The first-order statistical moment of the seismic moment tensor

Y. Y. Kagan and L. Knopoff Institute of Geophysics,
University of California, Los Angeles, California 90024, USA

Accepted 1984 October 30. Received 1984 October 25; in original form 1984 June 13

Summary. If a complex earthquake is assumed to be a set of individual, randomly oriented elementary pure double couple sources, the solution for the seismic moment of the complex event projected on the mean trend of the fault will perforce be comprised of sources of both double couple and compensated linear vector dipole (CLVD) types. We investigate the statistical properties of these two components of seismic sources in terms of the invariants of the seismic moment tensor of a realistic set of synthetic earthquakes. It is very likely that the size of the CLVD component is two to three orders of magnitude smaller than that of the double couple component.

1 Introduction

We have studied the geometry of earthquake fracture zones as a statistical problem, by imagining that the fracture zone is delineated, in part, by the locations of hypocentres. The hypocentres have been treated as scalar point quantities (Kagan & Knopoff 1980; Kagan 1981a, b). Even though this gives a highly restricted description of the geometry of earthquake fault zones, we find evidence for spatial scale-invariance of this geometry, and for its non-planar character. If we try to simulate slip in a complex earthquake event as a series of infinitesimal dislocations (Kagan 1982), these dislocations are perforce non-planar. If each subevent has a slip direction and a fault plane that is close to, but slightly different from its neighbours, then the sum of these infinitesimal, slightly disoriented double couples (Burridge & Knopoff 1964), will have a small component of the compensated linear vector dipole (CLVD) (Knopoff & Randall 1970), when all elemental double couples are referred to a coordinate system fixed in the plane of the general trend of the fault. We assume that a study of the relative orientations of the focal mechanisms of individual earthquakes can yield information regarding the spatial orientations of the microfeatures that make up a fault system. The fault plane solutions for individual earthquakes give evidence for the variations in alignment of the respective fracture surfaces and hence we suppose that these solutions describe the variations in angular orientation of portions of an extended fault system.

We take the degree of misfit among the fault plane solutions to be an indicator of the
non-planar character of a fault. In general, we do not have a preferred coordinate system for the projection of fault plane solutions, although we may have a rough idea of its orientation; if we choose the directions of this coordinate system inappropriately, we shall find it difficult to recognize the degree of misfit to pure double couple sources. Of course, we might try to rotate the sum of the projections on an arbitrary set of axes to a coordinate system in which the non-double couple source terms are minimized, but this too has the undesirable property that we are unable to separate the contributions of local roughness or irregularity of fault surfaces from those of large-scale curvature of a fault system; the presence of large-scale curvature would seem to require that we use an ever-rotating coordinate system to evaluate the local properties of irregularity of fault surfaces.

We prefer an alternative approach, which is to study the spatial correlations (and temporal correlations as well) of earthquake source solutions. In the lowest order, we can project all the solutions as above on to a common point, which we can take to be the origin of the coordinate system. In the next order, we can compare the relationships between pairs of solutions as a function of their separations in space (and time if we desire). In the latter case, we can compare the relationships between pairs of focal solutions for widely separated parts of the world, even if the faults on which two sets of earthquake pairs occur are completely misaligned. At this level of correlation, we project one earthquake on another; this procedure eliminates the need for the use of an absolute coordinate system. Indeed, large-scale curvature of faults emerges as a natural consequence of the variations of such correlations with distance between the pairs of earthquakes.

In this work we sacrifice two assumptions that are commonly used to describe earthquake focal mechanisms (Backus 1977a, b; Aki & Richards 1980; Doornbos 1982; Stump & Johnson 1982): (1) that the elementary sources comprising a finite source are spatially coherent and (2) that the finite source is a plane.

We make two simplifying assumptions: First, we assume that the elastic properties of the medium as well as the pre-stress are isotropic. The assumption regarding isotropy is probably a good approximation, at least, for length-scales of the order of metres or kilometres that we consider, but it is physically unrealistic in the near source region; we retain it for reasons of simplicity. We can hope that the results of the isotropic case will be applicable toward the solution of the general problem. Second, we assume that individual elementary fractures are not interactive, i.e. that the stress fields at the moment of the rupture are not influenced by other ruptures that take place within time intervals of the order of the travel times of elastic waves. These latter problems fall within the province of the dynamical theory of fractures, which is, once again, a model that is too complicated for the quality of the data we are dealing with.

The full problem of source mechanism has five degrees of freedom in three-dimensional space. The spatial correlations of these functions will be susceptible to errors in both the determinations of the focal mechanisms and the incompleteness of the data. The influence of these errors can be determined by a bootstrap process. We generate a stochastic model of the earthquake process which uses infinitesimal dislocations of double couple type as elementary building blocks. Then we postulate a seismic moment distribution and estimate the errors in the simulation by varying the parameters of the source distribution. The inaccuracies of the procedure are heavily dependent on the appropriateness of the model.

Having indicated the inadequacy of studies of the lowest-order moments of the correlations between focal mechanisms, i.e. those involving the projection of the sum of presumably randomly oriented solutions on to an arbitrarily chosen coordinate system at its origin, we propose to ignore our own warning by proposing, in this paper, to discuss
the properties of the one-point correlation or the lowest-order moment. We do so mainly for pedagogic reasons; we wish to learn about the properties of the seismic moment tensor under rotations of coordinates for isolated events and superpositions of solutions for events, before attacking the large-scale problem of two-point correlation studies. In addition, we have been able to construct realistic synthetic catalogues of earthquakes that have been based on our earlier statistical studies. It is our intention to study the one-point correlation function for earthquakes in these catalogues, generated under rigorously controlled conditions, before attempting to attack the more difficult problem of the two-point correlation and its application to real catalogues; the latter task is the subject of our next contribution (Kagan & Knopoff 1985). Unfortunately, the accuracy of modern catalogues of fault mechanisms of real earthquakes is too low to allow us to study the one-point statistical properties by the methods discussed in this paper; the two-point correlations are more accessible in real catalogues.

2 Seismic moment tensor

2.1 TENSOR INVARIANTS

The diagonalized source-moment deviatoric (traceless) tensor is

\[ m = M_0 \cdot \text{diag}[1, -1, 0] \tag{1} \]

in the case of a double couple source (Burridge & Knopoff 1964). This tensor is more easily recognized by its usual description if (1) is rotated by 45° counter-clockwise around \( x_3 \) to yield

\[
\begin{pmatrix}
0 & 1 & 0 \\
1 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}
\]

The compensated linear vector dipole (CLVD) of Knopoff & Randall (1970) has the moment tensor representation in diagonal form

\[ m = 3^{-1/2} \cdot M_0 \cdot \text{diag}[2, -1, -1]. \tag{2} \]

where we have normalized this moment distribution to the same norm as (1).

The invariants of a symmetric tensor \( m_{ij} \) are (cf. Love 1944, p. 83)

\[
I_1 = \text{tr}(m_{ij}) = m_{11} + m_{22} + m_{33} = \lambda_1 + \lambda_2 + \lambda_3 \\
I_2 = m_{11}m_{22} + m_{11}m_{33} + m_{22}m_{33} - m_{12}^2 - m_{23}^2 - m_{13}^2 = \lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_1\lambda_3 \\
I_3 = \det(m_{ij}) = \lambda_1\lambda_2\lambda_3
\]

where the \( \lambda \)s are the ordered eigenvalues of \( m_{ij} : |\lambda_1| > |\lambda_2| > |\lambda_3| \). For a traceless tensor \( I_1 = 0, I_2 < 0 \). In the case of the double couple, \( I_3 = 0 \), while in the case of the CLVD it is not. Since the invariants are not changed under rotation, these two sets of solutions cannot be transformed into one another by a 3-D rotation.

2.2 TRANSFORMATIONS OF DIFFERENT SOURCES

Despite these remarks about the inability to transform the double couple into a CLVD and vice versa, nevertheless we can easily decompose the CLVD into a sum of two equal
double couples, diag \([1, -1, 0]\) and diag \([1, 0, -1]\). Conversely the double couple can be represented as the non-zero sum of two equal CLVDs of opposite sign. Thus, if
\[
m = m_1 + R \cdot m_2 \cdot R^T, \tag{3}
\]
then \(\det(m) = 0\); here \(R\) is any orthonormal matrix and \(R^T\) is its transpose, and \(m_1\) and \(m_2\) are CLVD sources with opposite signs: \(m_1 = 3^{-1/2} \cdot M_0 \cdot \text{diag}[2, -1, -1]\) and \(m_2 = 3^{-1/2} \cdot M_0 \cdot \text{diag}[-2, 1, 1]\). As a simple example, a sum of two equal orthogonal CLVDs of opposite sign if is a double couple. To demonstrate the general case, suppose that the first CLVD has its eigenvector with its largest component along the 1-axis. Let the largest component of the eigenvector for the second CLVD be in the 1–2 plane and let the angle between the two set of axes in the 1–2 plane be \(\phi\). Then, if \(R\) is the usual rotation operator around the 3-axis
\[
R = \begin{pmatrix}
\cos \phi & -\sin \phi & 0 \\
\sin \phi & \cos \phi & 0 \\
0 & 0 & 1
\end{pmatrix} \tag{4}
\]
we obtain
\[
m = 3^{1/2} \sin \phi \cdot M_0 \cdot \begin{pmatrix}
\sin \phi & -\cos \phi & 0 \\
-\cos \phi & -\sin \phi & 0 \\
0 & 0 & 1
\end{pmatrix}
\]
which is the moment tensor for a double couple source; this expression can be diagonalized by a rotation of the axes in the 1–2 plane counter-clockwise through the angle \((\pi/2 - \phi)/2\). The scalar seismic moment of this source is \(3^{1/2} \sin \phi \cdot M_0\).

The above result might have been anticipated. The total number of degrees of freedom of the double couple source is four, which includes one for the norm of the tensor and three for the orientation of the coordinate system. On the other hand, the CLVD source only requires three parameters because of axial symmetry. Thus, two CLVD sources of equal norm have five degrees of freedom, one more than necessary to represent a double couple source. For this reason, the parameter \(\phi\) (or \(M_0\)) can be taken as a free parameter of the representation.

By means of a simple rotation we can also write the CLVD as the moment matrix
\[
\begin{pmatrix}
0 & 1 & 1 \\
1 & 0 & 1 \\
1 & 1 & 0
\end{pmatrix}
\]
Hence a CLVD can also be written as the sum of three equal double couples.

2.3 CLVD INDEX

We wish to calculate the amount of CLVD present in our moment tensors in comparison with the amount of double couple. Dziewonski, Chou & Woodhouse (1981) have proposed the use of the ratio \(\lambda_3/\lambda_1\) to represent the amount of CLVD component. This characterization has two disadvantages: (1) we would have to calculate the eigenvalues of each moment tensor solution, and (2) for a deviatoric source this ratio will be always negative; thus the sign of \(\lambda_1\) has to be communicated separately. A number of authors have proposed an alternative solution which is the subdivision of the deviatoric tensor into a sum of a
'major' and 'minor' double couples (Gilbert 1981; Kanamori & Given 1981; Dziewonski et al. 1981); similar objections apply in this case as well.

In this paper we calculate the dimensionless index $\Gamma$ as a measure of the content of the CLVD component in a deviatoric second-order tensor:

$$\Gamma = \frac{1}{2} \cdot I_3 \left( \frac{1}{3} \cdot I_2 \right)^{-3/2}.$$  

If $\Gamma = 0$, the tensor is a double couple, since $I_3 = 0$; if $\Gamma = -1$ or $\Gamma = 1$, we have a pure CLVD source. For given $\Gamma$, the eigenvalues $\lambda_k (k = 1, 2, 3)$ of the seismic moment tensor normalized as in (1) or (2), are

$$\lambda_k = \frac{2}{\sqrt{3}} \cos \left[ \frac{\arccos(\Gamma) + 2\pi(k - 1)}{3} \right],$$

where $\Sigma \lambda_k^2 = 2$ (cf. Sattinger 1983, p. 69). If we use the above-mentioned measure

$$\mu = -\frac{\lambda_3}{\lambda_1}, \quad 0.5 > \mu > 0,$$

then

$$\Gamma = \text{sgn}(\lambda_1) \frac{\sqrt{2} \mu(1 - \mu) \cdot 3^{3/2}}{(1 - \mu + \mu^2)^{3/2}}.$$  

For $|\mu| < 1, (|\lambda_1| > |\lambda_2| > |\lambda_3|), \mu = -2 \cdot 3^{-3/2} \cdot |\Gamma|$. 

There is an additional advantage of the use of the index $\Gamma$ which we discuss below.

3 One-point simulations

Up to now we have considered how the seismic moment tensor transforms under 3-D rotations and summations separately. In the real situation the total seismic moment tensor of an earthquake is a consequence of both of these operations. At great distance from a finite source region, we can consider the moment $M$ of a distributed source to be represented by the sum of its elementary seismic moment tensors $m(n), M = \Sigma_n m(n)$ all located at the same point. As indicated above, we expect that some of these elementary subevents $m(n)$ will be disoriented (rotated) with regard to the others. The composite moment $M$ must have $\text{tr}(M) = 0$, if all the individual sources are deviatoric. As illustrated by the examples in the previous section, a sum of traceless tensors, each with zero determinant, does not necessarily have a zero determinant; the opposite is also true. In other words, the set of double couples and axial sources considered separately, is not closed with regard to 3-D rotation and summation; taken together, they form a complete set (the 5-D vector space of traceless symmetric matrices). If we assume that earthquake sources have a self-similar or scale-invariant property, and if a finite source is shown to be a mixture of the two classes, then the elementary component subevents will also have this property.

3.1 3-D ROTATION OF SEISMIC MOMENT TENSOR

Any matrix $M$ can be represented in the rotated coordinate system as $M' = RMR^T$, where $R$ is the rotation matrix

$$R = \begin{bmatrix}
\cos \Phi + l^2(1 - \cos \Phi) & ln(1 - \cos \Phi) - n \sin \Phi & ln(1 - \cos \Phi) + m \sin \Phi \\
ln(1 - \cos \Phi) + n \sin \Phi & \cos \Phi + m^2(1 - \cos \Phi) & mn(1 - \cos \Phi) - l \sin \Phi \\
ln(1 - \cos \Phi) - m \sin \Phi & mn(1 - \cos \Phi) + l \sin \Phi & \cos \Phi + n^2(1 - \cos \Phi)
\end{bmatrix}$$  

and $(l, m, n)$ are the direction cosines of the axis of rotation and $\Phi$ the rotation angle.
We calculate the ensemble average value of $m_{11}$ for an arbitrary choice of axis of rotation with fixed rotation angle $\Phi$

$$\langle m_{11} \rangle = \frac{1}{4\pi} \int_{\Omega} (R_{11}^2 + R_{12}^2 + R_{13}^2)\,d\Omega.$$ 

The result of the integration is

$$\langle m_{11} \rangle = \frac{1}{15} \cdot [(2\lambda_1 - \lambda_2 - \lambda_3)(4\cos^2 \Phi + 2\cos \Phi - 6) + 15\lambda_1],$$

with a similar result for the other diagonal components. The off-diagonal components of $m_{ij}$ will be zero on the average. If the tensor $m$ has zero trace,

$$\langle m_{11} \rangle = \frac{\lambda_1}{15} (12\cos^2 \Phi + 6\cos \Phi - 3).$$

If the rotation $\Phi$ is a uniform random variable in the range $-\delta < \Phi < \delta$, we get for the ensemble average of $m_{11}$

$$\overline{m_{11}} = A^{-1} \frac{1}{2\pi} \int_{-\delta}^{\delta} \langle m_{11} \rangle (1 - \cos \Phi)\,d\Phi,$$

where $A$ is the normalizing coefficient

$$A = \frac{1}{2\pi} \int_{0}^{\delta} (1 - \cos \Phi)\,d\Phi.$$ 

The Haar measure, $(1 - \cos \Phi)\,d\Phi$, for the 3-D rotation is taken from Kendall & Moran (1963). Performing the integration, we obtain

$$\overline{m_{11}} = \frac{\lambda_1}{15} \left( (\delta - \sin \delta)^{-1} \cdot \sin \delta (1 + 3\cos \delta - 4\cos^2 \delta) \right).$$

If $\delta = \pi$, $\overline{m_{11}} = 0$, as expected. Hence the average of the rotated tensor is the zero tensor. For small values of $\delta$ we obtain

$$\overline{m_{11}} \approx \lambda_1 \cdot \left( 1 - \frac{3}{5} \cdot \delta^2 \right).$$

Thus the second invariant in case of small rotations and additions of traceless sources is

$$\langle I_2 \rangle \approx M_0^2 \cdot \left( 1 - \frac{6}{5} \cdot \delta^2 \right).$$

### 3.2 Summation of Sources with Uniform Random Orientation

First, we consider the sum of infinitesimal deviatoric sources when each of them is rotated perfectly randomly, independent of the others. A similar problem for the strain due to randomly oriented sources has been considered by Molnar (1983) and Kostrov (1974). We can solve this problem analytically for the case of two CLVD sources. In this case, (3) is the expression for the source, with the difference that the summands are CLVDs of the same sign. If the rotation matrix is that of (4), we obtain

$$\Gamma = \text{sgn}(\Gamma_m)\{(9\cos^2 \phi - 1)/(3\cos^2 \phi + 1)^{3/2}\},$$

(8)
where $\Gamma$ is the index of the resultant source, and $\Gamma_m$ is the index of the primitive sources. The distribution density of the angle $\phi$ is $(\sin \phi)/2$, $\pi > \phi > 0$. If we set $\xi = \cos \phi$, after some manipulations, we obtain the following expressions for the cumulative distribution function of $\Gamma$

$$F(\Gamma) = \xi \quad \text{for} \quad \Gamma_m = 1,$$

and

$$F(\Gamma) = 1 - \xi \quad \text{for} \quad \Gamma_m = -1,$$

(9)

where $\Gamma$ is defined by equation (8) and $1 > \xi > 0$. The inversion of equation (8) gives a direct expression for the cumulative distribution function of $\Gamma$. This equation is too cumbersome to be quoted here.

Similar calculations for a larger number of sources are rather complicated. To study this problem, we have recourse to a Monte Carlo simulation. The simulation of a random 3-D rotation is discussed in the Appendix. In Table 1 and Fig. 1 we show the distribution of $\Gamma$ for a source composed of elementary seismic tensors, each subjected to a random rotation. In these simulations we have used $10^7$ elementary events, subdivided into groups of $N$ each. The value of $\Gamma$ was determined for each group and the distributions of these values of $\Gamma$ as well as of the other invariants were then obtained. If the number of subevents $N$ in each cluster is small, the results depend both on $N$ and whether we consider clusters of CLVDs or double couples. For large $N$ the results of the simulations are the same for both types of initial elementary subevents. The addition of double couples converges rapidly; the sum of five tensors has a distribution that is similar to that for the sum of 1000 events; for the CLVD sources the convergence is much slower (Fig. 1). In these simulations we used CLVD sources of the same sign with $\Gamma = -1$. The values computed for the case of two CLVD sources using (9) are in agreement with those for the simulations (Fig. 1b, curve 1).

The value of the CLVD index $\Gamma$ for these composite sources is distributed uniformly in the interval $[-1, 1]$, i.e. the probability of $\Gamma$ belonging to any subinterval $[\Gamma, \Gamma + \Delta \Gamma]$ depends only on $\Delta \Gamma$ and is equal to $\Delta \Gamma/2$. The uniform distribution in Fig. 1 corresponds to the cumulative curve being a diagonal of the rectangle. The standard deviation of $\Gamma$ is that of a uniform distribution which is $3^{-1/2}$ (see Table 1). The mean of the simulated values of $\Gamma$ is close to zero for large $N$; the mathematical expectation is, of course, zero. Thus, on average, the total seismic moment of the sum tends to that of a double couple, although

<table>
<thead>
<tr>
<th>Case number</th>
<th>Number of summands in group $N$</th>
<th>$\Gamma$ average</th>
<th>Standard deviation of $I_1$</th>
<th>Standard deviation of $I_1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Double-couple source</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>2</td>
<td>-0.0001</td>
<td>0.5700</td>
<td>0.4472 $\times$ N</td>
</tr>
<tr>
<td>2</td>
<td>5</td>
<td>-0.0003</td>
<td>0.5770</td>
<td>0.5660 $\times$ N</td>
</tr>
<tr>
<td>3</td>
<td>10</td>
<td>-0.0001</td>
<td>0.5768</td>
<td>0.6001 $\times$ N</td>
</tr>
<tr>
<td>4</td>
<td>1000</td>
<td>0.0035</td>
<td>0.5793</td>
<td>0.6429 $\times$ N</td>
</tr>
<tr>
<td>CLVD source</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>2</td>
<td>-0.2812</td>
<td>0.6963</td>
<td>0.4472 $\times$ N</td>
</tr>
<tr>
<td>2</td>
<td>5</td>
<td>-0.2081</td>
<td>0.5721</td>
<td>0.5656 $\times$ N</td>
</tr>
<tr>
<td>3</td>
<td>10</td>
<td>-0.1458</td>
<td>0.5728</td>
<td>0.6010 $\times$ N</td>
</tr>
<tr>
<td>4</td>
<td>1000</td>
<td>-0.0149</td>
<td>0.5797</td>
<td>0.6303 $\times$ N</td>
</tr>
</tbody>
</table>
individual realizations could range from a purely negative CLVD source to a positive one. The sum of a small number of CLVD sources has a mean value of $\Gamma$ that retains a 'memory' of the value of this index for an initial event (Table 1): it is clearly negative. The property of uniformity of the distribution of $\Gamma$ for a sum of randomly rotated traceless sources will be of value when we compare the results for other types of rotational distributions. In fact, this is a major reason why we have used the index $\Gamma$ to characterize these traceless tensors.

The second invariant $I_2$ of the resultant sum of sources is equal to $N$. The standard error is approximately $0.63 \cdot N$. For the normalized (average) tensor the second invariant goes to zero as $1/N$, in agreement with the result above (6). We note that in the case of the addition of perfectly coherent tensors, the second invariant should be equal to the product of $N^2$ and the invariant $I_2$ for an elementary event; the standard deviation is, of course, equal to zero in this case.
3.3 Non-uniform 3-D rotation of the sources

Next we study a truncated uniform rotational distribution, as described in the Appendix. We consider rotations $\Phi$ limited by a quantity $\epsilon$ such that $\cos(\Phi/2) \leq 1 - \epsilon$. In the simulations we take values of $\epsilon$ varying from 0.01 to 0.1. For small values of $\epsilon$, $\Phi^2 \approx 8\epsilon$; thus (7) is written as

$$\langle I_2 \rangle \approx -M_0^2 \cdot \left(1 - \frac{48}{5} \epsilon\right).$$

(10)

We obtained close agreement with this expression in the simulation for both classes of source moment tensors.

The simulation of the truncated distribution for small values of $\epsilon$ is very inefficient in the use of computer time. For this reason, the major part of our analysis was carried out with the use of the rotational Fisher-type distribution (see Appendix). A similar distribution was used by Silver & Jordan (1982) for modelling errors in the estimation of seismic moment tensor components. We have not used a distribution identical to that of Silver & Jordan (1982) for two reasons: first, we impose the condition $\text{tr}(m) = 0$, so that the components of our tensors belong to a 5-D subspace of the 6-D vector space of the symmetric tensor; second, if we were to allow the components of the tensor to change arbitrarily, we would not be able to take into account the axial symmetry of the CLVD source and the zero value of the determinant for the double couple source matrices. We have imposed a Fisher-type distribution on the rotations themselves (see Appendix); any constraints on the properties of the tensors are automatically taken into account.

We used $\sigma_u$ values (equation A3 of the Appendix) that vary from $10^{-4}$ to $10^{-1}$ in our simulations. The results can be summarized as follows. For both types of sources and for $N$ sufficiently large ($N > 30$)

$$\langle I_2 \rangle \approx -M_0^2 \cdot (1 - 22\sigma_u^2) \pm 20\sigma_u^2 \cdot N^{1/2}.$$  

(11)

If we compare (10) and (11), we see that if we take $\sigma_u = 0.66\sqrt{\epsilon}$, the results for small values of $\sigma_u$ and $\epsilon$ should be comparable, which was indeed the case both for $I_2$ and $\Gamma$. For the index $\Gamma$ of a set of additive double couples, we obtain

$$\langle \Gamma \rangle \approx 0 \pm 21\sigma_u^2 \cdot N^{-1/2}.$$  

(12)

For CLVD sources, the result is

$$\langle \Gamma \rangle \approx (1 - 450\sigma_u^4 \cdot N^{-1}) \pm 500\sigma_u^4 \cdot N^{-1}.$$  

(13)

3.4 Synthetic earthquake sequences

The above calculations were carried out as though each elementary event is independent of every other. To simulate a fault we have used synthetic earthquake sequences, which have been created by the model we have developed earlier (Kagan & Knopoff 1981; Kagan 1982). These simulated sequences reproduce the spatial and temporal statistics of the earthquake catalogues studied so far.

The method of generation of the synthetic sequences is summarized as follows. An earthquake fault is assumed to consist of a set of infinitesimal, identical, elementary dislocations (subevents). Each of these subevents can beget a Poissonian number of offspring, i.e. dependent events; the mean number of offspring of any parent is one. All offspring are born after the parent; the time difference between the birth of a parent and its clone offspring is determined from the probability density $t^{-3/2}$. Each event is identified in a phase space of time, three coordinates of space, and the three parameters of oriented
slip on the dislocation. The position of each secondary crack is shifted randomly from the location of the preceding shock, along the plane of the parent dislocation. The orientations of the fault plane and slip vector of the secondary dislocations are rotated according to a 3-D Cauchy distribution which has one intrinsic scaling parameter (see Appendix, equation A1). In their turn, the secondary dislocations produce new dependent shocks according to the same law; the process cascades indefinitely until all events have been 'born'. The identical subevents are clumped into individual earthquakes of different sizes according to a model which depends on the rate of occurrence of these events; when the rate falls below certain threshold, an earthquake is said to have terminated; the size of a given earthquake depends on the peak rate of occurrence of subevents within the cluster defining the earthquake (Kagan & Knopoff 1981; Kagan 1982). We have generated four sequences of these elementary events. The value of the parameter $\phi_0$ which controls the degree of branching was set to $5 \times 10^{-6}$. As discussed in Kagan (1982), this value of $\phi_0$ corresponds to the degree of branching or non-planarity of real earthquake faults. The total numbers of elementary events in the four synthetic sequences was about 1, 15, 90, and 127 millions.

### 3.5 CLVD INDEX FOR SIMULATED EARTHQUAKES

We made two types of calculations. In the first we subdivided each of the four long sequences of elementary events described above, into subsets of $N$ events in each with $N= 25, 100, 250$; the average values and the standard deviations of $I_2$ and $\Gamma$ of the sum of the seismic moment tensors of all events in the subset were calculated. All of the elementary events were presumed to have double couple mechanisms.

The calculations were carried out as follows. We take a sample of 10 000 elementary events from the synthetic sequence; we then compute the positions of the unit $P$- and $T$-axes for each of the elementary double couple sources. We then calculate the seismic moment tensor matrix:

$$m_{ij} = (T_i + P_i)(T_j - P_j).$$

The absolute value of the second invariant $I_2$ for the sum of $N$ events, varies from 0.95 to 0.99 from one synthetic sequence (realization) to another. The value of the standard deviation of $\Gamma$ is not proportional to $N^{-1/2}$ as in the previous case (12) where all events were independent, but is instead,

$$\sigma_\Gamma \approx 0.1 \cdot N^{-0.29}$$

i.e. the values of $\sigma_\Gamma$ change from 0.02 to 0.04 over the above range of $N$. This result is undoubtedly due to the interdependence of the events in a synthetic fault sequence (Kagan 1982); the $N$ summed events effectively correspond to a much smaller number of independent events.

#### Table 2. $\sigma_\Gamma$ from synthetic sequences.

<table>
<thead>
<tr>
<th>Case number</th>
<th>Number of elementary events</th>
<th>$\sigma_\Gamma$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>11–100</td>
<td>0.002</td>
</tr>
<tr>
<td>2</td>
<td>101–1000</td>
<td>0.022</td>
</tr>
<tr>
<td>3</td>
<td>1 001–10 000</td>
<td>0.015</td>
</tr>
<tr>
<td>4</td>
<td>10 001–100 000</td>
<td>0.038</td>
</tr>
<tr>
<td>5</td>
<td>100 001–1 000 000</td>
<td>0.021</td>
</tr>
</tbody>
</table>
In the second method, we simulate the sequences and then determine the values of the parameters for the sum of all of the elementary event tensors in the sequence. The values of $\sigma_{\Gamma}$ are tabulated as a function of the total number of events in a sequence in Table 2. There is no apparent order to the values of $\sigma_{\Gamma}$: the values vary widely. The estimates do not have the stability one would expect from averages taken over a large number of realizations. This is an indication that the underlying distribution has infinite statistical moments. We have noted (Kagan & Knopoff 1981; Kagan 1982), that the statistical distributions which control the earthquake process are stable distributions, and most of these distributions have infinite moments. In the experimental and simulated results, the infinities do not enter directly, of course, but they prevent the stabilization of the average estimates of the moments.

The values of $\sigma_{\Gamma}$ in Table 2 are mainly determined by a few ‘outliers’ which are always present in a distribution with infinite moments. In this case a better estimate of the variability of $\Gamma$ can be obtained, for example, from the quartiles of the cumulative distributions. More than 50 per cent of the values are concentrated in a span of $\Gamma$ that is less than $10^{-3}$ (Fig. 2). This means that the index $\Gamma$ is expected to differ from zero by a quantity of the order of $10^{-4} - 10^{-2}$ in most cases.

We draw one additional inference from Fig. 2. Although we tried to reproduce the scale-invariant property of earthquake fault geometry in our simulations (Kagan 1982), we conclude that the synthetic fault is not fully self-similar. If the simulated distribution were truly self-similar, the distributions of all geometrical quantities should be independent of the number of elementary events in the simulation. It is possible that the condition of isotropy which has been imposed on our synthetic fault (Kagan 1982) is the cause of this problem. A simulated fault with isotropy can propagate in any direction, whereas anisotropic prestress conditions or anisotropic alignment of fractures should force the fault to have a preferred direction.

The value of $\sigma_{\Gamma}$ for synthetic sequences is 0.01–0.05 or less by both methods of analysis. These sequences were obtained using a value for the branching coefficient $\phi_0$ (Appendix, 1.0

![Figure 2](https://academic.oup.com/gji/article-abstract/81/2/429/1241295056586/1241295056586) Cumulative distributions of the parameter $\Gamma$ (dimensionless index of CLVD content) for a synthetic source composed of different numbers of elementary events. Case 1, 11–100; case 2, 10–1000; case 3, 1001–10^4; case 4, 10^4–10^5; case 5, 10^5–10^6 events.
equation A1) which have been estimated to reproduce the geometrical properties of real earthquake faults (Kagan 1982). Thus, we may take the above result as an indication that, even if a complex earthquake sequence is composed of pure double couple sources, disorientations among these elementary sources should induce a CLVD component in the composite source of the order of at most a few per cent (Fig. 2). Of course, if the elementary sources already have a non-zero CLVD component, the index $\Gamma$ of a composite source would be at least as large as the index of the elementary source (see equation 13). Again we underscore that the mathematical expectation of $\Gamma$ in equation (12) is zero, so the non-zero values of the index arise from the random property and are random themselves.

A value of $\sigma_\Gamma$ which ranges from 0.001 to 0.05, is of the same order of magnitude (or smaller) as would be obtained if one added together a few sources whose disorientation is caused by random rotational errors. If for example, we set $\sigma_\theta = 0.1$, and $N = 10$ in equation (12), we obtain $\sigma_\Gamma \approx 0.07$. Evidently, the addition of independent tensors does not reproduce exactly the errors in the estimation of the seismic moment tensor; however, it may give a rough estimate of the uncertainty. At the present time, the average accuracy of fault mechanism solutions is close to the above value ($\sigma_\theta = 0.1$), at best; we substantiate this statement in our forthcoming paper on two-point seismic moment correlations (Kagan & Knopoff 1985).

As another measure of the present accuracy we determined the distribution of $\Gamma$ for 309 moment tensor solutions for earthquakes in the Harvard catalogue (Dziewonski et al. 1981). Only 12 per cent of the solutions have a value of $\Gamma$ constrained in the $0.05$ range, and only 48 per cent are concentrated in the interval $0.25 > \Gamma > -0.25$ (cf. Giardini 1984).

4 Conclusions

Tentatively, our results shows that the contribution of CLVD sources to the general seismic moment tensor, due to random misalignment of the seismic sources, is two to three orders of magnitude less than that of the double couple sources. The above result holds for the stress conditions which are prevalent for tectonic earthquakes. We are quite aware that geophysical environments that favour earthquake occurrence by mechanisms other than that of shear fracture are likely to generate different results. For example, some earthquakes of the Mammoth Lake sequence of 1980 have been reported to have a value of the CLVD component that predominates over the double couple component (Julian 1983; Barker & Langston 1983); we have not studied in detail the case in which any double component is the consequence of the misalignment of an assemblage of almost aligned pure CLVD sources; the latter problem is complementary to that considered in this paper.

The small value of the CLVD contaminant prevents the use of the CLVD content in a general seismic source (as measured, for example, by the CLVD index, proposed in Section 2) for characterization of the source complexity. As noted above, the accuracy of modern measurements of the general seismic moment tensor is too low to allow for such discrimination. With improved accuracy it might be possible to use our results for source-complexity analysis. It is possible that the methods described in this paper could be applied for those investigations.

Acknowledgments

This research was supported by Grants CEE-80-08588 and CEE-82-14203 of the ASRA program of the National Science Foundation. The authors gratefully acknowledge the usefulness of the algebraic programming system REDUCE compiled by A. C. Hearn.
References

Appendix: simulation of 3-D rotation

Kendall & Moran (1963) have given two methods for the simulation of a completely random (or uniformly random) 3-D rotation, SO(3). The first is based on a random choice of a rotation axis and of a rotation angle, and the second on a simulation of a random orthogonal matrix. In the first case we use a method of simulation based on the correspondence between a direction in 4-D space, in particular the space of normalized quaternions, and rotations in ordinary 3-D Euclidean space (Moran 1975). The unit quaternion is topologically isomorphic to the unitary group SU(2), widely used in quantum physics for a description of 3-D rotations. The advantage of this approach is that a uniform distribution of directions on the unit quaternion sphere corresponds to a uniform random 3-D rotation (Moran 1975). This enables us to apply an algorithm proposed by Marsaglia (1972) which computes a point on a hypersphere (which is a 3-D sphere in 4-D space).

In practical terms, the simulation is performed as follows. A point is randomly chosen on the hypersphere; we interpret the four values so obtained as components of a unit quaternion \( q = (q_0, q_1, q_2, q_3) \). Because the mapping of the unit quaternion in SO(3) is two-to-one (Moran 1975; Mermin 1979) we identify quaternions \( (q_0, q_1, q_2, q_3) \) and \( (-q_0, -q_1, -q_2, -q_3) \) which correspond to the same rotation. Thus, we need to consider only one hemisphere of the hypersphere.

To find the rotated source we first represent the source as a tensor product of one (for the CLVD source) or two (for the double couple source) normalized vectors. Thus, the CLVD source is represented as

\[
m = \pm 3^{-1/2} \cdot M_0 \cdot (3 \cdot \mathbf{v} \otimes \mathbf{v} - \mathbf{l}).
\]

where \( \mathbf{v} \) is a unit vector parallel to the axis of the dipole and \( \mathbf{l} \) is the identity tensor. In indicial notation

\[
m_{ij} = \pm 3^{-1/2} \cdot M_0 \cdot (3 \cdot \mathbf{v}_i \mathbf{v}_j - \delta_{ij}).
\]

For the double couple a similar representation is given as the symmetrized tensor product of two unit orthogonal vectors \( \mathbf{s} \) (slip vector) and \( \mathbf{n} \) (normal to the plane of rupture)

\[
m = M_0 \cdot (\mathbf{s} \otimes \mathbf{n} + \mathbf{n} \otimes \mathbf{s})
\]

(\textit{cf.} Aki & Richards 1980).

Then we calculate the rotated positions of these vectors in the above representations. The transformation is (\textit{cf.} Kagan 1982)

\[
\mathbf{v}' = q \mathbf{v} q^\top.
\]
where \( \bar{q} \) is a conjugate quaternion. The source matrices can be then found using the above equations. As an alternative we calculate the orthogonal matrix of the rotation transformation. If we parameterize the 3-D rotation as we have done in (5), the values of the angles can be found from (Moran 1975)

\[
q_0 = \cos \frac{\Phi}{2}, \quad q_1 = l \cdot \sin \frac{\Phi}{2}, \quad q_2 = m \cdot \sin \frac{\Phi}{2}, \quad q_3 = n \cdot \sin \frac{\Phi}{2},
\]

and the orthogonal matrix can be found from equation (5). Then we left- and right-multiply the moment tensor of the source, such as (1) or (2), by this matrix.

A simulation of a non-uniform rotation is complicated because of the paucity of appropriate models of rotation. One obvious model is to generate the uniform distribution described above, and then retain only the points on the hypersphere which lie in the interval \( 1 < q_0 < 1 - \epsilon \). This case corresponds to a rotation in which the position of the rotation axis is taken uniformly random (i.e. to be a random point on the surface of an ordinary sphere in 3-D space), but the rotation angle \( \Phi \) is limited by the inequality

\[
1 > \cos \frac{\Phi}{2} > 1 - \epsilon.
\]

This means that the rotation is uniform in the range

\[
2 \arccos(1 - \epsilon) > \Phi > 0.
\]

The second model is an analogue of the Cauchy distribution we have used to model the 3-D rotations of elementary dislocations comprising a complex earthquake source (Kagan 1982). The distribution density function can be written as

\[
f(\Phi, \theta, \psi) = \frac{1 + \cot^2(\phi/2)}{2\pi \phi_0^2} \sin \left( \frac{\theta d\Phi d\theta d\psi}{1 + \phi_0^2 \cot^2(\phi/2)} \right).
\]

Here \( \phi_0 \) is the only intrinsic parameter of the distribution. The Cauchy distribution has infinite moments and hence is less suitable for the modelling of the rotation errors which we want to simulate. These errors are probably distributed according to something like a Gaussian or normal distribution. [See discussion of errors in fault plane solutions by Knopoff (1961) and Brillinger, Udias & Bolt (1980) and in seismic moment tensor inversions by Stump & Johnson (1977) and Patton & Aki (1979).]

There is no direct analogue of the normal distribution for directional or rotational data (Mardia 1972). Two models can be considered as approximations. The first ‘Gaussian’ model is a generalization of Brownian motion to 3-D rotations (Perrin 1928; Roberts & Ursell 1960; Roberts & Winch 1984). In this model a time-dependent random rotation has a distribution density function

\[
g(\Phi, t) \sin^2 \Phi/2 \sin t d\Phi d\theta d\psi dt,
\]

where the function \( g(\Phi, t) \) is expressed as

\[
g(\Phi, t) = (\pi^2 \sin \Phi/2)^{-1} \sin \Phi/2 + \ldots
\]

\[
+ (2n + 1) \sin \left( (2n + 1)\Phi/2 \right) \exp \left[ -n(n+1)Rt \right] + \ldots
\]

We have used the notation of equation (5).

For our purposes, the major disadvantage of this model is the difficulty of execution. It is not clear how to simulate the infinite series in (A2). One possibility, of course, is to simulate it in the way that the usual Gaussian distribution is sometimes generated, i.e., as
a finite sum of several random variables uniformly distributed over some interval. This
method is very time-consuming and was not attempted here.

The second 'Gaussian' model we consider here, is an adaptation of the von Mises–Fisher
hyperspherical distribution of a higher number of dimensions (Downs 1966; Mardia 1972,
1981) to the distribution of 3-D rotations. This Fisher-type distribution has been extensively
used for palaeomagnetic data to model errors of direction in 3-D space (Mardia 1981). Recently
the 6-D Fisher distribution was used by Silver & Jordan (1982) to model the
errors in the estimates of the components of the seismic moment tensor.

To simulate 3-D rotations, again we use the correspondence between directions in 4-D
space and 3-D rotations (Moran 1975). The distribution function of the rotation can then be
expressed as (Mardia 1972; Downs 1966)
\[
f(\Phi, \theta, \psi) = C \cdot \exp(\kappa \cos \Phi/2) \cdot \sin^2 \Phi/2 \cdot \sin \theta \, d\Phi \, d\theta \, d\psi,
\]
where \( C^{-1} = (2\pi^2) \cdot I_1(\kappa)/\kappa; \) \( I_1 \) is the modified Bessel function of the first kind. Unfortunately, to simulate this distribution we need to solve for the cumulative distribution
\[
F(\phi) = \int_0^{2\phi} \exp(\kappa \cos \Phi/2) \cdot \sin^2 \Phi/2 \, d\Phi.
\]
This cumulative function cannot be given in terms of elementary functions for an even-
dimensional space. For the purpose of simulation we have adopted another approach,
similar to that of Silver & Jordan (1982): we first generate a 3-D normally distributed
random variable \( u \) \((u_1, u_2, u_3)\). The standard deviation of \( u \) is taken to be small \((\sigma_u < 0.1)\). Then we calculate the unit quaternion
\[
q_0 = (1 + u_1^2 + u_2^2 + u_3^2)^{-1/2}
\]
and
\[
q_i = u_i \cdot (1 + u_1^2 + u_2^2 + u_3^2)^{-1/2} \quad \text{for } i = 1, 2, 3. \tag{A3}
\]
The orthogonal matrix which corresponds to this quaternion, simulates a Fisher-type
rotational distribution for small values of \( \sigma_u \).