Fermionic Determinant and Chiral Anomalies

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We derive chiral anomalies as changes of a certain type of regularized determinants of Dirac operators defined on Euclidean space-time $R^{2n}$ under infinitesimal chiral transformations. We study a relation between chiral anomalies and the topology of the space of invertible Dirac operators. A possible modification of our regularized determinant is discussed in connection with the chiral Schwinger model.

§ 1. Introduction

Chiral anomalies have great importance on physics. Axial anomaly is crucial for understanding of $\pi^0 \rightarrow 2\gamma$ decay$^1$ and the resolution of $U(1)$ problem.$^2$ On the other hand, if gauge anomalies exist at fundamental level, it has been considered that the theory becomes inconsistent because of the breakdown of renormalizability or unitarity.$^3$ So we suppose that chiral anomalies are not merely by-products of technical calculations due to the fact that it has not been found any regularization procedure that preserves all classical symmetries, but they must have deep mathematical structures as well as physical meanings. Fujikawa$^4$ regarded an anomalous term as the change of fermionic measure under chiral transformation in path-integral formalism, and clarified the relation between axial $U(1)$ anomaly and index theorem for a Dirac operator.$^5$ In hamiltonian approach, non-abelian anomaly is understood from algebraic-topological viewpoints as a phase factor which characterizes ray representation. Physically it means that anomalous Schwinger terms appear in commutation relations among Gauss law constraints and physical state conditions cannot be imposed consistently.$^6$ We can regard chiral anomalies as changes of regularized fermionic determinants under a chiral transformation in formulating the effective action. They were calculated by the use of $\zeta$-function regularization$^7$ and heat kernel method.$^8$ Moreover Alvarez-Gaume and Ginsparg$^9$ examined the topological content of non-abelian anomaly on the footing of fermionic determinant. They thought that the anomaly appears as the phase change of fermionic determinant under a chiral gauge transformation, and determined the form of $2n$-dimensional non-abelian anomaly by using an adiabatic argument and index theorem for $(2n + 2)$-dimensional Dirac operator related to $U(1)$ anomaly. They also connected non-abelian anomaly to non-trivial topology of gauge orbit space. In this way, chiral anomalies show very interesting algebraic and topological structures. However the above arguments hold only in the case of theories on compact space-times such as $S^{2n}$. It is natural to try to extend these arguments to those on Euclidean space-time $R^{2n}$. Such an extension is useful to clarify whether the topology of space-time has anything to do with that of a space like a gauge orbit space or not. This trial, however, has
not been considered seriously. This might be due to the fact that Dirac operators on $\mathbb{R}^{2n}$ have non-compact resolvent operators, in other words, they have continuous spectrum. Hence well-known $\zeta$-function regularization or heat kernel method does not make fermionic determinant well-defined. Also there is no useful tool such as index theorem which plays a central role in Ref. 9).

In this paper, regularized fermionic determinants of Ref. 10) are adopted and non-abelian anomalies are derived as those infinitesimal changes. Then the relation between non-abelian anomalies and the topology of the space $\mathcal{D}$ of invertible Dirac operators on $\mathbb{R}^{2n}$ are studied. It is shown that the existence of non-abelian anomalies indicates non-triviality of fundamental group of the space $\mathcal{D}$. However, unlike in the case of compact space-time, the connection between $U(1)$ anomalies and the topology of $\mathcal{D}$ could not be discussed. This may be owing to subtleties of theories on non-compact space-time.

The content of this paper is as follows. In § 2, a definition of a certain type of regularized determinants for Dirac operators and some of its properties are given. In § 3, we derive non-abelian anomalies in 2- and 4-dimensional Euclidean space-times. We study a relation between chiral anomalies and the topology of the space $\mathcal{D}$ of invertible Dirac operators in § 4. In § 5, we propose an extended version of our regularized fermionic determinant and study the chiral Schwinger model by using it.

§ 2. Definition of a regularized determinant

We define a space of Dirac operators and a regularized determinant for the ratio of two Dirac operators in $\mathbb{R}^{2n}$, and state some of their properties in this section. For the detailed proof of properties, we refer to Ref. 10).

Let $H$ be the Hilbert space consist of wave functions of a Dirac particle in $2n$-dimensional Euclidean space-time with $N$-dimensional space of internal symmetry, i.e.,

$$H = C^N \otimes C^{2n} \otimes L^2(\mathbb{R}^{2n}).$$

(2.1)

The anti-selfadjoint operator $\mathcal{D}$ in $H$ is defined as

$$\mathcal{D} = \sum_{\mu=0}^{2n-1} 1 \otimes \gamma_\mu \otimes \frac{\partial}{\partial x_\mu},$$

(2.2)

where $\gamma_\mu (\mu = 0, 1 \cdots 2n - 1)$ are Dirac's $\gamma$-matrices that are settled to be hermitian and to satisfy $\gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu = 2\delta_{\mu\nu}I$. We consider Dirac operators of the form

$$\mathcal{D} + m + \sum_{i=1}^k T_i \otimes \Gamma_i \otimes f_i,$$

(2.3)

for some positive integer $k$, where each $f_i$ is the operator on $L^2(\mathbb{R}^{2n})$ of multiplication by the rapid decreasing $C^\infty$ function $f_i$, $T_i$ and $\Gamma_i$ are operators on $C^N$ and $C^{2n}$ respectively. Let $\mathcal{D}$ be the space of Dirac operators of the above type, i.e.,

$$\mathcal{D} = \{ \mathcal{D} + m + A | A = \sum_{i=1}^k T_i \otimes \Gamma_i \otimes f_i \text{ for some } k \in \mathbb{N} \}.$$
We regard $\mathcal{D}$ as a metric space with distance\(^{(11)}\)

$$
\begin{aligned}
\|\mathcal{D} + m + A, \mathcal{D} + m + B\| &= \|(A - B)(\mathcal{D} + m)^{-1}\|_{\infty} + \|(A - B)(\mathcal{D} + m)^{-2} - 1\|_1,
\end{aligned}
$$

where $\|\cdot\|_{\infty}$ and $\|\cdot\|$ are the operator norm and the trace norm, respectively. We denote the set of all operators in $\mathcal{D}$ which have bounded inverses by $\mathcal{D}$. $\mathcal{D}$ is an open subspace of $\mathcal{D}$. It can be shown that $\mathcal{D}$ is arcwise connected\(^{(10)}\), i.e., for any $D_0, D_1 \in \mathcal{D}$, there exists (at least) one smooth curve $D : I \rightarrow \mathcal{D}$ such that $D(0) = D_0$ and $D(1) = D_1$. (Here and hereafter we denote the closed interval $[0, 1]$ by $I$.) For an arbitrary smooth curve $D : I \rightarrow \mathcal{D}$ and an arbitrary $M > 0$, we can define the trace class operator

$$
L_M(D) = \int_0^1 ds\, ds^{-1} D(s)^{-1} \exp(D(s)^2/M^2).
$$

Let $D$ and $D'$ be two arbitrary smooth curves $I \rightarrow \mathcal{D}$ which satisfy $D(0) = D'(0)$ and $D(1) = D'(1)$, then there exists an integer $l$ such that

$$
Tr L_M(D) - Tr L_M(D') = 2\pi il.
$$

Therefore $\exp(Tr L_M(D))$ depends on $D(0)$ and $D(1)$ but not on whole curve $D$. In the previous paper\(^{(10)}\), it is proposed that $\exp(Tr L_M(D))$ is taken for a regularization of the Fredholm determinant of $D(1) - D(0)^{-1}$ and it is denoted by $D_M(D(1) ; D(0))$.

Some properties of the regularized determinant are as follows.

1) If $D_0, D_1, D_2 \in \mathcal{D}$, then

$$
D_M(D_2 ; D_1) D_M(D_1 ; D_0) = D_M(D_2 ; D_0),
$$

and in particular

$$
D_M(D_0 ; D_0) = 1.
$$

2) Let $D(t) = \mathcal{D} + m + A(t)$ be an arbitrary smooth curve $(-\varepsilon, \varepsilon) \rightarrow \mathcal{D}$, then

$$
\begin{aligned}
\frac{d}{dt} D_M(D(t) ; D_0)|_{t=0} &= \text{Tr} \left[ \frac{dA}{dt}(0)(\mathcal{D} + m + A(0))^{-1}
\right.
\times \exp((\mathcal{D} + m + A(0))^2/M^2) \left. \right] D_M(\mathcal{D} + m + A(0) ; D_0),
\end{aligned}
$$

3) $D_M(D_2 ; D_1)$ is gauge invariant in the following sense. That is,

$$
D_M(UD_2 U^{-1} ; D_1) = D_M(D_2 ; D_1),
$$

where $U = e^A$ with a bounded operator $A$.

§ 3. Derivation of chiral anomalies

In this section, we shall get non-abelian anomalies in 2- and 4-dimensional space-
times by using our regularized determinant $D_M(D(1); D(0))$. The method which we use is essentially the same as that of Ref. 10), but for completeness we shall sketch the derivation. We take a model where only one left-handed fermi field couples to a non-abelian gauge field whose Lagrangian density is

$$L = \bar{\psi}(x)i\gamma^\mu\partial_\mu\psi(x) + \frac{1}{2g^2}\text{Tr} F^{\mu\nu}(x)F_{\mu\nu}(x), \quad (3.1)$$

suitably defined on Euclidean space-time. The form of Dirac operator is

$$\mathcal{D} = \partial + m - i\lambda(x)\frac{1 - \gamma_5}{2}, \quad (3.2)$$

where $\lambda(x) = \gamma^\mu A_\mu(x) T^\alpha$ and $\gamma_5 = i^n \Pi_{\mu=0}^{\mu=n-1} \gamma^\mu$. The effective action is formally defined by

$$e^{r(\lambda)} = \text{det} \mathcal{D}. \quad (3.3)$$

But 'det $\mathcal{D}$' is not a well-defined quantity, so we adopt our regularized determinant in place of it. Non-abelian anomalies can be obtained by examining the change of a regularized determinant under the local chiral gauge transformation

$$\psi(x) \rightarrow \exp\left( i \frac{1 - \gamma_5}{2} \epsilon \phi(x) \right) \psi(x),$$

$$\bar{\psi}(x) \rightarrow \bar{\psi}(x) \exp\left( -i \frac{1 + \gamma_5}{2} \epsilon \phi(x) \right), \quad (3.4)$$

where $\epsilon$ is an infinitesimal parameter and $\phi(x)$ is an arbitrary function, so

$$i \int d^{2n}x \text{(Non-abelian anomaly)}$$

$$= N_{reg} \left. \frac{d}{d\epsilon} D_M\left( \exp\left( -i \frac{1 + \gamma_5}{2} \epsilon \phi(x) \right) D_M\exp\left( i \frac{1 - \gamma_5}{2} \epsilon \phi(x) \right) \right) \right|_{\epsilon=0}$$

$$= \text{Tr}(\gamma_5 \phi(x) \exp(\mathcal{D}^2/M^2)), \quad (3.5)$$

where $N_{reg} = D_M(\mathcal{D}; \partial + m)^{-1}$. Equation (3.5) is deformed as follows by the use of Dunford integral and cyclic property of trace,

$$\text{Tr}(\gamma_5 \phi(x) \exp(\mathcal{D}^2/M^2))$$

$$= \text{Tr} \left[ \frac{1}{2\pi i} \int d\lambda \gamma_5 \phi(x) \exp((\lambda + m)^2/M^2) \lambda \left( \lambda^2 - \left( \partial - iA(x)\frac{1 - \gamma_5}{2} \right)^2 \right)^{-1} \right]. \quad (3.6)$$

The evaluations of general 2\text{nd}-dimensional cases are straightforward but tedious. So we treat only 2-dimensional and 4-dimensional cases here.

(1) 2-dimensional case

We take the notation

$$\sigma_{\mu\nu} = -\varepsilon_{\mu\nu} \gamma^5 \quad (\varepsilon_{01} = 1, \gamma_5 = i\gamma^0 \gamma^1)$$

and define $B$ and $C$ as
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\[
\left( \partial - iA(x)\frac{1-\gamma_5}{2} \right)^2 = \Delta - \frac{i}{2} (\partial \cdot A + A \cdot \partial + i\varepsilon_{\mu\nu}(\partial^\nu A^\mu - A^\mu \partial^\nu)) \\
- \frac{1}{2} \gamma_5 (\varepsilon_{\mu\nu}(\partial^\nu A^\mu) - i(\partial \cdot A)) = \Delta - B - \frac{1}{2} \gamma_5 C.
\]

By using an expansion
\[
(p - q)^{-1} = p^{-1} + p^{-1} q p^{-1} + \cdots + (p^{-1} q)^{n-1} + (p^{-1} q)^{n+1} (p + q)
\]
and the equalities,
\[
(\lambda^2 - A + B)^{-1} = (\lambda^2 - A)^{-1} - (\lambda^2 - A)^{-1} B (\lambda^2 - A + B)^{-1},
\]
\[
(\lambda^2 - A)^{-1} \frac{\gamma_5}{2} C = \frac{\gamma_5}{2} C (\lambda^2 - A)^{-1} + (\lambda^2 - A)^{-1} \frac{\gamma_5}{2} (A C + 2(\partial_\mu C) \cdot \partial^\mu) (\lambda^2 - A)^{-1},
\]
we can reach the following well-known result,
\[
(Eq. 3\cdot5) = \frac{1}{4\pi} \left[ \frac{1}{2\pi i} \int \frac{d\lambda}{\lambda} \exp((\lambda + m)^2/M^2) \right] \times \phi(x)(i(\partial \cdot A) - \varepsilon_{\mu\nu}(\partial^\nu A^\mu)) \rightarrow \frac{1}{4\pi} \int d^2 x \text{tr} \phi(x)(i\partial_\mu A^\mu - \varepsilon_{\mu\nu}(\partial^\nu A^\mu)).
\]

(II) 4-dimensional case

Applying the same calculations, we get the following result:
\[
(Eq. 3\cdot5) = \text{Tr} \left[ \gamma_5 \phi(x) \exp \left( (\partial - iA(x)\frac{1-\gamma_5}{2} + m)^2 / M^2 \right) \right] \\
\rightarrow M \to \infty \frac{1}{48\pi^2} \int d^4 x \text{tr} \phi(x) \left[ i(3\sqrt{\pi}(M^2-2m^2)+12m^2)(\partial_\mu A^\mu) \right. \\
+ i\partial_\mu \partial^\nu(A_\nu A^\mu) + [\partial_\nu \partial^\nu A_\mu, A^\mu] + 2[A_\mu, \partial_\nu \partial^\nu A^\mu] \\
+ i\partial_\mu(\partial^\nu A_\nu A^\mu - A_\nu A^\nu A^\mu - A^\mu A_\nu A^\nu) \\
- \frac{1}{24\pi^2} \int d^4 x \text{tr} \left( \phi(x) \varepsilon_{\mu\nu\rho\sigma}(A^\nu \partial^\rho A^\sigma - \frac{i}{2} A^\nu A^\sigma A^\rho) \right). 
\]

The non-abelian anomalies (3·7) and (3·8) involve extra terms besides what we obtain from so-called “decent equations”.12) This occurs due to a definition of regularization which we choose. If we follow the idea by Bardeen,13) these terms are removed by adding local counterterms to the lagrangian density. However these extra terms also satisfy Wess-Zumino consistency conditions.12)

§ 4. Topology of the space of Dirac operators

In this section, we study a relation between chiral anomalies and the topology of the space \( \mathcal{D} \) of invertible Dirac operators on Euclidean space-time \( R^{2n} \). Let \( D : I \to \mathcal{D} \) be operator-valued map which is transported round a closed path in the space of
invertible Dirac operators, i.e., $D(1)=D(0)$. Equation (2.9) (or (2.7)) leads to a kind of quantization condition,

$$\exp \left( \frac{\text{Tr}}{\int_0^1} \frac{dD(s)}{ds} D(s)^{-1} \exp(D(s)^2/M^2)ds \right) = 1$$

or

$$\text{Tr} \int_0^1 \frac{dD(s)}{ds} D(s)^{-1} \exp(D(s)^2/M^2)ds = 2\pi i l \quad (4.1)$$

for some integer $l$. This means that the left-hand side of Eq. (4.1) is homotopy invariant. In fact, $\text{Tr}(dD(s)/ds)D(s)^{-1} \exp(D(s)^2/M^2)$ in Eq. (4.1) can be considered as the trace of regularized Maurer-Cartan form $dg \cdot g^{-1}$. In finite dimensional classical groups, the trace of Maurer-Cartan form satisfies the condition that for every loop $C$ there exists an integer $l$ such that the relation

$$\int_C \text{tr} dg \cdot g^{-1} = 2\pi i l \quad (4.2)$$

holds. Here $l$ is the winding number of the loop $C$. We give an example to indicate that the analogy of this fact occurs with $l \neq 0$ in the case of infinite dimensional space, the space $\mathcal{D}$ of invertible Dirac operators. In other words, the space $\mathcal{D}$ is not simply connected.

We consider the chiral $SU(2)$ gauge theory in 2-dimensional space-time, where we set

$$\mathcal{D} = \mathcal{A} - i\mathcal{A}(x) \frac{1 - \gamma_5}{2}.$$  

For the operator-valued map $D(s)$, we get

$$D(s) = \mathcal{U}(x, s)(\mathcal{A} - i\mathcal{A}(x) \frac{1 - \gamma_5}{2} + m) \mathcal{U}(x, s), \quad (4.3)$$

where

$$\mathcal{U}(x, s) = \frac{1 + \gamma_5}{2} + \frac{1 - \gamma_5}{2} u(x, s),$$

$$\mathcal{U}(x, s) = \frac{1 - \gamma_5}{2} + \frac{1 + \gamma_5}{2} u^{-1}(x, s),$$

$$u(x, 0) = u(x, 1) = 1. \quad (4.4)$$

We may take as an example of $u(x, s)$,

$$u(x, s) = \exp \left( iF(r) \frac{x_1 \tau_1 + x_2 \tau_2 + \tan \pi \left( s - \frac{1}{2} \right) \cdot \tau_5}{r} \right),$$

where $r = (x_1^2 + x_2^2 + \tan^2 \pi \left( s - \frac{1}{2} \right))^{1/2}$ and $F(r) \to 0$ \hspace{1cm} ($r \to \infty$) \hspace{1cm} $\to n\pi$ \hspace{1cm} ($r \to 0$).
The left-hand side of \((1/2\pi i) \times \text{Eq. (4.1)}\) is
\[
\frac{1}{2\pi i} \text{Tr} \int_0^1 ds \frac{\partial}{\partial s} \left( \bar{U}(x, s)(\not{\partial} + m)U(x, s) \right) \left( \bar{U}(x, s)(\not{\partial} + m)U(x, s) \right)^{-1} \\
\times \exp \left( (\bar{U}(x, s)(\not{\partial} + m)U(x, s))^2 / M^2 \right)
\]
\[
= -\frac{1}{2\pi i} \text{Tr} \gamma_5 \int_0^1 ds \left( u^{-1} \partial_s u \right) \exp \left( \left( \not{\partial} - i\bar{A}(x) \frac{1-\gamma_5}{2} + \bar{m} \right)^2 / M^2 \right), \tag{4.5}
\]
where \(u = e^{i\phi}\),
\[
\bar{A}'(x) = u^{-1}(A'(x) + i\partial^\mu)u \quad \text{and} \quad \bar{m} = me^{-i\phi}. \tag{4.6}
\]

The right-hand side of (4.5) must be an integer for all \(M\). By continuity, the integer is independent of \(M\). Hence we obtain
\[
\frac{1}{2\pi i} \text{Tr} \int_0^1 \frac{dD(s)}{ds} D(s)^{-1} \exp(D(s)^2 / M^2) ds \\
= \lim_{M \to \infty} \left[ -\frac{1}{2\pi i} \text{Tr} \gamma_5 \int_0^1 ds \left( u^{-1} \partial_s u \right) \exp \left( \left( \not{\partial} - i\bar{A}(x) \frac{1-\gamma_5}{2} + \bar{m} \right)^2 / M^2 \right) \right] \\
= -\frac{1}{4\pi} \int_0^1 ds \int d^2x \text{tr} \left( (u^{-1} \partial_s u)(\partial^\mu \bar{A}^\mu + \epsilon_{\mu\nu} \partial^\nu \bar{A}^\mu) \right) \\
= \frac{1}{8\pi^2} \int_0^1 ds \int d^2x \text{tr} \left( \epsilon_{\mu\nu} u^{-1} \partial^\mu uu^{-1} \partial^\nu uu^{-1} \partial_s u \right), \tag{4.7}
\]
by explicit calculation using Eqs. (3.7) and (4.6). The fourth member of the above equations is the winding number of the map \(u : S^3 \to SU(2)\). The first member of them is the winding number of the map \(D : S^1 \to \mathcal{G}\). Thus we get the relation between the third homotopy group on the space-time \(\mathbf{R}^2\) added to parameter space and the first homotopy group on the space of invertible Dirac operators, i.e., \(\Pi_3(\mathcal{G}) \supset \Pi_1(SU(2)) = \mathbb{Z}\). A similar argument can be applied to 4-dimensional case, i.e., \(\Pi_1(\mathcal{G}) \supset \Pi_1(SU(4)) = \mathbb{Z}\), \(G = SU(N), N \geq 3\), and also higher-dimensional cases. From the above examples, we find that there are some "holes" in an appropriate closed path \(D(s)\) which is generated by performing successive infinitesimal transformations, and point out that the space of invertible Dirac operators is not simply connected \((\Pi_3(\mathcal{G}) \supset \mathbb{Z})\). So chiral transformations can be thought of a kind of touchstone to get some information about a non-trivial global structure. In other words, chiral anomalies can be regarded as the local reminiscences of "holes" in the space of invertible Dirac operators even on the flat space-time \(\mathbf{R}^{2n}\).

Finally we give a short remark about the relation between the topological approach by Alvarez-Gaume and Ginsparg and ours. As we have stated in the Introduction, they conjectured that on \(S^{2n}\) non-abelian anomaly appears as the change of the phase factor of fermionic determinant under chiral gauge transformation, and obtained it by using an adiabatic argument and \(2n + 2\)-dimensional index theorem. On the other hand, our argument may be complementary to that of them in the following respect. We have obtained chiral anomalies directly as the change of regularized fermionic determinant defined on Euclidean space-time, which is non-
compact, under chiral gauge transformation. And by using them with homotopy invariant quantity \((4\cdot1)\), we have shown that the space of invertible Dirac operators is not simply connected in spite of the fact that the space-time has trivial topology. There exists no corresponding finite loop, i.e., \(D : S^1 \rightarrow \mathcal{D}\), in the case of axial \(U(1)\) transformation. So, contrary to the theories on compact space-time, an interesting relation between \(U(1)\) anomaly and non-abelian anomaly is not found. The difficulty is attributed to non-compactness of space-time.

§ 5. Extension of determinants and chiral Schwinger model

As stated in the Introduction, it has been considered that the existence of gauge anomaly threatens the consistency of the theory with the breakdown of renormalizability or unitarity. Some interesting attempts, however, have been done to construct consistent anomalous gauge theories. Faddeev and Shatashvili proposed\(^4\) the model with a Wess-Zumino term, which is gauge invariant at quantum level. Jackiw and Rajaraman examined\(^5\) the consistency of the chiral Schwinger model. It is shown that this model contains an arbitrary parameter \(\alpha\) which reflects a regularization ambiguity. The theory can be made unitary with the appropriate choice of the parameter, though it is not gauge invariant.

In this section, we propose a possible modification of our regularized determinant \(D_\mu(D_1, D_0)\) and study the chiral Schwinger model by using it. It is found that the modified one shows an interesting topological structure.

The chiral Schwinger model is the 2-dimensional space-time model which only left-handed fermi field couples to \(U(1)\) gauge field. An application of our regularized determinant defined in § 2 leads to

\[
el^{\Gamma(A)} = D_\mu(D(1); D(0))
\]

\[
= \exp \left( \text{Tr} \int_0^1 ds \frac{dD(s)}{ds} D(s)^{-1} \exp(D(s)^2/M^2) \right)
\]

\[
= \exp \left( \int d^2x \left( -iA^\mu(\delta_{\mu\nu} - i\varepsilon_{\mu\nu}) \tilde{\partial}^\nu \tilde{\partial}^\alpha (\delta_{\alpha\beta} + i\varepsilon_{\alpha\beta}) A^\beta \right) \right),
\]

\[(5\cdot1)\]

where \(D(s) = \mathcal{D} - is\mathcal{A}(1 - \gamma_5)/2\). The effective action \(\Gamma(A)\) does not make the chiral Schwinger model consistent. However, we can extend the regularization procedure with an arbitrary parameter \(\alpha\) in order to make the model unitary. We define the regularized effective action by using the modified fermionic determinants as follows,

\[
el^{\tilde{\Gamma}(A)} = D_\mu(X(1); X(0))
\]

\[
= \exp \left( \text{Tr} \int_0^1 ds \frac{dX(s)}{ds} X(s)^{-1} \exp(X(s)/M^2) \right).
\]

\[(5\cdot2)\]

If we set

\[
X(s) = D(s)D_\alpha
\]

\[
= \left( \mathcal{D} - is\mathcal{A} \frac{1 - \gamma_5}{2} \right) \left( \mathcal{D} - i\mathcal{A} \frac{1 - \gamma_5}{2} - i\frac{\alpha}{2} \mathcal{A} \frac{1 + \gamma_5}{2} \right),
\]
As is already done in Ref. 16), we get

$$e^{P(A)} \equiv \exp \left( \int d^2 x (i a A^a - i A^a (\delta_{\mu\nu} - i \varepsilon_{\mu\nu}) \nabla^a \nabla^a (\delta_{ab} + i \varepsilon_{ab}) A^b) \right). \tag{5.3}$$

This result coincides with that of Ref. 15). Here we discuss the topological structure concerning this modified version and study the relations with $D_m(D(1); D(0))$. We introduce, for example, four points $P_0 = D(0)^2$, $P_1 = D(1)^2$, $Q_0 = D(0)D_a$, $Q_1 = D(1)D_a$ on the space of product of two invertible Dirac operators (see Fig. 1). Fermionic determinants for the path $P_0 \to P_1$ and $Q_0 \to Q_1$ are written as

$$\bar{D}_m(P_1; P_0) = \exp(\mathcal{A}(A)),$$

$$\bar{D}_m(Q_1; Q_0) = \exp(\mathcal{A}_a(A, a)), \tag{5.4}$$

where $\mathcal{A}(A)$ is the twice of $\Gamma(A)$. By introducing an additional parameter $t$, we can connect $P_0$ with $Q_0$ and $P_1$ with $Q_1$.

$$\bar{D}_m(D(0)D_a; D(0)^2)$$

$$= \exp \left( \text{Tr} \int_0^1 dt \frac{d}{dt} (D(0)D_0(t))(D(0)D_0(t))^{-1} \exp(D(0)D_0(t)/M^2) \right)$$

$$= U_a(0),$$

$$\bar{D}_m(D(1)D_a; D(1)^2)$$

$$= \exp \left( \text{Tr} \int_0^1 dt \frac{d}{dt} (D(1)D_1(t))(D(1)D_1(t))^{-1} \exp(D(1)D_1(t)/M^2) \right)$$

$$= U_a(1), \tag{5.5}$$

where $D_0(t) = iD_a + (1-t)D(0)$ and $D_1(t) = tD_a + (1-t)D(1)$. The property of Eq. (2.9) holds as it is and if we use it,

$$\exp(\mathcal{A}_a(A, a)) = U_a(1) \exp(\mathcal{A}(A)) U_a(0)^{-1}. \tag{5.6}$$

The local form of Eq. (5.6) is

$$\delta \mathcal{A}_a(A, a)$$

$$= U_a \delta \mathcal{A}(A) U_a^{-1} + \delta U_a U_a^{-1}, \tag{5.7}$$

where $\delta$ denotes the variation with respect to $A$. It is found that there exists a hidden 'gauge' structure where the different regularized determinants are related by the 'gauge' transformation. Thus the modified one $\bar{D}_m(X(1); X(0))$ may be regarded as an extension.
of \( D_M(D(1) ; D(0)) \). The above structure holds in general case and it is supposed that \( \Gamma(A) \) is not necessarily identical with the half of \( \mathcal{A}_a \). Actually in the case of the chiral Schwinger model, \( (1/2)\mathcal{A}(A) \) is the effective action obtained by the regularization method in a narrow sense and \( \mathcal{A}_a(A, a) \) corresponds to the results in Ref. 15.

§ 6. Summary

We have derived non-abelian anomalies as changes of our regularized determinant \( D_M(D(1) ; D(0)) \) defined on Euclidean space-time \( \mathbb{R}^{2n} \) under infinitesimal chiral transformations. We have shown that the space of invertible Dirac operators is not simply connected in spite of the fact that the space-time has trivial topology by using chiral transformation and homotopy invariant quantity \( D_M(D ; D) = 1 \). The extended version \( \tilde{D}_M(X(1) ; X(0)) \) of our regularized determinant has been proposed. There exists a hidden 'gauge' structure in which regularized effective action is regarded as 'gauge field'. This determinant can make chiral Schwinger model unitary.

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References