The collective rotation of a nuclear system having the quadrupole-quadrupole interaction is described by the dynamical nuclear field theory (DNFT). We use the one-body harmonic oscillator potential and restrict the discussion to the $\Delta N=0$ transitions. Energy eigenvalues of the resulting $SU_3$ Hamiltonian are obtained by using the eigenstates of the cranked harmonic oscillator. Both the low and high spin states are studied by the perturbative DNFT, reproducing successfully the diagonalization results. In spite of the simple rotational spectrum, the nuclear shape is seriously influenced by the rotational disturbances. Similarities with our previous analyses of the pair rotation are pointed out. Especially, the $SU_3$ rotation in the odd mass system decouples with the particle motion just as the pair rotation does in the single-j limit.

§ 1. Introduction

Nuclear collective rotation is an old but still expanding topic both in the range of related experimental data and in the sophistication of the theory.

In this paper we study this problem by the dynamical nuclear field theory (DNFT), which was applied successfully to describe the pair rotation of a system of nucleons interacting via the pairing force. We consider a nuclear system having the quadrupole-quadrupole interaction (QQ) and assume the existence of the Hartree field with a static deformation. The model reduces to the $SU_3$ one if we take the spherical harmonic oscillator potential and ignore $\Delta N=2$ components from the quadrupole moment and angular momentum operators. The exact solution of the $SU_3$ model is well known group-theoretically. As the driving Hamiltonian of the DNFT, we have the Hartree field Hamiltonian of the cranking model (in the present paper, the cranked harmonic oscillator is abbreviated as CHO). Many authors since Valatin carried out the diagonalization of the CHO. We calculate the eigenvalues of the $SU_3$ Hamiltonian by taking the expectation values with respect to the CHO eigenstates (see also Ripka et al.). We regard the reproduction of this result by the perturbative DNFT as guaranteeing its applicability for studying the collective rotation of a system with more general potentials. As another merit of the $SU_3$ model, we can point out the similarity with the pair rotation problem, especially in the single-j limit. In both cases, we find the complete attenuation of the coupling between the rotation and the odd particle (RPC).

In § 2, we give the general formulation of the DNFT. We impose the Hartree self-consistency condition between the quadrupole moment and the potential deformation. Section 3 is devoted to deriving the eigenvalues of the $SU_3$ Hamiltonian by the
diagonalization of the CHO. In § 4, we study the low spin states by using the perturbative DNFT and compare the results with the cranking model.\textsuperscript{4) We consider in § 5 the high spin states as superposing a small rotation to the system rotating already with a large angular velocity. Focusing on the high spin states of the odd mass system in § 6, we can show that the RPC is completely attenuated. Finally in § 7 we summarize the results.

§ 2. General formulation

We consider a nuclear system with the Hamiltonian

\[ H = H_{\text{sp}} - \frac{1}{2} \chi \sum_{\mu} \bar{Q}_{\mu} \bar{Q}_{\mu}, \tag{2\cdot1} \]

in which \( H_{\text{sp}} \) is a one-body Hamiltonian with a spherical symmetry and \( \bar{Q}_{\mu} \) is the quadrupole moment operator. Corresponding time-dependent Hamiltonian for a deformed field is given by

\[ H_{\text{def}}(t) = H_{\text{sp}} - \sum_{\mu} a_{\mu}(t) \bar{Q}_{\mu}, \tag{2\cdot2} \]

in the laboratory frame. Collective parameters \( a_{\mu}(t) \) are transformed to those in an intrinsic frame, \( a_{\nu}(t) \), by

\[ a_{\mu}(t) = \sum_{\nu} D^{(2)(\nu)}_{\mu}(\phi(t), \theta(t), \psi(t)) a_{\nu}(t). \tag{2\cdot3} \]

Ignoring the vibrational excitations, we take \( a_{\nu}(t) \) to be real and independent of time:

\[ a_{\nu}(t) = a_{\nu} = \text{real}. \tag{2\cdot4} \]

In terms of the angular momentum operator \( \hat{L}_{\nu} \), we express the rotation operator \( R \) as

\[ R = \exp(-i\phi \hat{L}_{\nu} / \hbar) \exp(-i\theta \hat{L}_{\nu} / \hbar) \exp(-i\psi \hat{L}_{\nu} / \hbar). \tag{2\cdot5} \]

Then Eq. (2·2) contains the following combination of \( \bar{Q}_{\mu} \):

\[ \sum_{\mu} D^{(2)\dagger}_{\nu}(\phi, \theta, \psi) \bar{Q}_{\mu} = \bar{Q}_{\nu}(t) = \bar{Q}_{\nu}(r'(t)) = R \bar{Q}_{\nu}(r) R^{-1}. \tag{2\cdot6} \]

Now we transform the time-dependent Schrödinger equation

\[ i\hbar \frac{\partial \Psi(t)}{\partial t} = H_{\text{def}}(t) \Psi(t) \tag{2\cdot7} \]

into the body-fixed frame. The wave function is transformed as

\[ \Psi = R \phi, \tag{2\cdot8} \]

in which \( \phi \) is the stationary eigenstates of \( \tilde{H} \)

\[ \tilde{H} \phi = \tilde{E} \phi, \quad \tilde{H} = R^{-1} H_{\text{def}}(t) R - i\hbar R^{-1} \frac{\partial R}{\partial t}. \tag{2\cdot9} \]

We use two relations
where

\[ \Omega_x = -\dot{\phi} \sin \theta \cos \phi + \dot{\theta} \sin \phi, \]

\[ \Omega_y = \dot{\phi} \sin \theta \sin \phi + \dot{\theta} \cos \phi, \]

\[ \Omega_z = \dot{\phi} \cos \theta + \dot{\phi}, \]

are the angular velocities associated with the rotation of the coordinate axes, referred to the body-fixed frame. Then we obtain

\[ \vec{H} = H_{sp} - \sum_{\nu=-2}^{2} a_{\nu} \vec{Q}_{\nu} - \sum_{\kappa=x,y,z} \Omega_{\kappa} \vec{L}_{\kappa}. \]

Although \( \Omega_{\kappa} \) are the functions of \( t \) in general, uniform rotations are possible, for example, when any one of three Euler's angles depends on time linearly while the others are independent of \( t \). The deviation from the uniform rotation can be treated by regarding it as the wobbling motion.

We decompose \( \vec{H} \) into \( H_{sp} + V \), in which \( V \) is composed of the Coriolis coupling given by the last term of Eq. (2.13) and higher order correction terms to be specified in later sections. We take into account the effect of \( V \) by using the effective operator \( \mathcal{O}_{\text{eff}} \) for any operator \( \mathcal{O} \). Perturbative expansions of \( \mathcal{O}_{\text{eff}} \) in powers of \( V \) (\( \vec{Q} \) in the present case) were given in Ref. 1). We take the Hartree self-consistency condition

\[ a_{\nu} = \chi \vec{Q}_{\nu,\text{eff}}, \]

which determines the deformation parameters \( a_{\nu} \) for any quadrupole strength. This condition guarantees minimization of the energy as will be discussed later. Similarly, the effective angular momentum operator \( \vec{L}_{\kappa,\text{eff}} \) along the \( \kappa \)-th principal axis will be derived explicitly and the moment of inertia \( \mathcal{J}_{\kappa} \) is then defined by the relation

\[ \mathcal{J}_{\kappa} \Omega_{\kappa} = \vec{L}_{\kappa,\text{eff}}. \]

In the following, we take the intrinsic \( x \) axis as the cranking axis and write \( \Omega_{x} \) as \( \Omega \) simply. In Appendix B, we will consider the case of a tilted rotation.

§ 3. Diagonalization of the \( SU_3 \) Hamiltonian

In the following we restrict ourselves to the \( SU_3 \) model for \( H \), in which we neglect the effect of \( \Delta N = 2 \) transitions. The one-body Hamiltonian \( H_{sp} \) with the spherical symmetry is expressed as

\[ H_{sp} = \sum_{i=1}^{A} (h_{sp})_i, \quad h_{sp} = \frac{p^2}{2m} + \frac{1}{2} m \omega^2 r^2. \]

When we use the stretched coordinates
and introduce the boson operators $c_x, c_x^\dagger$ by

$$
x' = \frac{1}{i\sqrt{2}}(c_x^\dagger - c_x), \quad y' = \frac{1}{\sqrt{2}}(c_y^\dagger + c_y), \quad z' = \frac{1}{i\sqrt{2}}(c_x^\dagger - c_x),$$

$$
p_x' = \frac{1}{\sqrt{2}}(c_x^\dagger + c_x), \quad p_y' = \frac{i}{\sqrt{2}}(c_y^\dagger - c_y), \quad p_z' = \frac{1}{\sqrt{2}}(c_x^\dagger + c_x),$$

we can write $h_{sp}$ as

$$h_{sp} = \hbar \omega_0 \sum_{k=x,y,z} \left( c_k^\dagger c_k + \frac{1}{2} \right).$$

From now on, we put $\hbar = 1$. The quadrupole moment operator $\mathcal{Q}_v = \sum_{i=1}^3 q_v^i$ can be expressed as

$$q_0 = 2c_x^\dagger c_x - c_x^\dagger c_x - c_y^\dagger c_y,$$

$$q_1 = -q_1^i = \sqrt{\frac{3}{2}}[c_y^\dagger c_x - c_x^\dagger c_y - (c_x^\dagger c_x + c_y^\dagger c_y)],$$

$$q_2 = q_2^i = \sqrt{\frac{3}{2}}[c_x^\dagger c_x - c_y^\dagger c_y + (c_x^\dagger c_y + c_y^\dagger c_x)].$$

These correspond to the quadrupole operators in the $SU_3$ model: $\mathcal{Q}_v = [\mathcal{Q}_v(r') + \mathcal{Q}_v(p')]/2$, associated with the stretched coordinates. Similarly, we can write the angular momentum operator $\mathcal{L}_v = \sum_{i=1}^3 I_v^i$ as

$$l_x = y'p_z' - z'p_y' = c_y^\dagger c_x + c_x^\dagger c_y,$$

$$l_y = i(c_x^\dagger c_x - c_y^\dagger c_y), \quad l_x = -(c_x^\dagger c_y + c_y^\dagger c_x).$$

Group-theoretically, the $SU_3$ Hamiltonian is known to have the eigenvalues

$$E(\lambda, \mu, L) = \omega_0(\Sigma_0 + \Sigma_2 + \Sigma_3) - 2\chi(\lambda^2 + \lambda\mu + \mu^2 + 3\lambda + 3\mu) + \frac{3}{2}L(L+1).$$

Here $\Sigma_0, \Sigma_2$ and $\Sigma_3$ are the total number of quanta in the $x, y$ and $z$ direction, respectively. The labels $\lambda$ and $\mu$ of an irreducible representation in the $SU_3$ model are expressed as

$$\lambda = \Sigma_3 - \Sigma_2, \quad \mu = \Sigma_2 - \Sigma_1,$$

assuming a configuration fulfilling the inequality $\Sigma_0 \geq \Sigma_2 \geq \Sigma_1$. The eigenvalue of angular momentum $L$ is restricted by

$$L_{\text{min}} \leq L \leq L_{\text{max}},$$

where

$$L_{\text{min}} = K, \quad L_{\text{max}} = K + \lambda, \quad K = \mu, \mu - 2, \cdots, (0, 1),$$

$L = \text{even only if } K = 0$. 
In the rest of this section, we derive the energy, which is given by the expectation value of $H$ with the basis which diagonalizes the CHO Hamiltonian $\tilde{H}$.

As usual, the intrinsic frame is chosen to coincide with the principal axes of the deformed nucleus, so that

$$a_1 = a_{-1} = 0, \quad a_2 = a_{-2}.$$  \hfill (3.12)

The driving Hamiltonian $\tilde{H} = \sum_{j=1}^{N} \tilde{h}_j$ is given by

$$\tilde{h}_j = h_{2p} - \left[ a_0 q_0 + a_2 (q_2 + q_2^*) \right] - Q \Omega \xi$$

$$= \sum_{\epsilon = x, y, z} \omega_{\epsilon} \left( c_{\epsilon}^* c_{\epsilon} + \frac{1}{2} \right) - \Omega \left( c_{y}^* c_{x} + c_{x}^* c_{y} \right), \tag{3.13}$$

where

$$\omega_x = \omega_0 + a_0 - \sqrt{6} a_2, \quad \omega_y = \omega_0 + a_0 + \sqrt{6} a_2, \quad \omega_z = \omega_0 - 2 a_0. \tag{3.14}$$

We can diagonalize $\tilde{h}$ as

$$\tilde{h} = \sum_{\sigma = 1}^{3} \omega_{\sigma} \left( c_{\sigma}^* c_{\sigma} + \frac{1}{2} \right) \tag{3.15}$$

by the transformation

$$c_x = c_1, \quad c_y = i c_2 + v c_3, \quad c_z = - v c_2 + i c_3. \tag{3.16}$$

The transformation coefficients $u$ and $v$ are given by

$$u = \sqrt{\frac{1}{2} (1 + q)}, \quad v = \sqrt{\frac{1}{2} (1 - q)} \tag{3.17a}$$

with

$$q = (\omega_y - \omega_z)/d, \quad d = (\omega_y - \omega_z)^2 + 4 \Omega^2. \tag{3.17b}$$

The eigen-frequencies $\omega_{\sigma}$ take the values

$$\omega_1 = \omega_x,$$

$$\omega_2 = \frac{1}{2} (\omega_y + \omega_z) + \frac{1}{2} q (\omega_y - \omega_z) + \sqrt{1 - q^2} \Omega = \frac{1}{2} (\omega_y + \omega_z) + \frac{1}{2} d,$$

$$\omega_3 = \frac{1}{2} (\omega_y + \omega_z) - \frac{1}{2} d. \tag{3.18}$$

In terms of $c_{\sigma}^*$ and $c_{\sigma}$, we can express $q_{\nu}$ and $l_{\xi}$ as

$$q_0 = (2u^2 - v^2)c_3^* c_3 + (2v^2 - u^2)c_2^* c_2 - c_1^* c_1 - 3uv (c_2^* c_3 + c_3^* c_2),$$

$$q_1 = \sqrt{\frac{3}{2}} [c_2^* c_3 - c_3^* c_2 - u (c_1^* c_3 + c_3^* c_1) + v (c_1^* c_2 + c_2^* c_1)],$$
\[ q_x = \sqrt{\frac{3}{2}} \left[ c_1 c_1 - u^2 c_2 c_2 - v^2 c_2 c_2 - u v (c_2 c_3 + c_3 c_2) + u (c_1 c_2 - c_2 c_1) + v (c_1 c_3 - c_3 c_1) \right], \quad (3.19) \]

\[ l_x = 2 u v (c_1 c_2 - c_2 c_1) + (u^2 - v^2)(c_2 c_3 + c_3 c_2), \]

\[ l_y = i [u (c_1 c_3 - c_3 c_1) - v (c_1 c_2 - c_2 c_1)], \]

\[ l_z = - [u (c_1 c_2 + c_2 c_1) + v (c_1 c_3 + c_3 c_1)]. \quad (3.20) \]

Nonzero expectation values of \( \bar{Q}_y \) and \( \bar{L}_x \) in the eigenstates \((\lambda, \mu)\) of \( \bar{H} \) are

\[ \langle \bar{Q}_y \rangle = \frac{1}{2} \left[ \lambda + 2 \mu + 3 q \lambda \right], \quad \langle \bar{Q}_x \rangle = \frac{1}{2} \sqrt{\frac{3}{2}} \left[ - (\lambda + 2 \mu) + q \lambda \right], \quad (3.21a) \]

\[ \langle \bar{L}_x \rangle = \sqrt{1 - q^2}, \quad (3.21b) \]

where Eq. (3.8) relates \( \lambda \) and \( \mu \) to \( \Sigma_a \) given by

\[ \Sigma_a = \left\langle \sum_{a} (c_a^+ a_c + \frac{1}{2}) \right\rangle. \quad (3.22) \]

The deformation parameters \( a_\nu \) and the frequencies of the deformed field \( \omega_x \) affected by the rotation are uniquely determined by the Hartree self-consistency condition,

\[ a_\nu = \chi \langle \bar{Q}_\nu \rangle, \quad \nu = 0, 2, \quad (3.23a) \]

\[ \omega_x = \omega_0 + 2 \chi (\lambda + 2 \mu), \]

\[ \omega_y = \omega_0 - \chi (\lambda + 2 \mu) + 3 q \chi \lambda, \]

\[ \omega_z = \omega_0 - \chi (\lambda + 2 \mu) - 3 q \chi \lambda. \quad (3.23b) \]

By using the result \( \omega_y - \omega_x = 6 \chi q \lambda \) in Eq. (3.17b), we can write \( q \) as

\[ q = \begin{cases} \sqrt{1 - \left( \frac{Q}{Q_{\text{crit}}} \right)^2} & \text{for } 0 \leq Q \leq Q_{\text{crit}} = 3 \chi \lambda, \\ 0 & \text{for } Q \geq Q_{\text{crit}}. \end{cases} \quad (3.24) \]

If we note the dependence of \( a_\nu \) and \( \omega_x \) on the rotation expressed very simply in terms of \( q \), we can easily find the well-known change of the nuclear shape with the increase of \( Q \).

It is surprising to realize that the eigen-frequencies \( \omega_x \) of \( \bar{h} \), namely, of the normal modes in the rotating body-fixed frame turn out to be constant for any angular velocity

\[ \omega_1 = \omega_0 + 2 \chi (\lambda + 2 \mu), \]

\[ \omega_2 = \omega_0 + 2 \chi (\lambda - \mu), \]

\[ \omega_3 = \omega_0 - 2 \chi (2 \lambda + \mu). \quad (3.25) \]

Many level crossings occur as \( Q \) increases when the deformation is given and fixed. However, no level crossing occurs for a fixed configuration \((\lambda, \mu)\), when the equilibrium deformation is determined by the Hartree condition for the consistency between the density distribution and the mean field, and consequently when the effect of the shape change with increasing \( Q \) is taken into account. Whether or not a unique and simple feature similar to the above result may persist in the intrinsic
system under strong rotational disturbances in a more realistic case is yet to be investigated.

By equating the angular momentum $\langle L_x \rangle$ (3.21b) with $\mathcal{J}_\Omega$ by the aid of Eq. (3.24), we have the moment of inertia

$$\mathcal{J} = \frac{1}{3\chi}$$

(3.26)

agreeing with that in Eq. (3.7). The lower and upper limits on $\langle L_x \rangle$ are exactly given by Eq. (3.10) with $K = \langle \hat{L}_x \rangle$ equal to zero.

Next we calculate the energy $E$. The condition that the expectation value of $H$ is minimum under a fixed angular momentum $I = \langle L_x \rangle$ in the laboratory frame is equivalent to that of the minimization of the expectation value of $H'$ (3.28) in the intrinsic frame. The eigenstate $|a_\nu, \Omega \rangle$ of $\tilde{H}$ of Eq. (3.13) has the eigenvalue

$$\tilde{E}(a_\nu, \Omega) = \omega_1 \Sigma_1 + \omega_2 \Sigma_2 + \omega_3 \Sigma_3.$$  

(3.27)

First, we take the expectation value of $H'$ defined by

$$H' = H - \Omega \tilde{L}_x$$

(3.28)

in the $|a_\nu, \Omega \rangle$ state, which is written as

$$E'(a_\nu, \Omega) = \langle a_\nu, \Omega | H' | a_\nu, \Omega \rangle$$

$$= \tilde{E}(a_\nu, \Omega) + \sum_\nu a_\nu \langle \tilde{Q}_\nu \rangle - \frac{\chi}{2} \sum_\nu \langle \tilde{Q}_\nu \rangle^2.$$  

Its derivative with respect to $a_\nu$ is

$$\frac{\partial}{\partial a_\nu} E'(a_\nu, \Omega) = \frac{\partial}{\partial a_\nu} \tilde{E}(a_\nu, \Omega) + \left\{ \langle \tilde{Q}_\nu \rangle + a_\nu \frac{\partial}{\partial a_\nu} \langle \tilde{Q}_\nu \rangle \right\} - \chi \langle \tilde{Q}_\nu \rangle \frac{\partial}{\partial a_\nu} \langle \tilde{Q}_\nu \rangle$$

$$= (a_\nu - \chi \langle \tilde{Q}_\nu \rangle) \frac{\partial}{\partial a_\nu} \langle \tilde{Q}_\nu \rangle,$$

because the derivative $\partial \tilde{E}/\partial a_\nu$ is equal to $-\langle \tilde{Q}_\nu \rangle$, due to Feynman's theorem. Thus the Hartree self-consistency condition (2.14) minimizes $E'(a_\nu, \Omega)$, reducing it to

$$E'(a_\nu, \Omega) = \tilde{E}(a_\nu, \Omega) + \frac{\chi}{2} \sum_\nu \langle \tilde{Q}_\nu \rangle^2.$$  

(3.29)

We now use the relations

$$\omega_1 \Sigma_1 + \omega_2 \Sigma_2 + \omega_3 \Sigma_3 = \omega_0 (\Sigma_1 + \Sigma_2 + \Sigma_3) - \chi [(\lambda + 2 \mu)^2 + 3q^2 \lambda^2] - \Omega \langle \tilde{L}_x \rangle,$$  

(3.30a)

$$\sum_\nu \langle \tilde{Q}_\nu \rangle^2 = (\lambda + 2 \mu)^2 + 3q^2 \lambda^2,$$  

(3.30b)

and extract the dependence on $\langle \tilde{L}_x \rangle^2$ (3.21b) from the $q^2 \lambda^2$ term. The expectation value $E$ of $H$ (2.1) is then written as

$$E = E'(a_\nu, \Omega) + \Omega \langle \tilde{L}_x \rangle = E_0 (\lambda, \mu) + \frac{3}{2} \chi \langle \tilde{L}_x \rangle^2,$$  

(3.31)
where the intrinsic energy is given by

\[ E_0(\lambda, \mu) = \omega_0(\Sigma_1 + \Sigma_2 + \Sigma_3) - 2\chi(\lambda^2 + \lambda \mu + \mu^2). \]  

(3.32)

We note the difference \( \Delta E = -6\chi(\lambda + \mu) \) from the exact intrinsic energy in Eq. (3.7). In Appendix A, we will discuss this discrepancy by including the fluctuation effect (Fock term) to the Hartree approximation.

§ 4. Low spin states

In this section, we suppose that the nucleus is rotating slowly. Taking the perturbative approach, we put the condition that the intrinsic configuration must not be altered by the rotation:

\[ \Sigma_1 = \Sigma_2, \quad \Sigma_2 = \Sigma_3, \quad \Sigma_3 = \Sigma_1, \]  

(4.1)

in which \( \Sigma_\kappa (\kappa = x, y, z) \) denotes \( \langle \Sigma_\kappa A (c_\kappa^+ c_\kappa / 2) \rangle \) averaged over the Fermi vacuum of \( c_\kappa \). We expand \( q \) (3.17b) in powers of \( \Omega \)

\[ q \approx 1 - 2\Omega^2/\Delta \omega^2, \quad \Delta \omega = \omega_y - \omega_z. \]  

(4.2)

Expectation values of \( \hat{Q}_\nu \) and \( \hat{L}_x \) (3.21) are written as

\[ \langle \hat{Q}_0 \rangle \approx \hat{Q}_0 - 3\chi\lambda\Omega^2/\Delta \omega^2, \quad \hat{Q}_0 = 2\lambda + \mu, \]  

\[ \langle \hat{Q}_2 \rangle \approx \hat{Q}_2 - \sqrt{\frac{3}{2}}\chi\lambda\Omega^2/\Delta \omega^2, \quad \hat{Q}_2 = -\sqrt{\frac{3}{2}}\mu, \]  

\[ \langle \hat{L}_x \rangle \approx 2\chi\lambda\Omega/\Delta \omega. \]  

(4.3)

(4.4)

Correspondingly, \( a_\nu \) (3.23a) and \( \omega_\kappa \) (3.18) are expanded to second order, as

\[ a_0 \approx \hat{a}_0 - 3\chi\lambda\Omega^2/\Delta \omega^2, \quad \hat{a}_0 = \chi(2\lambda + \mu), \]  

\[ a_2 \approx \hat{a}_2 - \sqrt{\frac{3}{2}}\chi\lambda\Omega^2/\Delta \omega^2, \quad \hat{a}_2 = -\sqrt{\frac{3}{2}}\chi\mu, \]  

\[ \omega_2 \approx \omega_y + \Omega^2/\Delta \omega, \quad \omega_3 \approx \omega_z - \Omega^2/\Delta \omega. \]  

(4.5)

(4.6)

In the conventional cranking model, “the energy” is defined by a sum of single particle energies in the mean field. “The energy in the rotating frame” is

\[ E \approx \omega_x\Sigma_x + \omega_y\Sigma_y + \omega_z\Sigma_z - \lambda\Omega^2/\Delta \omega. \]  

(4.7)

By equating Eq. (4.4) with \( \partial \Omega \), we find the moment of inertia

\[ \partial = 2\lambda/\Delta \omega + O(\Omega^2). \]  

(4.8)

Only when we add the relation (3.14) between \( \omega_\kappa \) and the deformation parameters, \( \partial \) can reduce approximately to the \( SU_3 \) value \( 1/3\chi \). Thereby we obtain “the energy in the laboratory frame”

\[ E = E + \Omega \langle \hat{L}_x \rangle \approx \omega_x\Sigma_x + \omega_y\Sigma_y + \omega_z\Sigma_z + \frac{1}{2}\partial \langle \hat{L}_x \rangle^2. \]  

(4.9)
However, the mean field is derived from the quadrupole interaction and its deformation parameters are determined by the Hartree self-consistency condition (2.14) at equilibrium. The nuclear shape changes with the rotation even if the configuration is fixed. Taking this fact into account with the relation (3.14), "the intrinsic energy"

\[ \omega_x \Sigma_x + \omega_y \Sigma_y + \omega_z \Sigma_z = \omega_0 (\Sigma_x + \Sigma_y + \Sigma_z) - 4 \chi (\lambda^2 + \mu + \mu^2) + \frac{1}{2} \beta <\bar{L}_x>^2 \]  

\[
\text{(4.10)}
\]
does not agree with the exact value (3.7) or its approximation (3.32) and also contains the extra rotational energy. Thus we can also point out another discrepancy that the moment of inertia given by the total rotational energy is different from that obtained from the angular momentum relation by a factor of 2. This discrepancy can be removed naturally if we calculate the expectation value of the original Hamiltonian as in § 3.

We now study the low spin behavior of the nucleus by the DNFT. In order to estimate the effective quadrupole moment \( \tilde{Q}_{\text{eff}} \) up to second order in \( \Omega \), we must follow the discussion presented in the case of the pair rotation. Namely, we follow the Hartree self-consistency condition (3.23a) more rigorously by taking into account the dependence of the collective parameters \( a_\nu \) on \( \Omega \), and impose the additional condition that \( \bar{L}_{\text{c, eff}} \) does not contain terms higher than first order in \( \Omega \). This condition turns out to be fulfilled automatically, when we simplify the treatment by utilizing the exact dependence of \( a_\nu \) on \( \Omega \).

According to the DNFT, we here expand \( q \) given by Eq. (3.24) instead of Eq. (3.17b) in powers of \( \Omega \). In this case, \( \Delta \omega \) reduces to \( 2\Omega_{\text{crit}} = 6\chi \lambda \). Using the expansions (4.5), we can divide \( \widetilde{h} \) as

\[
\widetilde{h} = h_0 + \nu, \quad \nu = v_1 + v_2, \quad \text{(4.11a)}
\]
where

\[
h_0 = h_0 - \left[ a_{0} q_0 + a_2 (q_2 + q_2^*) \right], \quad \text{(4.11b)}
\]

\[
v_1 = -\Omega l_x = -\Omega (c_y^* c_z + c_z^* c_y), \quad \text{(4.11c)}
\]

\[
v_2 = \frac{3}{4} \chi \lambda \Omega^2 / \Omega_{\text{crit}}^2 \left( q_0 + \frac{1}{\sqrt{6}} (q_2 + q_2^*) \right)
\]

\[
= \frac{3}{2} \chi \lambda \Omega^2 / \Omega_{\text{crit}}^2 (c_x^* c_y - c_y^* c_x). \quad \text{(4.11d)}
\]

Since the diagonal operator \( v_2 \) does not contribute to any effective operator in the first order perturbation, it is not necessary to take into account the \( \Omega \) dependence of \( a_\nu \). We will see in § 5.2 that the \( v_2 \) term quadratic in \( \Omega \) can contribute in second order for high spin states.

As the effective operators in zeroth order, we obtain
\[ \tilde{Q}_{\nu,\text{eff}}[0] = \tilde{Q}_\nu, \quad (\tilde{Q}_1 = 0), \quad \tilde{L}_{x,\text{eff}}[0] = 0, \]
\[ H_{0,\text{eff}}[0] = \omega_x \Sigma_x + \omega_y \Sigma_y + \omega_z \Sigma_z. \quad (4.12) \]

Their first order terms are
\[ \tilde{Q}_{\nu,\text{eff}}[1] = 0, \quad \tilde{L}_{x,\text{eff}}[1] = \lambda \Omega / \Omega_{\text{crit}} = \mathcal{J} \Omega, \]
\[ H_{0,\text{eff}}[1] = 0. \quad (4.13) \]

In second order, the diagonal parts of \( \tilde{Q}_0 \) and \( \tilde{Q}_2 \) contribute
\[ \tilde{Q}_{0,\text{eff}}[2] = -\frac{3}{4} \frac{1}{\Omega^2} / \Omega_{\text{crit}}^2, \quad \tilde{Q}_{2,\text{eff}}[2] = -\frac{1}{4} \sqrt{\frac{3}{2}} \lambda \Omega^2 / \Omega_{\text{crit}}^2, \]
\[ \tilde{Q}_{1,\text{eff}}[2] = 0, \quad \tilde{L}_{x,\text{eff}}[2] = 0, \]
\[ H_{0,\text{eff}}[2] = \frac{1}{2} \lambda \Omega^2 / \Omega_{\text{crit}}^2 = \frac{1}{2} \mathcal{J} \Omega^2. \quad (4.14) \]

Finally we can get the effective Hamiltonian \( H_{\text{eff}} \) in the form
\[ H_{\text{eff}} = H_{0,\text{eff}} - \frac{1}{2} \chi \sum_{\nu} (\tilde{Q}_{\nu,\text{eff}})^2 = E_0(\lambda, \mu) + \frac{1}{2} \mathcal{J} (\tilde{L}_{x,\text{eff}})^2 \quad (4.15) \]

up to the order considered. Here \( \mathcal{J} = 1/3 \chi \) and the intrinsic energy \( E_0(\lambda, \mu) \) coincides with Eq. (3.32). In cases more general than the present model, we cannot expect that the effective angular momentum operator \( \tilde{L}_{x,\text{eff}} \) involves only the linear term in \( \Omega \). Similarly, the effective quadrupole moment operator \( \tilde{Q}_{\nu,\text{eff}} \) and the deformation parameter \( a_\nu(\Omega) \) include higher order terms in \( \Omega \), and thus the rotational energy can deviate from that of a pure rotor. It should be noted that the importance of the additional coupling term of Eq. (4.11) is worth while to be investigated in realistic cases.

§ 5. High spin states

5.1. First-order approximation

We now turn our discussion to the region of large angular momenta. We exert an infinitesimal rotation with the angular velocity \( \delta \Omega \) to the nucleus which is already rotating with \( \Omega \) about the same intrinsic \( x \) axis. The analogy with the case of the pair rotation is clear, if we make the correspondence between \( L_\nu \), \( \Omega \) and \( a_\nu \) with the nucleon number \( N_\nu \), the chemical potential \( \lambda \) and the energy gap \( \Delta \), respectively. We suppose the deformations and the frequencies to change by the additional rotation \( \delta \Omega \). We divide the one-body driving Hamiltonian as
\[ \tilde{h}(\Omega + \delta \Omega) = h_0 + v, \quad (5.1a) \]
where
\[ h_0 = \tilde{h}(\Omega) = h_{\text{sp}} - [a_0(\Omega)q_0 + a_2(\Omega)(q_2 + q_2^*)] - \Omega l_x \]
\[ = \frac{3}{2} \omega_\phi \left( c_\phi^* c_\phi + \frac{1}{2} \right), \quad (5.1b) \]
\[ v = \hat{h}(\Omega + \delta \Omega) - \hat{h}(\Omega). \]  

(5.1c)

The deformation parameters determined at the angular velocity \( \Omega \) by the Hartree self-consistency condition are given already by Eqs. (3.23) and (3.24). The eigen-frequencies \( \omega_\alpha \) are independent of \( \Omega \) as given by Eq. (3.25). The whole effects arising from the additional rotation \( \delta \Omega \) are contained in \( v \).

We can write \( v \) up to first order in \( \delta \Omega \) as

\[ v_1 = -\delta \Omega \left[ l_x + \sum_v \frac{\partial}{\partial \Omega} a_v(\Omega) q_v \right]. \]  

(5.2)

From Eqs. (3.21a) and (3.23a) we get

\[ \frac{\partial a_0(\Omega)}{\partial \Omega} = -\frac{\sqrt{1-q^2}}{2q}, \quad \frac{\partial a_2(\Omega)}{\partial \Omega} = \frac{1}{\sqrt{6}} \frac{\partial a_0(\Omega)}{\partial \Omega}. \]  

(5.3)

Substituting these into Eq. (5.2), we find

\[ v_1 = -\frac{1}{q} \delta \Omega (c_3^+ c_3 + c_3^+ c_3). \]  

(5.4)

In general, we do not know the analytical forms of \( a_\nu(\Omega) \). From the Hartree self-consistency conditions required in higher order, we can, however, obtain the equations for determining \( \frac{\partial a_\nu(\Omega)}{\partial \Omega} \). We also note that the first term of Eq. (5.2) is just the conventional cranking term due to the change in the angular velocity, while the second arises from the change in deformation parameters \( a_\nu \) with the additional rotation under the self-consistency. Due to the second term, the cranking term is seriously modified. As compared with the original coupling having both the diagonal and nondiagonal components with the coefficient \( q \), the new cranking term has only a nondiagonal component with the coefficient \( 1/q \).

We can express the effective angular momentum \( \hat{L}_{\xi, \text{eff}} \) as

\[ \hat{L}_{\xi, \text{eff}}[0] = J_0 = \sqrt{1-q^2} \lambda = \mathcal{J} \Omega, \quad \hat{L}_{\xi, \text{eff}}[1] = R = \mathcal{J} \delta \Omega, \]  

(5.5)

where

\[ \mathcal{J} = \frac{2\lambda}{\omega_2 - \omega_3} = \frac{1}{3 \chi}. \]  

(5.6)

Here \( J_0 \), same as given by Eq. (3.21b), is the angular momentum of the nucleus rotating with \( \Omega \) while \( R \) is the additional one induced by the increase of the angular velocity by \( \delta \Omega \). We obtain the following effective forms for \( \hat{Q}_\nu \):

\[ \hat{Q}_{0, \text{eff}}[0] = \frac{1}{2} \left[ \lambda + 2 \mu + 3 q \lambda \right], \quad \hat{Q}_{2, \text{eff}}[0] = \frac{1}{2} \sqrt{\frac{3}{2} \left[ - \left( \lambda + 2 \mu \right) + q \lambda \right]}, \]

\[ \hat{Q}_{0, \text{eff}}[1] = -\frac{3}{2} \sqrt{1-q^2} \mathcal{J} \delta \Omega = \sqrt{6} \hat{Q}_{2, \text{eff}}[1]. \]
where $\bar{Q}_{\nu,\text{eff}}[0]$ is equivalent to Eq. (3.21a). The effective Hamiltonian for the single particle motion in the CHO is given by

$$H_{\text{eff}}[0] = \omega_1 \Sigma_1 + \omega_2 \Sigma_2 + \omega_3 \Sigma_3, \quad H_{\text{eff}}[1] = 0,$$

$$H_{\text{eff}}[2] = \frac{1}{2q^2} \mathcal{G} (\delta \Omega)^2. \quad (5.8)$$

Since the deformation parameters are determined at $Q$, we write the effective total Hamiltonian $H_{\text{eff}}$ as

$$H_{\text{eff}} = H_{\text{eff}}[0] + \left( \sum_{\nu} \mathcal{G}(Q_{\nu,\text{eff}}[0]) \right) - \frac{1}{2} \mathcal{G} \left( \sum_{\nu} (Q_{\nu,\text{eff}}[0])^2 \right) + \Omega L_{x,\text{eff}} \quad (5.9)$$

in the Hartree approximation. In the second term of the right-hand side, we find the cancellation of the first order term in $\delta \Omega$, so that we have

$$\frac{1}{2} \mathcal{G} \sum_{\nu} (Q_{\nu,\text{eff}}[0])^2 - \frac{1}{2q^2} \mathcal{G} (\delta \Omega)^2. \quad (5.10)$$

We then find

$$H_{\text{eff}} = E_0(Q) + \mathcal{G} \delta \Omega + \frac{1}{2} \mathcal{G} (\delta \Omega)^2 = E_0(J_0) + \frac{1}{\mathcal{G}} \mathcal{G} R + \frac{1}{2} \mathcal{G} R^2, \quad (5.11)$$

where $E_0(Q)$, expressed also as $E_0(J_0)$, is the energy of the nucleus rotating with $Q$:

$$E_0(Q) = \omega_1 \Sigma_1 + \omega_2 \Sigma_2 + \omega_3 \Sigma_3 + \frac{1}{2} \mathcal{G} \sum_{\nu} (Q_{\nu,\text{eff}}[0])^2 + \Omega L_{x,\text{eff}}[0]$$

$$= E_0(\lambda, \mu) + \frac{1}{2} \mathcal{G} \Omega^2. \quad (5.12)$$

### 5.2. Second-order effects

We here evaluate $Q_{\nu,\text{eff}}$ and $L_{x,\text{eff}}$ up to second order in $\delta \Omega$ with the aim to show that the second order term in $L_{x,\text{eff}}$ is equal to zero and $H_{\text{eff}}$ is not affected by including the effect from the second order term if we invoke the self-consistency condition.

By using the second derivatives of $a_\nu$ with respect to $Q$ given by

$$\frac{\partial^2 a_\nu(Q)}{\partial Q^2} = -\frac{1}{2q^5}, \quad \frac{\partial^2 a_\nu(Q)}{\partial Q^2} = -\frac{1}{\sqrt{6}} \frac{\partial^2 a_\nu(Q)}{\partial Q^2}, \quad (5.13)$$

we get

$$v_2 = -\frac{1}{2} \left[ \frac{\partial^2 a_0}{\partial Q^2} q_0 + \frac{\partial^2 a_2}{\partial Q^2} (q_2 + q_2^*) \right] (\delta \Omega)^2$$

$$= \mathcal{G} \frac{1}{2\lambda} \frac{1}{q^5} (\delta \Omega)^2 \left[ q(c_3^+ c_3 - c_2^+ c_2) - \sqrt{1 - \frac{\mathcal{G}}{q^5}} (c_2^+ c_3 + c_3^+ c_2) \right]. \quad (5.14)$$

By means of the nondiagonal term, $v_2$ can contribute in second order in $\delta \Omega$ in contrast with the situation of § 4.
In second order, any effective operator is composed of the part obtained by acting \( n_1 \) twice and that having \( n_2 \) once. Writing in this order, \( \tilde{L}_{x,\text{eff}}[2] \) turns out to be zero:

\[
L_{x,\text{eff}}[2] = -\left( \frac{\delta Q}{q} \right)^2 (-2\sqrt{1-q^2}) \frac{\lambda}{(\omega_2-\omega_3)^3} + \sqrt{1-q^2} \frac{\mathcal{J}(\delta Q)^2}{q^2} = 0 .
\]  

(5·15)

The result that \( \tilde{L}_{x,\text{eff}}[2] \) is zero corresponds to our postulate made in the DNFT treatment for the pair rotation, in which the effective number operator is zero in second order.\(^1\)

In a similar manner, we can calculate \( \tilde{Q}_{v,\text{eff}}[2] \) as

\[
\tilde{Q}_{0,\text{eff}}[2] = \left( \frac{\delta Q}{q} \right)^2 (-3q) \frac{\lambda}{(\omega_2-\omega_3)^3} \frac{3(1-q^2)}{2q^2} \frac{\mathcal{J}(\delta Q)^2}{\omega_2-\omega_3} = \frac{\mathcal{J}}{4\lambda q^3} (\delta Q)^2 .
\]

(5·15)

\[
\tilde{Q}_{2,\text{eff}}[2] = \frac{1}{\sqrt{6}} \tilde{Q}_{0,\text{eff}}[2] , \quad \tilde{Q}_{1,\text{eff}}[2] = 0 .
\]

(5·15)

Thus, \( \alpha_{v}(\Omega + \delta \Omega) \) coincides with \( \chi \tilde{Q}_{v,\text{eff}} \) up to second order in \( \delta \Omega \), satisfying the self-consistency condition (2·14) to the same order. We note that \( H_{\text{eff}} \) remains unchanged from Eq. (5·11), because \( \tilde{Q}_{v,\text{eff}}[2] \) does not contribute in this order.

§ 6. High spin states in odd mass nucleus

—— Attenuation of the rotation-particle coupling ——

We consider in this section the odd mass nucleus with the spin \( I \), in which an extra particle occupies the valence orbitals outside the even-even core. We suppose that whereas the nucleus (or the core) is rotating with \( \Omega \) before adding the extra particle, the angular velocity of the total nucleus changes to \( \Omega + \delta \Omega \) after its presence. The number of quanta of the filled oscillator orbits in each direction changes from \( \Sigma_\nu \) of the core to \( \Sigma_\nu + \sigma_\nu \), in which \( \sigma_\nu \) is the contribution from the particle. We define \( \lambda_\nu \) and \( \mu_\nu \) by

\[
\lambda_\nu = \sigma_\nu - \sigma_\xi , \quad \mu_\nu = \sigma_\xi - \sigma_\mu .
\]

(6·1)

For \( \lambda_\nu \) and \( \mu_\nu \), we take the self-consistent values to the core. We can use \( \tilde{h} \) given in § 5 for evaluating the effective operators up to second order.

In the zeroth order, the effective angular momentum is written as

\[
\tilde{L}_{x,\text{eff}}[0] = J_0 + j ,
\]

(6·2a)

where \( J_0 \) represents the contribution from the core, while the effect of the extra particle is contained in \( j \), corresponding to the number of the extra particles in the case of the pair rotation.\(^2\)

\[
J_0 = \sqrt{1-q^2} \lambda = \mathcal{J} \Omega , \quad j = \sqrt{1-q^2} \frac{\lambda_\nu}{\lambda} = \mathcal{J} \frac{\lambda_\nu}{\lambda} \Omega , \quad \mathcal{J} = \frac{1}{3\chi} .
\]

(6·2b)

The first order term of the effective angular momentum is also written as \( R \)

\[
\tilde{L}_{x,\text{eff}}[1] = \tilde{R} = \left( \frac{\delta Q}{q} \right) \left( \frac{-2q}{\omega_2-\omega_3} \right) (\lambda + \lambda_\nu) = \mathcal{J} \delta \Omega ,
\]

(6·3)
where the term depending on $\lambda_0$ corresponds to the exchange term in the pair rotation problem. Due to the exchange term, we now have $J_t$, the moment of inertia of the total nucleus including the extra particle

$$J_t = J \left(1 + \frac{\lambda_0}{\lambda}\right). \tag{6.4}$$

As in § 5.2, we find that the second order term of the effective angular momentum vanishes:

$$\hat{L}_{x,\text{eff}}[2] = 0. \tag{6.5}$$

Summing up all contributions, the effective angular momentum is composed of the contribution from the core, the external particle and the additional rotation

$$L_{x,\text{eff}} = I = J_0 + j + R. \tag{6.6}$$

Since both $J_0$ and $j$ correspond to the rotation with the angular velocity $\Omega$, we put together them as $J = J_0 + j$. The additional rotation $R$ originates from the coupling between the core and the extra particle and is thus associated with the change in the total angular velocity. It represents the effect of readjustment of the rotational motion of the total system.

In a similar manner, we can express the zeroth order term in the effective quadrupole moment as the sum of the core and the particle parts

$$Q_{\nu,\text{eff}}[0] = Q_{\nu,c} + Q_{\nu,p}, \quad \nu = 0, 2, \tag{6.7}$$

where

$$Q_{0,c} = \frac{1}{2} [\lambda + 2\mu + 3q\lambda], \quad Q_{2,c} = \sqrt{\frac{3}{8}} \left[-(\lambda + 2\mu) + q\lambda\right], \tag{6.8a}$$

$$Q_{0,p} = \frac{1}{2} [\lambda_0 + 2\mu_0 + 3q_0\lambda_0], \quad Q_{2,p} = \sqrt{\frac{3}{8}} \left[-(\lambda_0 + 2\mu_0) + q_0\lambda_0\right]. \tag{6.8b}$$

We obtain higher order terms as

$$\hat{Q}_{0,\text{eff}}[1] = -\frac{3}{2} \frac{\sqrt{1-q^2}}{q} J_t \delta \Omega, \quad \hat{Q}_{2,\text{eff}}[1] = \frac{1}{\sqrt{6}} \hat{Q}_{0,\text{eff}}[1], \tag{6.9a}$$

$$\hat{Q}_{0,\text{eff}}[2] = -\frac{1}{4 \sqrt{2}q} J_t (\delta \Omega)^2, \quad \hat{Q}_{2,\text{eff}}[2] = \frac{1}{\sqrt{6}} \hat{Q}_{0,\text{eff}}[2]. \tag{6.9b}$$

The effective forms of $H_0$ are given by

$$H_{0,\text{eff}}[0] = \sum_{\sigma} \omega_\sigma (\Sigma_\sigma + \sigma_\sigma), \quad H_{0,\text{eff}}[2] = \frac{1}{2q^2} J_t (\delta \Omega)^2. \tag{6.10}$$

We note that Eqs. (6.9) and (6.10) contain $J_t$ rather than $J$.

By using these expressions, we can decompose $H_{\text{eff}}$ as
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\[ H_{\text{eff}} = H_{0,\text{eff}} + \sum_{\nu} \alpha_{\nu}(\Omega) \frac{\dot{Q}_{\nu,\text{eff}}}{2} + \Omega \dot{L}_{x,\text{eff}} \]
\[ = \tilde{H}_{\text{core}} + \tilde{H}_{\text{ext}} + \tilde{H}_{\text{rot}}, \tag{6.11a} \]

where

\[ \tilde{H}_{\text{core}} = \omega_1 \sigma_1 + \omega_2 \sigma_2 + \omega_3 \sigma_3 + \frac{\chi}{2} \sum_{\nu} (Q_{\nu,c})^2 + \Omega j, \tag{6.11b} \]
\[ \tilde{H}_{\text{ext}} = \omega_1 \sigma_1 + \omega_2 \sigma_2 + \omega_3 \sigma_3 - \frac{\chi}{2} \sum_{\nu} (Q_{\nu,p})^2 + \Omega j, \tag{6.11c} \]
\[ \tilde{H}_{\text{rot}} = H_{0,\text{eff}}[2] - \chi \sum_{\nu} Q_{\nu,p} \dot{Q}_{\nu,\text{eff}}[n \geq 1] - \frac{\chi}{2} \sum_{\nu} (\dot{Q}_{\nu,\text{eff}}[n \geq 1])^2 + \Omega \dot{L}_{x,\text{eff}}[1]. \tag{6.11d} \]

In Eq. (6.11d), we denote the terms linear and quadratic in $\delta \Omega$ as $\tilde{H}_{\text{rot}}[n]$ ($n=1, 2$), given by

\[ \tilde{H}_{\text{rot}}[1] = (\Omega + \frac{j}{2}) \mathcal{J}_{s} \delta \Omega = \frac{1}{2} \mathcal{J} R, \]
\[ \tilde{H}_{\text{rot}}[2] = \frac{1}{2} \mathcal{J} (\mathcal{J}_{s} \delta \Omega)^2 = \frac{1}{2} \mathcal{J} R^2. \tag{6.12} \]

We can regard $\tilde{H}_{\text{rot}}[2]$ as the collective rotation constructed on the rapidly rotating nucleus with $\Omega$. We obtain the usual rotation-particle coupling $- \mathcal{J} / \mathcal{J}$ by substituting the relation $R = I - J$ (Eq. (6.6)) to Eq. (6.12). However, we have the counter term in $\tilde{H}_{\text{rot}}[1]$, which is the result by considering the coupling between the total rotation and the extra particle and comes mainly from $Q_{\nu,p} \dot{Q}_{\nu,\text{eff}}[1]$. Thus Eq. (6.11d) reduces to

\[ \tilde{H}_{\text{rot}} = \frac{1}{2} \mathcal{J} (I^2 - J^2). \tag{6.13} \]

Other terms of $H_{\text{eff}}$ can be rewritten as

\[ \tilde{H}_{\text{core}} = \omega_0 (\sigma_1 + \sigma_2 + \sigma_3) - 2 \chi (\lambda^2 + \lambda \mu + \mu^2) + \frac{1}{2} \mathcal{J} J_0^2, \tag{6.14a} \]
\[ \tilde{H}_{\text{ext}} = \omega_0 (\sigma_1 + \sigma_2 + \sigma_3) - 2 \chi (\lambda_p^2 + \lambda_p \mu_p + \mu^2) + \frac{1}{2} \mathcal{J} J^2 \]
\[ - 2 \chi (2 \lambda \mu_p + 2 \mu \lambda_p + 2 \mu \mu_p) + \frac{1}{\mathcal{J}} J_0 j. \tag{6.14b} \]

We take the sum of the parts depending on the angular momenta and denote it as $\tilde{H}_{\text{rot}}$:

\[ \tilde{H}_{\text{rot}} = \frac{1}{2} \mathcal{J} J_0^2 + \frac{1}{2} \mathcal{J} J_0 j + \frac{1}{2} \mathcal{J} J^2 + \frac{1}{2} \mathcal{J} (I^2 - J^2), \]

which turns out to be

\[ \tilde{H}_{\text{rot}} = \frac{1}{2} \mathcal{J} I^2. \tag{6.15} \]

We find that the coupling between the rotation and the particle is completely attenu-
ated in the present $SU_3$ model, in parallel with the case of the pair rotation in the single-$j$ limit.\(^3\)

We denote the rest of $H_{\text{eff}}$ as $\tilde{H}_{\text{int}}$, which is given by

\[
\tilde{H}_{\text{int}} = \omega_0 \sum_{a=1}^{g} (\sigma_a + \sigma_a^\dagger) - 2\chi \left[ (\lambda + \lambda_p) \right] \left[ (\lambda + \lambda_p)(\mu + \mu_p) + (\mu + \mu_p) \right].
\]

Thus the total Hamiltonian

\[
H_{\text{eff}} = \tilde{H}_{\text{int}} + \tilde{H}_{\text{rot}}
\]

coincides with the results of § 3 for the total system including the external particle. It should be noticed that the total rotational energy is expressed by using $\tilde{J}$ rather than $J$ as the moment of inertia, which is consistent with the exact solution (3·7).

§ 7. Summary and discussion

With the DNFT we have studied the collective rotation of the nucleus which has the harmonic oscillator potential and the quadrupole interaction with $\Delta N = 0$ only. Expectation values of the $SU_3$ Hamiltonian with respect to the eigenstates of the driving CHO Hamiltonian are the good approximation to the exact solution. The perturbative DNFT reproduces the energy for both the low and the high spin states of the even-even nucleus and the high spin states of the odd mass nucleus, which guarantees the applicability of the DNFT to the rotational mode of the system with a more general Hamiltonian. We must note that although the simple rotational pattern of the $SU_3$ model is reproduced, the shape of the mean field is affected by the rotational disturbances.

Similarity of the ordinary rotation in the $SU_3$ model and the pair rotation is very clear when we consider high spin states. The Hartree self-consistency condition corresponds to the equation for the energy gap. Both the effective angular momentum and number operators have the components only up to first order in fluctuations. Coupling of the extra particle with the $SU_3$ rotation or the pair rotation in the single-$j$ limit attenuates completely. It will be interesting to study the Coriolis attenuation problem in the general case following our approach.

The Hartree self-consistency condition has played important roles in obtaining the expressions for the intrinsic and rotational energies and the moment of inertia. We also note that the eigen-frequencies in the rotating body-fixed frame remain constant with increasing $Q$ for a fixed configuration. Moreover, the rigorous fulfillment of this condition is essential when the angular velocity fluctuates around a fixed high angular velocity, and when an extra particle is added to the core system. Complete attenuation of the RPC is attained by considering the coupling between the total rotation and the extra particle, whose dynamics has been included in the effective interaction.

The “Hartree” self-consistency can be achieved for any interaction strength. On the other hand, the “nuclear” self-consistency between the shape of the potential and that of the density distribution\(^6,11\) is realized only at the self-consistent strength.\(^{12}\) The minimization of the total energy of the deformed harmonic oscillator under the
saturation (constant volume) condition also yields the nuclear self-consistency. Moreover, the effective interaction in deformed nuclei satisfying the nuclear self-consistency rigorously is the multipole-multipole interaction expressed in the doubly stretched coordinates, which represents a big improvement over the use of the \((Q \cdot Q)\) interaction adopted here. Analysis of a system with refined self-consistent effective interactions in a similar manner as the present paper will be desirable.

We have included the results of the tilted rotation in Appendix B. Cuypers argued that the tilted rotation is unstable in the CHO model. Since this model serves as useful for predicting the equilibrium shape for each spin, without assuming any two-body interaction, it will be interesting to extend the DNFT treatment of the \(SU_3\) model to such a direction.

We have here considered the angular velocity as the classical collective parameter. In a forthcoming paper, we will study the quantized \(SU_3\) model in which \(Q_e\) is replaced by \(I_e/J_e\) with the total angular momentum operator \(I_e\).

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Appendix A

—— Inclusion of the Fluctuation Effect to the Hartree Approximation ——

The intrinsic energy \((3.32)\) obtained in § 3 by use of the eigenstates of the CHO differs from the exact value \((3.7)\) by \(\Delta E = -6\chi(\lambda + \mu)\). We can show that this difference can be dissolved by including the fluctuation part or the Fock term to the Hartree approximation in which \(\langle Q \cdot Q \rangle\) is replaced by \(\langle \langle Q \rangle \cdot \langle Q \rangle \rangle\).

To make easy the estimate of the effect of the non-diagonal part of \(Q_\nu\) on the energy, we consider the Casimir operator \(\tilde{C} = (\tilde{Q} \cdot \tilde{Q}) + 3(\tilde{L} \cdot \tilde{L})\). We write the decomposition of \(q_\nu, Q_\nu, l_\nu\) and \(L_\nu\) into the diagonal and non-diagonal parts as

\[
q_\nu = q_\nu(d) + q_\nu(nd), \quad \text{etc.} \tag{A·1}
\]

A similar expression of \(\tilde{C}\) is denoted as

\[
\tilde{C} = C(d \cdot d) + C(d \cdot nd) + C(nd \cdot nd). \tag{A·2}
\]

Defining \(S_a\) and \(S_{ab}\) by

\[
S_a = \sum_{i=1}^A (c_a^\dagger c_a)_i, \quad S_{ab} = \sum_{i=1}^A (c_a^\dagger c_b)_i, \tag{A·3}
\]

we can write the first term of Eq. (A·2) as

\[
C(d \cdot d) = (Q_\nu(d))^2 + 2(Q_\nu(d))^2 + 3(L_\nu(d))^2
= 4\sum_{a=1}^2 S_a - \sum_{a < b} S_a S_b = (2S_1 - S_2 - S_3)^2 + 3(S_3 - S_2)^2. \tag{A·4}
\]
The contribution to the second term of Eq. (A·2) from the \((Q \cdot Q)\) part in \(\bar{C}\) is given by

\[
\bar{Q}_0(d)\bar{Q}_0(nd) + \bar{Q}_2'(d)\bar{Q}_2(nd) + \bar{Q}_2(d)\bar{Q}_2'(nd) = 6uu(v^2 - u^2)(S_2 - S_0)(S_{23} + S_{20}).
\]  
(A·5)

But the corresponding \((L \cdot L)\) part yields the same result except for the change of the sign, so that we have \(C(d' \cdot nd) = 0\). For the last term of Eq. (A·2) involving \(Q_L, L_y\) and \(L_z\), we find

\[
C(nd \cdot nd) = 6 \sum_{\sigma < \beta} (S_{\sigma \beta}S_{\beta \sigma} + S_{\sigma \delta}S_{\delta \beta}).
\]  
(A·6)

Its expectation value is equal to \(12(\lambda + \mu)\).

The Casimir operator \(\bar{C}\) is expressed originally by the quadrupole moment and the angular momentum operators which are written in the \(c_\kappa (\kappa = x, y, z)\) representation. The above results show just the following relation:

\[
\bar{C} = (\bar{Q}^{(a)} \cdot \bar{Q}^{(a)}) + 3(\bar{L}^{(a)} \cdot \bar{L}^{(a)})
\]  
(A·7)

with the operators now in the \(c_\alpha (\alpha = 1, 2, 3)\) space. The conservation of the structure of \(\bar{C}\) is quite natural because the transformation from \(c_\kappa\) to \(c_\alpha\) is unitary. However, we must note the meaning attached to it. Namely, the invariant \(\bar{C}\) guarantees the occurrence of a pure rotational spectrum although the nuclear shape changes by the rotational disturbances.

Substituting \(\bar{C}\) into the interaction energy and replacing \(\langle \bar{L}_x(d) \rangle = \bar{g} \Omega = L\), we get

\[
-\frac{1}{2} \chi \langle (Q \cdot Q) \rangle = -\frac{1}{2} \chi \langle \bar{C} \rangle + \frac{3}{2} \chi \langle (L \cdot L) \rangle
\]

\[
= -\frac{1}{2} \chi [(\lambda + 2\mu)^2 + 3\lambda^2] - 6\chi(\lambda + \mu)
\]

\[
+ \frac{1}{2} \bar{g} \bar{L}^2 + \frac{1}{2} \bar{g} \langle \bar{L}_x^2(nd) + \bar{L}_y^2 + \bar{L}_z^2 \rangle.
\]  
(A·8)

The second term which arises from Eq. (A·6) accounts for the energy deficit \(\Delta E\). The last term is associated with the fluctuation of the kinetic energy of the extra degree of freedom. In the exact treatment of the \(SU_3\) model, its sum with the third term is put to be equal to \(L(L + 1)/2 \bar{g}\) as Eq. (3·7). On the other hand, the present analysis with the cranking model equates only the third term to \(L(L + 1)/2 \bar{g}\), so that we have to discard the last term. Some discussions were done on the effect of the fluctuation.\(^7,^{15,16}\) Ripka et al.\(^7\) argued that this energy can be removed by the angular momentum projection. In the Hartree-Fock calculations of light nuclei, one subtracts this term from the Hamiltonian to be used in the variational equation.\(^15\) Marshalek and Goodman\(^16\) discussed the problem of the angular momentum fluctuations in the backbending phenomena and suggested a method based on the RPA.

Finally we note that we can also account for the energy deficit by the rotation-particle coupling if we quantize the rotational motion.\(^17\)
Appendix B

**SU₃ Model for the Tilted Rotation with Nonzero K**

In § 3, we have assumed the cranking axis along the intrinsic $x$ axis. Here we tilt the angular velocity so as to have two components $\Omega_x$ and $\Omega_z$. Diagonalization of the CHO with the tilted rotation was attempted by Vassanji and Harvey¹⁸) and Cuypers.¹⁴) In order to elucidate a physical meaning by minimizing the numbers of the transformation coefficients, we take a different approach. Since it will turn out that an $a_1$ deformation is induced by the rotation, we include the corresponding term in the driving Hamiltonian from the start. Namely, we extend $\tilde{h}$ (3.13) to the form

$$\tilde{h} = h_{sp} - [a_0 q_0 + a_1 (q_1 + q_1^*) + a_2 (q_2 + q_2^*)] - [\Omega_x l_x + \Omega_z l_z]$$

with the quadrupole and the angular momentum operators are given by Eq. (3.5) and (3.6), respectively. This Hamiltonian cannot be brought to a diagonalized form by a rotation of the coordinate with the D-function of a definite rank because the tensors with the ranks equal to one and two are mixed up. We introduce another set of quadrupole operators $p_\nu$ with phases different from Eq. (3.3)

$$p_0 = 2 c_x' c_x - c_x^* c_x - c_y' c_y,$$

$$p_1 = -\sqrt{3} [c_x' c_x + c_x^* c_x + i(c_y' c_x + c_x^* c_y)] = -p_1^1,$$

$$p_2 = \sqrt{3} [c_x' c_x - c_y' c_y + i(c_x^* c_y + c_y^* c_x)] = p_2^1.$$

In terms of $p_\nu$, both $q_\nu$ and $l_\nu$ can be expressed as the tensors with rank 2, namely,

$$q_0 = p_0, \quad q_1 + q_1^* = p_1 + p_1^*; \quad q_2 + q_2^* = p_2 + p_2^*,$$

$$l_x = \frac{i}{\sqrt{6}} (p_1 - p_1^*), \quad \quad \quad \quad \quad l_z = \frac{i}{\sqrt{6}} (p_2 - p_2^*).$$

Then, we can write $\tilde{h}$ as

$$\tilde{h} = h_{sp} - \sum_{\nu = -2}^{2} X_{\nu}^* p_\nu, \quad h_{sp} = \omega_0 \left( c_x' c_x + c_y' c_y + c_x^* c_x + \frac{3}{2} \right),$$

where the coefficients $X_{\nu}^*$ are given by

$$X_{\nu}^* = a_\nu + i \Omega_\nu = (-)^\nu X_{-\nu} \quad (\nu = 0, 1, 2)$$

with

$$\Omega_0 = 0, \quad \Omega_1 = \frac{1}{\sqrt{6}} \Omega_x, \quad \Omega_2 = \frac{1}{\sqrt{6}} \Omega_z.$$

We can now diagonalize $\tilde{h}$ (B.4) by rotating the coordinate axes with the Euler angles $\phi$, $\vartheta$ and $\phi$, which transforms $c_{\nu}'$ to $b_{\nu}'$ (both are their standard components; $\mu$ and $\nu = 0, \pm 1$).
where the relations between the standard and the Cartesian components of $c^t$ and $b^t$ are

\begin{align}
    c_1^t &= -\frac{1}{\sqrt{2}}(c_x^t + ic_y^t), \\
    c_{-1}^t &= \frac{1}{\sqrt{2}}(c_x^t - ic_y^t), \\
    b_1^t &= -\frac{1}{\sqrt{2}}(b_1^t + ib_2^t), \\
    b_{-1}^t &= \frac{1}{\sqrt{2}}(b_1^t - ib_2^t),
\end{align}

in a compact form

\begin{align}
    c_{\nu}^t &= \sum_{s=x}^{3} e_{\nu s} c_s^t, \\
    b_{\nu}^t &= \sum_{s=x}^{3} e_{\nu s} b_s^t. \\
\end{align}

We can express $p_{\nu}$ again in a compact form

\begin{align}
    p_{\nu} = \sum_{i,j} E_{2\nu}(i,j) c_i^t c_j, \\
\end{align}

where the coefficient $E_{2\nu}(i,j)$ is given by

\begin{align}
    E_{2\nu}(i,j) &= \sqrt{6} \sum_{i\nu j} (-)^{\nu s} (1 \nu_1 - \nu_2 | 2 \nu) e_{\nu i} e_{\nu j}. \\
\end{align}

Substituting these relations into Eq. (B-4), we obtain

\begin{align}
    \vec{h} = h_{sp} - \sum_{\nu=-2}^{2} Y_{\nu}^* \rho_{\nu}(b), \\
    h_{sp} &= \omega_0 \sum_{s=1}^{3} (b_s^t b_s + \frac{1}{2}), \\
\end{align}

with

\begin{align}
    Y_{\nu}^* &= \sum_{i} X_{\nu} D_{i\nu}^{(2)} = (-)^{\nu} Y_\nu. \\
\end{align}

If we impose the conditions

\begin{align}
    Y_{1}^* &= Y_{-1}^* = 0, \\
    Y_{2}^* &= Y_{-2}^*, \\
\end{align}

or written in a different way

\begin{align}
    \text{Re}[Y_{1}^*] &= \text{Im}[Y_{1}^*] = 0, \\
    \text{Im}[Y_{2}^*] &= 0, \\
\end{align}

$\vec{h}$ contains only the terms $h_{sp}, \rho_0(b)$ and $p_2(b) + p_2^*(b)$ and is thus diagonal in this representation. We can determine the eigen-frequencies $\omega_{\alpha}(\alpha = 1, 2, 3)$ by rewriting $\vec{h}$ as

\begin{align}
    \vec{h} &= h_{sp} - \sum_{i,j} (\sum_{\nu} Y_{\nu}^* E_{2\nu}(i,j)) c_i^t c_j = \sum_{s=1}^{3} \omega_s (b_s^t b_s + \frac{1}{2}), \\
\end{align}

and demanding that

\begin{align}
    \omega_{\alpha} &= \omega_0 - \sum_{\nu} Y_{\nu}^* E_{2\nu}(\alpha, \alpha), \\
    0 &= \sum_{\nu} Y_{\nu}^* E_{2\nu}(\alpha, \beta) \quad (\alpha \neq \beta), \\
\end{align}
with the understanding that $E_{2\nu}(a, \beta)$ is obtained from $E_{2\nu}(i, j)$ by replacing the indices $i$ or $j = x, y$ and $z$ by $a$ or $\beta = 1, 2$ and $3$, respectively. Equation (B·17b) is equivalent to Eq. (B·15a) or (B·15b). Three equations in Eq. (B·15b) can be written explicitly as

\[
\sqrt{\frac{3}{8}} a_0 \sin 2\theta - \cos 2\theta (a_1 \cos \phi - \Omega_1 \sin \phi) - \frac{1}{2} \sin 2\theta (a_2 \cos 2\phi - \Omega_2 \sin 2\phi) = 0, 
\]
\[
\cos \theta (a_1 \sin \phi + \Omega_1 \cos \phi) + \sin \theta (a_2 \sin 2\phi + \Omega_2 \cos 2\phi) = 0, 
\]
\[
\left[ \sqrt{\frac{3}{2}} a_0 - \frac{3 \cos^2 \theta - 1}{\sin 2\theta} (a_1 \cos \phi - \Omega_1 \sin \phi) \right] \sin 2\phi - \frac{\cos 2\phi}{\sin \theta} (a_1 \sin \phi + \Omega_1 \cos \phi) = 0. 
\]

Now we calculate the expectation values of $\hat{L}_x$ and $\hat{Q}_\nu$ at the ground state. Using Eqs. (B·3), (B·7) and (B·11), we obtain the following results:

\[
\langle L_x \rangle = -\frac{2}{3} A \text{Im}[D_{1\nu}^0] + \sqrt{\frac{2}{3}} B \text{Im}[D_{1\nu}^{(2)} + D_{1\nu}^{(3)}], 
\]
\[
\langle L_y \rangle = -\frac{2}{3} A \text{Im}[D_{1\nu}^{(2)}] + \sqrt{\frac{2}{3}} B \text{Im}[D_{1\nu}^{(3)} + D_{1\nu}^{(2)}], 
\]
\[
\langle Q_\nu + Q_\nu^* \rangle = \sqrt{\frac{8}{3}} A \text{Re}[D_{1\nu}^{(3)}] - 2B \text{Im}[D_{1\nu}^{(2)} + D_{1\nu}^{(3)}], 
\]
\[
\langle L_y \rangle = 0, \quad \langle Q_1 - Q_1^* \rangle = 0, \quad \langle Q_2 - Q_2^* \rangle = 0 
\]

with

\[
A = \sqrt{\frac{3}{2}} (2\lambda + \mu), \quad B = \sqrt{\frac{3}{2}} \mu. 
\]

We can thereby write the Hartree self-consistency condition (2·14) as follows:

\[
\frac{\sqrt{6}}{x} a_0 = A (3 \cos^2 \theta - 1) - 3 B \sin^2 \theta \cos 2\phi, 
\]
\[
\frac{2}{x} a_1 = A \sin 2\theta \cos \phi - 2 B \sin \theta (\sin \phi \sin 2\phi - \cos \theta \cos \phi \cos 2\phi), 
\]
\[
\frac{2}{x} a_2 = A \sin^2 \theta \cos 2\phi - B [(1 + \cos^2 \theta) \cos 2\phi \cos 2\phi - 2 \cos \theta \sin 2\phi \sin 2\phi]. 
\]

Substituting these $a_\nu$ into Eqs. (B·18b) and (B·18c), we obtain the following equations for $\Omega_1$ and $\Omega_2$:

\[
\frac{2}{x} \Omega_1 = -A \sin 2\theta \sin \phi - 2 B \sin \theta (\cos \phi \sin 2\phi + \cos \theta \sin \phi \cos 2\phi), 
\]
\[
\frac{2}{x} \Omega_2 = -A \sin^2 \theta \sin 2\phi + B [(1 + \cos^2 \theta) \sin 2\phi \cos 2\phi + 2 \cos \theta \cos 2\phi \sin 2\phi]. 
\]
Comparison of these with $\langle \hat{L}_x \rangle$ and $\langle \hat{L}_z \rangle$ calculated by use of Eqs. (B·19a) and (B·19b) gives

$$
\langle \hat{L}_x \rangle = \mathcal{J} \Omega_x, \quad \langle \hat{L}_z \rangle = \mathcal{J} \Omega_z
$$

with $\mathcal{J} = 1/3\chi$ given by Eq. (3·26).

The remaining one of the diagonalization condition, Eq. (B·18a), is shown to be satisfied identically, if we use Eqs. (B·21) for $a_0$, $a_1$ and $a_2$ and Eqs. (B·22) for $\Omega_1$ and $\Omega_2$. This means that due to the Hartree self-consistency condition (B·21), one of three Euler angles becomes redundant, which we take to be $\phi$ and set $\phi = -\pi/2$. From Eq. (B·18c) and then Eq. (B·18b), we obtain

$$
a_1 \sin \phi = -\Omega_1 \cos \phi, \quad a_2 \sin 2\phi = -\Omega_2 \cos 2\phi.
$$

From Eqs. (B·22a) and (B·22b) rewritten as

$$
\frac{2}{\chi} \Omega_1 = -(A - B) \sin 2\theta \sin \phi,
$$

$$
\frac{2}{\chi} \Omega_2 = -[A \sin^2 \theta + B(1 + \cos^2 \theta)] \sin 2\phi,
$$

we can determine $\phi$ and $\theta$ for given $\Omega_1$ and $\Omega_2$. We can then fix $a_0$ from the self-consistency condition (B·21a) expressed as

$$
\frac{\sqrt{6}}{\chi} a_0 = A(3\cos^2 \theta - 1) + 3B \sin^2 \theta.
$$

For $a_1$ and $a_2$, we can use Eqs. (B·24) and (B·25). In the simple case $\Omega_2 = 0$, we can set $\phi = \pi/2$. With this choice, $(b_1, b_2, b_3)$ of Eq. (B·9) becomes equal to $(c_1, c_2, c_3)$ of §3 and the angle $\theta$ is related to the transformation coefficients (3·17a) as $(u, v) = (\cos \theta, \sin \theta)$.

Finally, let us calculate the energy. For $H'$ (3·28) we have

$$
E'(a_v, \Omega) = \omega_0 (\Sigma_1 + \Sigma_2 + \Sigma_3) - a_0 \langle \tilde{Q}_0 \rangle - a_1 \langle \tilde{Q}_1 + \tilde{Q}_1' \rangle - a_2 \langle \tilde{Q}_2 + \tilde{Q}_2' \rangle
$$

$$
- \Omega_2 \langle \hat{L}_x \rangle - \Omega_2 \langle \hat{L}_z \rangle + \frac{1}{2} \chi [\langle \tilde{Q}_0 \rangle^2 - \langle (\tilde{Q}_1 + \tilde{Q}_1')/2 \rangle^2 - \langle (\tilde{Q}_2 + \tilde{Q}_2')/2 \rangle^2]
$$

$$
= \omega_0 (\Sigma_1 + \Sigma_2 + \Sigma_3) - \sum_v \frac{1}{\chi} a_v^2 - \Omega_2 \langle \hat{L}_x \rangle - \Omega_2 \langle \hat{L}_z \rangle,
$$

where the second term can be reduced to

$$
- \chi \left[ \frac{1}{3} A^2 + B^2 - (\Omega_1/\chi)^2 - (\Omega_2/\chi)^2 \right] = -2\chi(\hat{\lambda}^2 + \lambda \mu + \mu^2) + \frac{1}{2 \mathcal{J}} (\langle \hat{L}_x \rangle^2 + \langle \hat{L}_z \rangle^2).
$$

We obtain the energy of the system

$$
E = E'(a_v, \Omega) + \Omega_2 \langle \hat{L}_x \rangle + \Omega_2 \langle \hat{L}_z \rangle = E_0(\lambda, \mu) + \frac{1}{2 \mathcal{J}} L(L + 1),
$$

where we have defined $L$ by
\[ \langle \hat{L}_x \rangle^2 = L(L+1) - K^2. \]  

(B\cdot 28b)

Thereby we find the energy for the rotation tilted in the \((x, z)\) plane to have the same expression as Eq. \((3\cdot 31)\).

References