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DISCUSSION

J. R. Shanebrook[3] and W. J. Sumner[4]

The flow field selected by the authors for comparison purposes provides an interesting test for the three-dimensional integral method since the flow is dominated by severe pressure gradients and the effects of cross-flow are substantial. More importantly, Johnston [12] determined the three-dimensional line of separation for this flow field from dye traces on the wall and smoke flow studies. We therefore ask the authors to comment on the applicability of the present integral method for predicting three-dimensional separation of a turbulent boundary layer in the context of the selected flow field. Also, another interesting comparison would be with the values of skin friction coefficient determined by Johnston [12] along the plane of symmetry, by a law of the wall technique. These results for skin friction then provide a convenient means for estimating the singular point of separation on the plane of symmetry. It should, perhaps, be mentioned that the prediction of separation in a three-dimensional flow field is of great practical importance since many devices impose pressure gradients on the boundary layer that tend to slide the fluid sideways along the surface and not necessarily away from the surface in the usual two-dimensional sense. An integral method that could adequately describe this mode of separation would be of immediate interest to many designers.

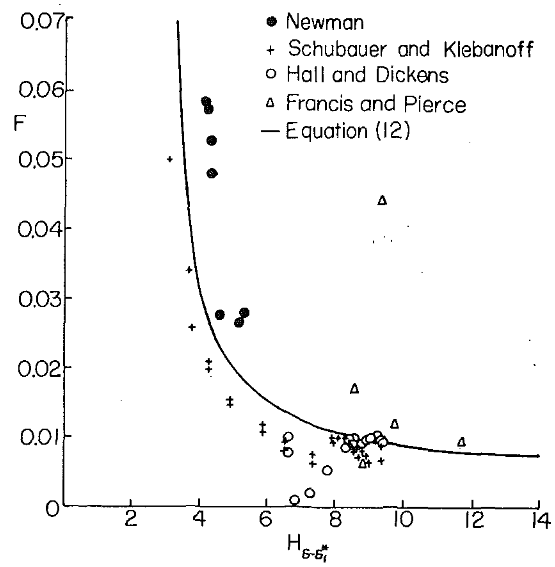

Fig. 9 Entrainment function compared with three-dimensional, turbulent boundary layer data of references [20] and [21]

Regarding the four experimental values shown in Fig. 8, we believe it would be beneficial if the authors could briefly outline the techniques and assumptions employed in reducing the data in order to obtain values for the entrainment function. That is, the equation used to calculate F along the plane of symmetry and the definition of the boundary layer thickness employed would be of interest. For example, near the end wall the assumption of an irrotational outer flow may not be adequate for accurate determination of F.

The authors have asked for more detailed information on a three-dimensional entrainment function. Fig. 9 compares four sources of data with Standen's [19][5] analytical approximation to Head's [14] original graphical correlation for $F(H_{\delta-\delta_1}{}^*)$ given by,

$$F = 0.0306(H_{\delta-\delta_1}{}^* - 3.0)^{-0.653} \qquad (12)$$

Head's original correlation was based on the two-dimensional data of Newman and of Schubauer and Klebanoff. We calculated the other values of F shown in this figure from the three-dimensional, incompressible data of Francis and Pierce [20] and the three-dimensional, compressible data of Hall and Dickens [21]. It is seen that, with few exceptions, the three-dimensional data lie within the range originally correlated by Head, indicating that two-dimensional correlation is probably adequate for at least the three-dimensional flow conditions of these experiments.

Finally, the authors do not mention if they employed Head's two-dimensional correlation for $H_{\delta-\delta_1}{}^*(H)$ in their calculations. Regardless, it would be interesting to know if the present data compare favorably with this correlation since it has been used in other three-dimensional methods. We found this correlation

[3] Associate Professor, Department of Mechanical Engineering, Union College, Schenectady, N. Y. Mem. ASME.

[4] Engineer, Knolls Atomic Power Laboratory, Schenectady, N. Y.

[5] Numbers 19–21 in brackets designate Additional References at end of discussion.

to adequately represent the three-dimensional data of references [20] and [21].

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Robert R. Bass[6]

This paper was indeed an interesting one from our viewpoint and we would like to share the results of a very similar work reported in 1968. See reference [22][7]. After studying the results of Pierce and Klinksiek and comparing them to our results (reference [22]) it becomes quite clear that the two different approaches predict the experimental data with equal accuracy, neither perhaps as good as one would hope for. People are still interested in what some would call "old fashioned" momentum integral techniques to predict what seems to be the "classical data set" from Johnston's MIT work.

Briefly, our approach was to use the eight equations in eight unknowns formulated in Johnston's thesis, assuming the following velocity profiles:

and
$$\frac{u}{U} = \left(\frac{\eta}{\delta}\right)^{1/7} \tag{13}$$

$$\frac{w}{U} = \left(1 - \frac{\eta}{\delta}\right)^3 \left(\frac{\eta}{\delta}\right)^{1/7} \text{ and the} \tag{14}$$

skin friction:

$$C_{f_x} = 0.246\,[\exp\,(-1.561\,H_x)]\,R_{\theta_x}{}^{-0.268} \tag{15}$$

The eight momentum integral and auxiliary equations were solved numerically by a complicated iterative scheme where the partial derivatives were handled by finite differencing. The potential flow solution from a complex variable analysis was used as input so that the terms like $\frac{\partial U}{\partial \phi}$ and $\frac{\partial U}{\partial \psi}$ could be evaluated. This approach was admittedly a brute force type solution with many of the assumptions built into the equations by Johnston a bit insecure; however the results are very similar to those of Pierce and Klinksiek.

Along the plane of symmetry shown as Fig. 3 of Pierce and Klinksiek our predictions agree with theirs. A better prediction of this data has been accomplished by Mellor, reference [23], by using the Mellor-Herring "2D" finite difference—turbulent boundary layer program with a "3D" cross-flow correction.

Fig. 10 of this discussion comes from reference [22] and is a plot of momentum thickness versus X ($\theta_x = \theta_{11}$) which can be compared to Fig. 4 of Pierce and Klinksiek for $Z = 2.5$ and Fig. 5 for $Z = 5.0$. Making this comparison shows that both methods are predicting about the same results. A similar comparison for the shape factor ($H_x = H$) can be made between Fig. 11 of this discussion and Figs. 4 and 5 of Pierce and Klinksiek. In this comparison our results for the $X = 21$ and 19 in. locations seem to be a little closer to the experimental data. One last comparison is offered and that is made by plotting our calculated results as ⊗'s in Fig. 6 of Pierce and Klinksiek which gives θ_{13} and in Fig. 7 which gives θ_{33}. This comparison is called Fig. 12 of this discussion.

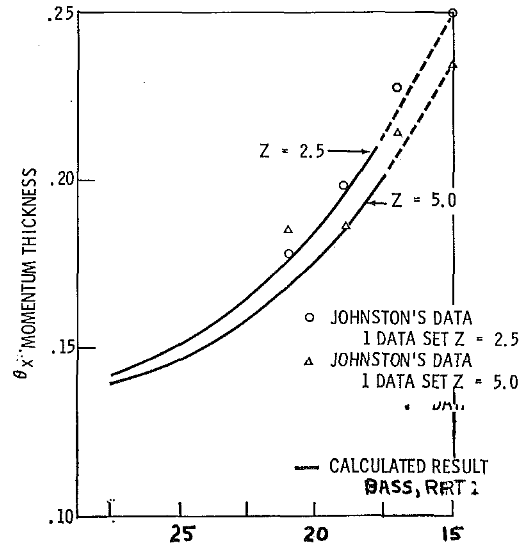

Fig. 10

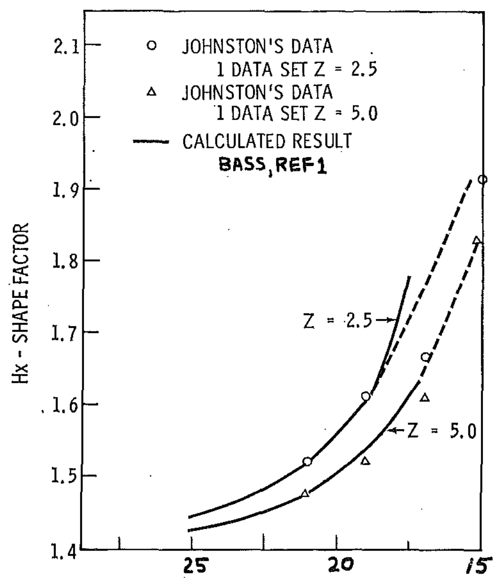

Fig. 11

In conclusion, it was interesting for us to find that results we obtained some five years ago compare completely favorably with those of Pierce and Klinksiek, although we used an approach which is possibly more restrictive mathematically.

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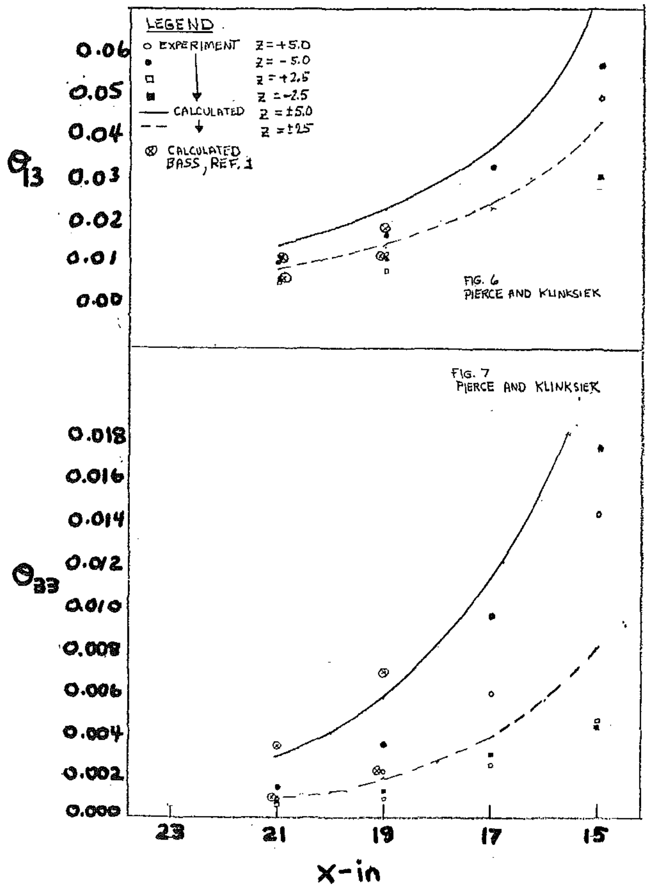

Fig. 12 Cross-flow parameters

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J. F. Nash[8]

There is very little in the paper that is new: Power-law streamwise profiles were used by Mager, parabolic cross-flow profiles were also used by Mager, the Ludwieg-Tillmann relation for the streamwise component of wall shear stress was used by Eichelbrenner and Peube, and by Smith, the entrainment function was used by Cumpsty and Head, and the resulting set of equations has been integrated by several people. Nevertheless, the paper provides an opportunity to review the basic assumptions made in this and other integral methods, and to discuss the likely future of integral methods in three-dimensional turbulent boundary layers generally. If I make certain criticisms of integral methods, I do not think that I shall hurt the authors' feelings too much because I know that they have also developed a very successful differential method!

To start on a more constructive note, however, the simple power law is found to be a better representation of a range of measured streamwise profiles, in three dimensions, than it is in two-dimensional flows. Three-dimensional profiles frequently appear to retain a characteristic roundness much closer to separation than do their counterparts in two dimensions, and in some cases the power law gives a better overall curve fit than Coles or Thompson profiles.[9] It is not immediately clear why this should be, but it may be associated with the tendency of the velocity vectors within the three-dimensional boundary layer to be deflected by the adverse pressure gradients rather than reduced in magnitude. In two dimensions, of course, the vectors cannot usually avoid being reduced in magnitude.

On the other hand, I am very suspicious of families of prescribed cross-flow profiles. The Mager profile is probably as good as the other simple ones. However, measured cross-flow profiles come in such a variety of shapes that it is impossible to curve-fit them all without a large number of disposable parameters,[9] and then every additional parameter demands an additional equation to solve for it. I think it would be very revealing if the authors would compare the cross-flow profiles *implied by their calculated results* with the measured ones, in addition to just comparing cross-flow momentum thicknesses. Comparisons between theoretical and measured cross-flow profiles look bad enough even when the *measured* integral parameters are used to derive them.

The assumption that the skin friction can be related to the streamwise velocity profile—whether by the Ludwieg-Tillmann expression or otherwise—is valid only for small cross-flows. This is because such a relationship implies that the law of the wall is satisfied by the streamwise components of velocity and the streamwise component of wall shear stress. If the law of the wall is valid at all in a three-dimensional boundary layer, it must apply in the direction of the *resultant* wall shear stress. The error involved in assuming that it applies in the streamwise direction also is not large, but it does increase with wall cross-flow angle.

I must emphasize that these remarks refer to integral methods in general, and not just to the present paper. My feeling is that, because of their dependence on prescribed profile models, they are currently too inflexible to treat a wide range of practical flows. Moreover, if sufficiently flexible profile models were to be incorporated, the number of equations would have to be increased to the point where the original simplicity and economy of the integral approach would be lost. I believe that three-dimensional boundary layers are complex enough to require the use of differential methods, and that such methods do not have to be prohibitively expensive to run.

Authors' Closure

The authors appreciate the thorough reviews and careful comments of Dr. Nash and Mr. Bass, and the additional information on the entrainment function from Dr. Shanebrook is especially welcome. As noted in the introduction, the purpose of this paper was not in presenting any new momentum integral solution method, but rather of verification of an existing method in a flow geometry which provided substantial transverse gradients. Such comparisons with experiment are not in abundance in the open literature, as most published works deal with the weak secondary flow assumption and/or flow over infinite yawed wings.

The authors also acknowledge and warn users of momentum integral methods in that while fair to good agreement may be obtained in gross flow parameters such as momentum and defect thicknesses, details of the flow field may very well remain a mystery. This is largely due to the general lack of flexibility in cross-flow velocity profile models. Shanebrook and Hatch [13] have presented a multiparameter family with substantial flexibility, but this flexibility introduces new parameters which require additional auxiliary equations, and at this time there appears to be little basis in fact or experiment to provide such additional equations for a mathematical closure of the more complex momentum integral formulation.

As noted by Dr. Nash the authors have also developed finite difference solution techniques to the full three-dimensional turbulent boundary layer equations [25, 26].[10] Such solutions while taking more computer time, provide excellent agreement with details of the flow fields where comparisons have been made with

[8] Scientist, Lockheed-Georgia Co., Marietta, Ga.

[9] Nash, J. F., and Patel, V. C., *Three-Dimensional Turbulent Boundary Layers*, SBC Technical Books, 1972.

[10] Numbers in brackets designate Additional References at end of Closure.

experiments. On the IBM system 360/65 the difference solution [26] to the boundary layer equations requires about nine times the time taken for a momentum integral solution (18 versus 2 minutes), for the geometry studied in this paper.

Skin friction values along the plane of symmetry agree closely with those reported by Johnston. This is inferred in the close agreement in θ and H values along this line, and on which values the Ludwieg-Tillmann shear law is based. This shear law was shown to agree well with the Clauser chart (Law of the Wall) results in Johnston's experiment. The calculated skin friction does not literally go to zero as the exponential nature of the shear law would require H to approach infinity. The calculation was terminated where C_f reached some artifically small value or was sufficiently small to cause a computer underflow. Thus the point of vanishing shear was never reached, though it is approached within some arbitrarily small value. Since this singular point of separation on the plane of symmetry occurs ahead of the region of three-dimensional separation, no calculations were made in this latter region. Regarding Fig. 8, the experimental values were obtained using the data of Johnston. Equation (3) was used (though in Cartesian coordinates to facilitate its use) and difference quotients were constructed. Because of the uncertainty in differentiating experimental data, similar calculations were made at $z \pm 5$ and $z = \pm 2.5$ and extrapolated looking for consistency with the values at $z = 0$.

It is gratifying to note that the problem geometry, as well as the general method of solution, suggested to Mr. Bass in his early visit with the first author in 1967 was in fact pursued to completion. Publication of these results in the open literature would surely have been of broad interest at that time, providing verification of the general technique for a flow with large traverse gradients. It was this lack of verification in the open literature that prompted the authors to undertake this study, albeit with a moderate time lag. It should be noted that the momentum integral equations cited by Mr. Bass are not in orthogonal curvilinear coordinates compatible with the potential free-stream flow, but rather are written along a "curved" x coordinate which is then translated in the transverse direction but with constant curvature. Additionally, the variation of shape parameter used in that analysis contains in it implicitely the single parameter Squire-Young skin friction formula which is questionable in its inconsistency with the two parameter Ludwieg-Tillmann formula listed as one of the eight equations in the system solved. Finally, some clarification of Figs. 9 and 10 is required in that the lines of z = constant are *not* streamlines as suggested by the abscissa label.

These two solutions of this flow geometry and their reasonably good agreement in predicting gross flow parameters points to the weakness of momentum integral methods. In addition to using different auxiliary equations (the authors used an entrainment function after Head while Mr. Bass used a Tetervin and von Doenhoff variation of shape parameter formulation) both solutions differ somewhat in the model used to represent the mean flow velocity in the stream wise and cross-flow directions. The authors used a variable n (power profile) and a quadratic variation of flow direction through the boundary layer while Mr. Bass uses a fixed n ($^1/_7$ root profile) and a cubic variation of flow direction. Thus while gross flow parameters are reasonably close in agreement, these different velocity profile models would likely give significant differences in the details of the flow fields they predict.

Nonetheless, the authors feel that momentum integral solutions should be explored in that some of the successful methods giving quick and approximate results for two-dimensional flows have their origin in momentum integral formulations. Such faster and approximate calculations, which might avoid computer use and be of wide utility to the practicing engineer, would be useful in three-dimensional circumstances, as there are many times when the engineer might be interested in skin-friction and displacement thicknesses at the sacrifice of details of the flow field. But there is no question that difference solutions to the full three-dimensional boundary layer equations do provide reasonably fast and significantly more accurate predictions of the details of a flow field.

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