left( \frac{j_i + m - 3}{2j_i + 1} \right) \left( \frac{j_i + m - 2}{2j_i + 2} \right) \left( \frac{j_i + m - 1}{2j_i + 3} \right) \left( \frac{j_i + m}{2j_i + 4} \right) \left( \frac{j_i + m + 2}{2j_i + 5} \right) \left( \frac{j_i + m + 3}{2j_i + 6} \right) \]

\[ m^2 = -2, \sqrt{\frac{6}{N}} \left( \frac{j_i + m - 3}{2j_i + 1} \right) \left( \frac{j_i + m - 2}{2j_i + 2} \right) \left( \frac{j_i + m - 1}{2j_i + 3} \right) \left( \frac{j_i + m}{2j_i + 4} \right) \left( \frac{j_i + m + 2}{2j_i + 5} \right) \left( \frac{j_i + m + 3}{2j_i + 6} \right) \]

\[ m^2 = -3, \sqrt{\frac{6}{N}} \left( \frac{j_i + m - 3}{2j_i + 1} \right) \left( \frac{j_i + m - 2}{2j_i + 2} \right) \left( \frac{j_i + m - 1}{2j_i + 3} \right) \left( \frac{j_i + m}{2j_i + 4} \right) \left( \frac{j_i + m + 2}{2j_i + 5} \right) \left( \frac{j_i + m + 3}{2j_i + 6} \right) \]

\[ j = j_i + 1 \]

\[ m^2 = 0, 2 \sqrt{\frac{30m}{N}} \left( \frac{j_i + m - 3}{2j_i + 1} \right) \left( \frac{j_i + m - 2}{2j_i + 2} \right) \left( \frac{j_i + m - 1}{2j_i + 3} \right) \left( \frac{j_i + m}{2j_i + 4} \right) \left( \frac{j_i + m + 2}{2j_i + 5} \right) \left( \frac{j_i + m + 3}{2j_i + 6} \right) \]

\[ m^2 = -1, \sqrt{\frac{15}{N}} \left( \frac{j_i + m - 3}{2j_i + 1} \right) \left( \frac{j_i + m - 2}{2j_i + 2} \right) \left( \frac{j_i + m - 1}{2j_i + 3} \right) \left( \frac{j_i + m}{2j_i + 4} \right) \left( \frac{j_i + m + 2}{2j_i + 5} \right) \left( \frac{j_i + m + 3}{2j_i + 6} \right) \]

\[ m^2 = -2, 2 \left( \frac{j_i + m - 3}{2j_i + 1} \right) \left( \frac{j_i + m - 2}{2j_i + 2} \right) \left( \frac{j_i + m - 1}{2j_i + 3} \right) \left( \frac{j_i + m}{2j_i + 4} \right) \left( \frac{j_i + m + 2}{2j_i + 5} \right) \left( \frac{j_i + m + 3}{2j_i + 6} \right) \]

\[ m^2 = -3, \sqrt{\frac{6}{N}} \left( \frac{j_i + m - 3}{2j_i + 1} \right) \left( \frac{j_i + m - 2}{2j_i + 2} \right) \left( \frac{j_i + m - 1}{2j_i + 3} \right) \left( \frac{j_i + m}{2j_i + 4} \right) \left( \frac{j_i + m + 2}{2j_i + 5} \right) \left( \frac{j_i + m + 3}{2j_i + 6} \right) \]

\[ j = j_i + 1 \]

\[ m^2 = 0, 2 \sqrt{\frac{30m}{N}} \left( \frac{j_i + m - 3}{2j_i + 1} \right) \left( \frac{j_i + m - 2}{2j_i + 2} \right) \left( \frac{j_i + m - 1}{2j_i + 3} \right) \left( \frac{j_i + m}{2j_i + 4} \right) \left( \frac{j_i + m + 2}{2j_i + 5} \right) \left( \frac{j_i + m + 3}{2j_i + 6} \right) \]

\[ m^2 = 1, \frac{-(j_i^2 + 10j_im - 15m^2 - 3j_i + 25m - 10)}{(2j_i + 1)(2j_i + 2)(2j_i + 3)(2j_i + 4)(2j_i + 5)} \]

\[ m^2 = 0, -2 \sqrt{\frac{3}{N}} \left( \frac{j_i^2 - 5m^2 + 2j_i}{2j_i + 1} \right) \frac{-(j_i + m - 1)(j_i + m + 1)}{(2j_i + 1)(2j_i + 2)(2j_i + 3)(2j_i + 4)(2j_i + 5)} \]
\[ m_s = -1, \quad -(j_1^2 - 10j_1m - 15m^2 - 3j_1 - 25m - 10) \]
\[ \sqrt{(j_1 - m + 1)(j_1 - m)} \]
\[ \sqrt{(2j_1 - 1)2j_1(2j_1 + 1)(2j_1 + 2)(2j_1 + 4)(2j_1 + 5)} \]
\[ m_s = -2, \quad \sqrt{10}(j_1 + 3m + 4) \sqrt{(j_1 - m + 1)(j_1 - m - 1)(j_1 + m + 2)} \]
\[ \sqrt{(2j_1 - 1)2j_1(2j_1 + 1)(2j_1 + 2)(2j_1 + 4)(2j_1 + 5)} \]
\[ m_s = -3, \quad \sqrt{15}(j_1 + m + 3)(j_1 + m + 2)(j_1 - m - 2)(j_1 - m - 1)(j_1 - m)(j_1 - m + 1) \]
\[ \sqrt{(2j_1 - 1)2j_1(2j_1 + 1)(2j_1 + 2)(2j_1 + 4)(2j_1 + 5)} \]

\[ j = j_1 \]

\[ m_s = 3, \quad -2\sqrt{5} \sqrt{(j_1 - m + 3)(j_1 - m - 1)(j_1 + m)(j_1 + m - 1)(j_1 + m - 2)} \]
\[ \sqrt{(2j_1 - 2)(2j_1 - 1)2j_1(2j_1 + 2)(2j_1 + 3)(2j_1 + 4)} \]

\[ m_s = 2, 2(m - 1) \sqrt{30} \sqrt{(j_1 - m + 2)(j_1 - m + 1)(j_1 + m)(j_1 + m - 1)} \]
\[ \sqrt{(2j_1 - 2)(2j_1 - 1)2j_1(2j_1 + 2)(2j_1 + 3)(2j_1 + 4)} \]

\[ m_s = 1, \quad 2\sqrt{2} [j_1(j_1 + 1) - 5m(m + 1) - 2] \]
\[ \sqrt{(2j_1 - 2)(2j_1 - 1)2j_1(2j_1 + 2)(2j_1 + 3)(2j_1 + 4)} \]

\[ m_s = 0, \quad -4m(3j_1^2 - 5m^2 + 3j_1 - 1) \]
\[ \frac{1}{\sqrt{(2j_1 - 2)(2j_1 - 1)2j_1(2j_1 + 2)(2j_1 + 3)(2j_1 + 4)}} \]

\[ m_s = -1, \quad -2\sqrt{2} [j_1(j_1 + 1) - 5m(m + 1) - 2] \]
\[ \sqrt{(2j_1 - 2)(2j_1 - 1)2j_1(2j_1 + 2)(2j_1 + 3)(2j_1 + 4)} \]

\[ m_s = -2, 2(m + 1) \sqrt{30} \sqrt{(j_1 - m)(j_1 - m - 1)(j_1 + m)(j_1 + m - 1)} \]
\[ \sqrt{(2j_1 - 2)(2j_1 - 1)2j_1(2j_1 + 2)(2j_1 + 3)(2j_1 + 4)} \]

\[ m_s = -3, 2\sqrt{5} \sqrt{(j_1 - m)(j_1 - m - 1)(j_1 + m)(j_1 + m - 1)(j_1 + m + 1)} \]
\[ \sqrt{(2j_1 - 2)(2j_1 - 1)2j_1(2j_1 + 2)(2j_1 + 3)(2j_1 + 4)} \]

\[ j = j_1 - 1 \]

\[ m_s = 3, \quad \sqrt{15} \sqrt{(j_1 - m + 3)(j_1 - m + 2)(j_1 - m + 1)(j_1 - m)(j_1 + m - 1)} \]
\[ \sqrt{(2j_1 - 3)(2j_1 - 2)2j_1(2j_1 + 1)(2j_1 + 2)(2j_1 + 3)} \]

\[ m_s = 2, -(j_1 + 3m - 3) \sqrt{10} \sqrt{(j_1 - m + 2)(j_1 - m + 1)(j_1 - m)(j_1 + m - 1)} \]
\[ \sqrt{(2j_1 - 3)(2j_1 - 2)2j_1(2j_1 + 1)(2j_1 + 2)(2j_1 + 3)} \]

\[ m_s = 1, \quad (-j_1^2 - 10j_1m + 15m^2 - 5j_1 - 15m + 6) \]
\[ \sqrt{(j_1 - m + 1)(j_1 - m)} \]
\[ \sqrt{(2j_1 - 3)(2j_1 - 2)2j_1(2j_1 + 1)(2j_1 + 2)(2j_1 + 3)} \]
\[
\begin{align*}
m_2 &= 0, \quad (j_1^2 - 5m^2 - 1)2\sqrt{3} N \sqrt{\frac{(j_1 - m)(j_1 + m)}{(2j_1 - 3)(2j_1 - 2)2j_1(2j_1 + 1)(2j_1 + 2)(2j_1 + 3)}} \\
m_2 &= -1, \quad (-j_1^2 - 10j_1m + 15m^2 - 5j_1 + 15m + 6) \\
m_2 &= -2, \quad (-j_1 + 3m + 3) \sqrt{10} N \sqrt{\frac{(j_1 - m - 1)(j_1 + m + 1)}{(2j_1 - 3)(2j_1 - 2)2j_1(2j_1 + 1)(2j_1 + 2)(2j_1 + 3)}} \\
m_2 &= -3, \quad -\sqrt{15} N \sqrt{10} \sqrt{\frac{(j_1 - m - 1)(j_1 - m - 2)(j_1 + m + 3)(j_1 + m + 2)(j_1 + m + 1)(j_1 + m)}{(2j_1 - 3)(2j_1 - 2)2j_1(2j_1 + 1)(2j_1 + 2)(2j_1 + 3)}} \\
\end{align*}
\]

\(j = j_1 - 2\)

\[
\begin{align*}
m_2 &= 3, \quad -\sqrt{6} N \sqrt{\frac{(j_1 - m + 3)(j_1 - m + 2)(j_1 - m + 1)(j_1 - m)(j_1 - m - 1)(j_1 + m - 2)}{(2j_1 - 4)(2j_1 - 2)(2j_1 - 1)2j_1(2j_1 + 1)(2j_1 + 2)}} \\
m_2 &= 2, \quad 2(2j_1 + 3m - 4) N \sqrt{10} \sqrt{\frac{(j_1 - m + 2)(j_1 - m + 1)(j_1 - m)(j_1 - m - 1)}{(2j_1 - 4)(2j_1 - 2)(2j_1 - 1)2j_1(2j_1 + 1)(2j_1 + 2)}} \\
m_2 &= 1, \quad -(j_1 + 3m + 2) \sqrt{10} N \sqrt{\frac{(j_1 - m + 1)(j_1 - m)(j_1 - m - 1)(j_1 + m - 1)}{(2j_1 - 4)(2j_1 - 2)(2j_1 - 1)2j_1(2j_1 + 1)(2j_1 + 2)}} \\
m_2 &= 0, \quad 2m \sqrt{30} N \sqrt{\frac{(j_1 - m)(j_1 - m - 1)(j_1 + m)(j_1 + m - 1)}{(2j_1 - 4)(2j_1 - 2)(2j_1 - 1)2j_1(2j_1 + 1)(2j_1 + 2)}} \\
m_2 &= -1, \quad (j_1 - 3m - 2) \sqrt{10} N \sqrt{\frac{(j_1 - m - 1)(j_1 + m + 1)(j_1 + m)(j_1 + m - 1)}{(2j_1 - 4)(2j_1 - 2)(2j_1 - 1)2j_1(2j_1 + 1)(2j_1 + 2)}} \\
m_2 &= -2, \quad -2(2j_1 - 3m - 4) N \sqrt{10} \sqrt{\frac{(j_1 + m + 2)(j_1 + m + 1)(j_1 + m)(j_1 + m - 1)}{(2j_1 - 4)(2j_1 - 2)(2j_1 - 1)2j_1(2j_1 + 1)(2j_1 + 2)}} \\
m_2 &= -3, \quad \sqrt{6} N \sqrt{\frac{(j_1 - m - 2)(j_1 + m + 3)(j_1 + m + 2)(j_1 + m + 1)(j_1 + m)(j_1 + m - 1)}{(2j_1 - 4)(2j_1 - 2)(2j_1 - 1)2j_1(2j_1 + 1)(2j_1 + 2)}} \\
\end{align*}
\]

\(j = j_1 - 3\)

\[
\begin{align*}
m_2 &= 3, \quad \sqrt{N} \sqrt{\frac{(j_1 - m + 3)(j_1 - m + 2)(j_1 - m + 1)(j_1 - m)(j_1 - m - 1)(j_1 - m - 2)}{(2j_1 - 4)(2j_1 - 3)(2j_1 - 2)(2j_1 - 1)2j_1(2j_1 + 1)}} \\
m_2 &= 2, \quad -\sqrt{6} N \sqrt{\frac{(j_1 - m + 2)(j_1 - m + 1)(j_1 - m)(j_1 - m - 1)(j_1 - m - 2)(j_1 + m - 2)}{(2j_1 - 4)(2j_1 - 3)(2j_1 - 2)(2j_1 - 1)2j_1(2j_1 + 1)}} \\
m_2 &= 1, \quad \sqrt{15} N \sqrt{\frac{(j_1 - m + 1)(j_1 - m)(j_1 - m - 1)(j_1 - m - 2)(j_1 + m - 1)(j_1 + m - 2)}{(2j_1 - 4)(2j_1 - 3)(2j_1 - 2)(2j_1 - 1)2j_1(2j_1 + 1)}} \\
\end{align*}
\]
\[ m_z = 0, \quad -\sqrt{20} \begin{pmatrix} (j_1 - m)(j_1 - m - 1)(j_1 - m - 2)(j_1 + m)(j_1 + m - 1)(j_1 + m - 2) \\ (2j_1 - 4)(2j_1 - 3)(2j_1 - 2)(2j_1 - 1)2j_1(2j_1 + 1) \end{pmatrix} \]

\[ m_z = -1, \quad \sqrt{15} \begin{pmatrix} (j_1 - m - 1)(j_1 - m - 2)(j_1 + m + 1)(j_1 + m)(j_1 + m - 1)(j_1 + m - 2) \\ (2j_1 - 4)(2j_1 - 3)(2j_1 - 2)(2j_1 - 1)2j_1(2j_1 + 1) \end{pmatrix} \]

\[ m_z = -2, \quad -\sqrt{6} \begin{pmatrix} (j_1 - m - 2)(j_1 + m + 2)(j_1 + m + 1)(j_1 + m)(j_1 + m - 1)(j_1 + m - 2) \\ (2j_1 - 4)(2j_1 - 3)(2j_1 - 2)(2j_1 - 1)2j_1(2j_1 + 1) \end{pmatrix} \]

\[ m_z = -3, \quad \sqrt{3} \begin{pmatrix} (j_1 + m + 3)(j_1 + m + 2)(j_1 + m + 1)(j_1 + m)(j_1 + m - 1)(j_1 + m - 2) \\ (2j_1 - 4)(2j_1 - 3)(2j_1 - 2)(2j_1 - 1)2j_1(2j_1 + 1) \end{pmatrix} \]

**References**

1) D. R. Hamilton, Phys. Rev. 58 (1940), 122.
3) G. Racah, Phys. Rev. 84 (1951), 910.
6) G. Goertzel, Phys. Rev. 70 (1946), 897.
7) D. S. Ling Jr. and D. L. Falkoff, Phys. Rev. 76 (1949), 1639.
11) M. Fuchs and E. S. Lennox, Phys. Rev. 79 (1950), 221.
15) This method is based on H. A. Bethe, Ann. d. Phys. 4 (1930), 443.