Kac Formulas for G/H Conformal Field Theories

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New Kac (spectral-) formulas for various conformal field theories are proposed. We use the G/H picture as a guiding principle. Our prescription is verified for all the known cases.

Introduction
Conformally invariant 2-dimensional quantum field theories describe universality classes of 2-dimensional critical phenomena, and they provide the building blocks of classical string ground states. Therefore the complete classification of all 2-dimensional conformal field theories is one of our major goals.

In 2 dimensions, the conformal group is infinite dimensional. The algebra consists of two copies of the Virasoro algebra, labeled by their central charge $c$,

$$[L_n, L_m] = (n - m)L_{n+m} + \frac{c}{12}n(n^2 - 1)\delta_{n+m,0}.$$ 

Among the whole set of conformally invariant models there are exactly solvable models, the so-called minimal models, in which the state space is a finite sum of representations of the Virasoro algebra. The central charge for the unitary minimal models has the following expression with a positive integer $k$,

$$c = 1 - \frac{6}{(k+2)(k+3)}.$$  \hspace{1cm} (1)

The first several models in this series correspond to the statistical models.\(^2\)

As for searching for other exactly solvable models, a remarkable progress has been achieved. The basic idea is that the Virasoro algebra by itself is not sufficiently restrictive. Now many exactly solvable models are known: the original minimal models with $c < 1$, superconformal minimal models,\(^3\) the Wess-Zumino-Witten (WZW) models with Kac-Moody algebra,\(^4\) $Z_n$-parafermion models,\(^5\),\(^6\) $S_3$ models,\(^7\) and $W_n$-algebra minimal models,\(^8\)\textsuperscript{–10} etc.

As another important observation on the minimal models with $c < 1$, the coset construction of the Virasoro algebra initiated by Goddard, Kent and Olive\(^1\) is known. In this picture the minimal models with $c < 1$ can be described by the following coset,

$$\frac{A_k \times A_1}{A_{k+1}},$$ \hspace{1cm} (2)

where $A_k$ denotes the $SU(2)$ Kac-Moody algebra with the level $k$. As for this direction a conjecture by Kyoto group,\(^1\) which states that any coset pair produces an integrable system, was a crucial step. Indeed a new unitary exactly solvable series of conformal field theories labeled by two integers $k$ and $l$ have been discovered by
several groups independently. These models have the following coset construction,

\[ \frac{A_1 \times A_1}{A_1^{k+1}}. \]  

Their central charge can be easily computed and given as

\[ c = 1 - \frac{6l}{(k+2)(k+l+2)} + \frac{2(l-1)}{l+2}. \]  

The first two terms in the expression (4) can be interpreted as the central charge for a bosonic field with a given background charge. The last term is the contribution from \( Z_n \) para-fermions. From this observation the authors of Ref. 13) generalized the Feigin-Fuchs construction\(^{14}\) to these models. The extension of this construction to higher rank algebras, say \( A_{k-1}^{\#} \times A_{k-1}/A_{k-1}^{\#} \), is yet to be studied. However, as a first step we shall propose Kac formulas for the extended Virasoro algebras corresponding to the above coset pairs.

**Unified coset construction** A unified point of view is highly desired for the classification problem. Recently Douglas proposed a unified construction\(^{15}\) which includes all the known examples. Let us first review his construction. It is essentially a generalization of the coset construction. To define the Hilbert space of the model which is based on a coset \( G/H \), we consider a highest weight module of \( G \) and decompose it into the representations of \( H \). This can be written as (we concentrate on the left moving part)

\[ L(A) = \sum_L L(\lambda) \otimes U(A, \lambda), \]  

where \( L(A) \) and \( L(\lambda) \) denote the representations of the Kac-Moody algebra built on the finite dimensional representations of \( G \) and \( H \), respectively. Here \( U(A, \lambda) \) is the Hilbert space of states in \( L(A) \) which correspond to the highest weight states of an \( H \) representation \( L(\lambda) \). We can rewrite this equation as the character relation by using an explicit formula

\[ \chi_A(z, \theta) = \text{Tr} z^{L_0} e^{2\pi i \theta f_0}, \]  

where \( f_0 \) are the elements of the Cartan subalgebra, \( L_0 \) is a density of the Sugawara form of the Kac-Moody algebra, and \( z \) and \( \theta_i \) are the parameters. The result corresponding to (5) is

\[ \chi_A(z, \theta) = \sum_{\lambda} \chi(\lambda) b_\lambda^A(z), \]  

where \( b_\lambda^A \) are called the branching functions. The idea is the identification of these branching functions with the left moving components of the \( G/H \) partition function. It is natural to expect that there exists an infinite-dimensional symmetry algebra associated with the pair \( G/H \). In this formulation primary fields fall into classes labeled by the choice of \( G \) and \( H \) representations for both left and right algebras, and they schematically factorize as

\[ g_i^A = h_i^A \phi_i^A, \]  

\[ (8) \]
where $g^A_i(h_i)$ are the primary fields for the $G(H)$-WZW model and $\phi_i^A$ are the primary fields for the $G/H$ theory. The simplest example is provided by the unitary minimal models whose central charge is given by (1). In this case $G$ and $H$ are chosen as $A_1^k \times A_1^l$ and diagonal $A_1^{k+l}$, respectively. The branching rule in this case can be explicitly written down,

$$\chi_{k,\ell}(z, \theta) = \sum_{q} \chi_{k+1,\ell-1/2}(z, \theta) \chi_{c,q}(z),$$

where $\chi_{k,\ell}(z, \theta)$ is the character for the level $k$ isospin $j(0 \leq j \leq k)$ representation of the $SU(2)$ Kac-Moody algebra and the branching functions $\chi_{c,q}(z)$ coincide with the character for the representation $(c, h)$ of the Virasoro algebra with central charge $c$. The conformal dimension $h$ is labeled by $j$ and $q$. If we label a representation by its dimension, $p=2j+1$ for $A_1^k$, the primary fields should be labeled by a pair of integers $(p, q)$, where $1 \leq p \leq k+1$ and $1 \leq q \leq k+2$ corresponding to $A_1^k$ and $A_1^{k+1}$ representations, respectively. The summation rule of the above relation (9) is a reflection of the $A_1^1$ representation, $2\varepsilon = 0$ or 1, depending on whether $p-q$ is even or odd.

The explicit construction of the coset primaries, $\phi_{(p,q)}$, proceeds as follows. Since the $G/H$ states are defined by the linear relation (8), one can automatically obtain the right $\phi_{(p,q)}$ when one considers the construction of an $H$ state in the representation $q$ from a $G$ state in $p$. From this view point, $\phi_{(p,q)}$ generally appears at the dimension higher than the simple difference of the dimensions of $A_1^k$ and $A_1^{k+1}$ primaries in the representations $p$ and $q$, respectively. The lowest case corresponds to the ground state, $\phi_{(p,p)}$, of the $G$ representation $p$. An H state in the representation $q$ can be obtained by applying some suitable current operators on the H state in $p$. Hence one needs only the lowest dimensional such operator to get the right dimension of $\phi_{(p,q)}$. The lowest dimensional operator in the representation, $q=2s+1$, cannot be a product of $s$ currents. By using the representation of $A_1^1$ currents in terms of a free boson $\phi$,

$$J^4(z) \sim \exp(\pm i\phi(z)),$$

$$J^3(z) \sim i\partial\phi(z),$$

one can immediately understand that the lowest dimensional operator is $:\exp(iQ\phi):$ with charge $Q=(p-q)/2$. Thus we obtain the right dimension for the field $\phi_{(p,q)}$

$$h = \frac{p^2-1}{4(k+2)} - \frac{q^2-1}{4(k+3)} + \frac{(p-q)^2}{4},$$

(10)

where we used the fact that conformal dimension of the $A_1^k$ WZW primaries is given by $j(j+1)/(k+2)$.

**Generalization** Let us generalize the above argument to the $A_1^k \times A_1^l/A_1^{k+l}$ models. These are studied by the authors in Ref. 13), however, we rederive Kac formulas for these cases to explain our prescription. It also supports the correctness of our proposal even though it is not proved. The central charges for these models are given by
where we take \(m=k+2\). In our general treatment \((l>1)\) the last term in (11) always appears and exactly corresponds to the central charge for the \(Z_l\)-parafermion theories. Hence one must take into account the parafermionic contributions. To get the right spectrum one also needs the lowest dimensional operator in our method. The representation of the higher level currents requires \(Z_l\) parafermion currents, \(\phi(z)\) and \(\phi^*(z)\), besides a free boson \(\phi(z)\).

\[
\begin{align*}
J^+(z) &\sim \phi(z) \exp[+i\phi(z)/\sqrt{I}], \\
J^-(z) &\sim \phi^*(z) \exp[-i\phi(z)/\sqrt{I}], \\
J^0(z) &\sim \partial \phi(z).
\end{align*}
\]

Hence we obtain the lowest dimensional operator with charge \(Q=(p-q)/2\)

\[
\sigma_t(z) \exp[iQ\phi(z)/\sqrt{I}];
\]

where \(\sigma_t\) are primaries of the \(Z_l\) parafermion theory, labeled by \(t\). Note that \(t\) is restricted so as to produce the \(A_{l-1}^{k+1}\) \(q\)-representation from the \(A_{l-1}^k\)-representation, \(t=|p-q(mod l)|\). As a result the coset field, \(\phi(p,q)\), has a dimension

\[
h=\frac{p^2-1}{4m} - \frac{q^2-1}{4(m+l)} + \frac{(p-q)^2}{4l} + \frac{t(l-t)}{2l(l+2)},
\]

where the last term is the dimension of \(\sigma_t\) and \(t=|p-q(mod l)|\). In this approach the extraction of the \(U(1)\) factor, \(\exp[i\phi(t)\sqrt{I}]\): crucial.

In fact, if we choose \(l=2\) in (11) and (13), one can immediately obtain the unitary discrete series for superconformal algebras,

\[
c=\frac{3}{2} \left(1-\frac{8}{m(m+2)}\right)
\]

and

\[
h=\frac{[(m+2)p-mq]^2}{8m(m+2)} - \frac{4}{16} + \frac{t(2-t)}{16},
\]

where \(t=0\) or 1 corresponds to the the Neveu-Schwarz and Ramond algebras, respectively.

The case \(l=4\) is known to correspond to the \(S^3\) model of Fateev-Zamolodchikov.\(^7\) We can also show that our formula (13) gives the right spectrum for this model,

\[
c=2(1-\frac{12}{m(m+4)})
\]

and

\[
h=\frac{[(m+4)p-mq]^2}{16m(m+4)} - 16 + \frac{t(4-t)}{48},
\]

where \(t=|p-q(mod 4)|\).
Though the Fock space and the structure of the extended Virasoro algebra for the general $A_{1}^{k} \times A_{1}^{l}/A_{1}^{k+l}$ models are not known owing to the complexity of the operator product expansion, we can always write down the Kac-spectrum and get the correlation functions in the context of $G/H$ conformal field theory. This approach respects the manifest unitarity, so our prescription does.

Next we generalize the above idea to all the $A_{N-1}$ coset theories. The central charge for the $A_{N-1}^{k} \times A_{N-1}^{l}/A_{N-1}^{k+l}$ models is given by

$$c = \frac{(N^2 - 1)l}{N + l} \left[ 1 - \frac{N(N + 1)}{m(m + l)} \right] = (N - 1) \left[ 1 - \frac{N(N + 1)}{m(m + l)} \right] + \frac{l(N^2 - 1)}{l + N} - (N - 1),$$

(14)

where we have chosen $m = N + k$. If $l = 1$, the last two terms cancel and the total central charge becomes the one for the $W_N$-algebra. In general, the last two terms survive and coincide with the central charge for the $Z_l$ parafermionic current algebra constructed on the $A_{N-1}$ root lattice. The extraction of the $U(1)$ factor of $A_{N-1}$ clarifies the role of parafermion and the discrete symmetries in it. Following the prescriptions above the Kac-formulas for these models are given by

$$h = \frac{(p, p + 2p)}{2m} \frac{(q, q + 2q)}{2(m + l)} + \frac{|p - q|^2}{2l} + h_{Z_l},$$

(15)

where the two Casimir's are expressed in terms of the dominant weights $p$ and $q$. $p$ is half of the sum of positive roots and $h_{Z_l}$ denotes the dimension of the primary field for the $Z_l$ parafermionic theory defined on the $A_{N-1}$ root lattice, $M_R$. The general expression for $h_{Z_l}$ has already been given in Ref. 6) for the parafermion theory constructed on the $A_{N-1}$ root lattice, and $h_{Z_l}$ becomes

$$h_{Z_l} = \frac{(A, A + 2\rho)}{2(l + g_c)} \frac{A^2}{2l},$$

(16)

where $A = |p - q|$ mod $l M_R^* (M_R^* \sim$ the weight lattice dual to $M_R$) is the highest weight representation of $A_{N-1}$ and $g_c = N$ for this case. Note that the $Z_l$ parafermion theory itself has the structure of the coset construction. The Kac-spectrum for this theory can be naturally understood as in (16), in the context of the unified coset construction. The choice of $l = 1$ in the above formula provides a simple derivation of the principal series of the $W_N$-minimal models. Fateev and Lukyanov derived this formula (15) for the case $l = 1$ under the conjecture that the generalized Virasoro operators with spin higher than two commute with the screening charges. However we do not need such a conjecture due to the manifest unitarity of our method.

We want to note that the derived formula (15) is applicable to any simple untwisted affine Kac-Moody algebra $\tilde{g}^k$ with level $k$. The generalized coset construction, $\tilde{g}^k \times \tilde{g}^l/\tilde{g}^{k+l}$, gives the central charge as

$$c = \text{rank } g - \frac{g_l \dim g}{m(m + l)} + \frac{l \dim g}{l + g_c} - \text{rank } g,$$

(17)

where $g_c$ is the dual Coxeter number of $\tilde{g}$. So one can use the formula (15) as the
Kac-formula for the $\tilde{g}^k \times \tilde{g}^i \tilde{g}^{k+i}$ models, where $m = k + g_c$.

**Discussion** Let us conclude this paper with several remarks. Comparing with the ordinary Feigin-Fuchs method, our method has an advantage that it preserves manifest unitarity. In the case of the Feigin-Fuchs method, the background charge must be chosen so as to ensure unitary. However, we have no general principle to determine the background charge because we do not know the structure of the extended Virasoro algebra. In a sense one could say that our method determines the background charge of the Feigin-Fuchs method. This approach does not give a method to determine the extended Virasoro algebra. Supposing that all the coset models are solvable, we expect that an effective method will be found from this approach. As for a future problem besides the above one, generalization to arbitrary Riemann surfaces must be studied.

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