Topological $CP^{n-1}$ Model and Topological Quantum Mechanics

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According to a way of the construction of topological quantum field theories in our previous paper, we present two topological models, topological $CP^{n-1}$ model in 2-manifold and topological quantum mechanics. In the topological $CP^{n-1}$ model we find topological invariants more systematically by using homogeneous coordinates than we do by using local coordinates as in Witten's model. The topological quantum mechanics is completely solvable.

Topological quantum field theories (TQFT) have recently been proposed by Witten and have been shown to generate topological invariants of space-time manifolds. His way of their construction seems to be rather artificial. So in a recent paper we have clarified a general way of the construction of TQFT. Namely we have found a huge local symmetry involved in TQFT and have shown that TQFT can be obtained by an appropriate gauge fixing of the symmetry. Invariant action carrying this symmetry is a topological index of field manifold (or space-time manifold). Evidently the index is invariant under almost arbitrary changes of fields. Accordingly our way of the construction of TQFT is to start with this invariant action involving a huge local symmetry and leading no classical dynamics.

In this paper we shall present two examples of TQFT, which are derived according to the procedures in Ref. 4). They are a topological $CP^{n-1}$ model in 2-manifold and a topological quantum mechanics. The topological $CP^{n-1}$ model based on homogeneous coordinates, not local coordinates used in Ref. 2) has a great similarity with the model in Ref. 1). BRS charge is nilpotent up to $U(1)$ gauge transformation, and topological invariants can be obtained systematically. They are described in moduli space of instantons. These instantons are well-known solutions in the case of the manifold $M = S^2$. On the other hand, the topological quantum mechanics is a quite simple, but non-trivial example of TQFT. We can diagonalize its Hamiltonian and obtain all of physical states. It shows explicitly some of typical features of TQFT.

In the case of sigma models on homogeneous spaces like this $CP^{n-1} \sim SU(n)/U(n-1)$, it is more convenient to take homogeneous coordinates (e.g., $Z^i$ used in this paper) than to take local coordinates as in Ref. 2). As the homogeneous coordinates are defined up to gauge transformations, the models keep gauge invariances and hence have a structure in BRS charge similar to the one in the topological gauge model. Consequently we can find topological invariants (physical quantities) systematically contrary to the case of Witten's model.

First we shall present the topological $CP^{n-1}$ model. Basic fields $Z_i$ take values in the space of $CP^{n-1} : Z_i (i = 1 \sim n)$. $\sum_{i} \bar{Z}_i Z_i = Z_i \bar{Z}_i = 1$ and $Z_i$ are defined up to $U(1)$ gauge transformation (we identify $Z_i$ with $Z_i e^{i\theta}$). $\bar{Z}_i$ is a complex conjugate of $Z_i$. We abbreviate the index $i$ in summation which is indicated with a dot. These fields
define a mapping from a compact 2-manifold \( M \) with metric \( g_{ab} \) to the space of \( CP^{n-1} \). In the case, we have a topological index of the fields.

\[
S=\frac{1}{2} \int_M F = i \int_M \sqrt{g} \varepsilon^{ab} D_aZ \cdot D_bZ d^2 \sigma
\]

(1)

with anti-symmetric tensor \( \varepsilon^{ab} \), where \( F \) is a 2-form \( F_{ab} \) and the covariant derivative \( D_a \) is defined such that \( D_aZ = (\partial_a - iA_a)Z_i, (iA_a = \bar{Z} \cdot \partial_a Z) \).

\( \sigma^a(a=1,2) \) are coordinates in \( M \). Since \( dF = 0 \), the index \( S \) is invariant under the transformation \( \delta Z_i = \varepsilon_i \) with the condition, \( \bar{\varepsilon} \cdot Z + Z \cdot \varepsilon = 0 \) (we call it topological symmetry). We take this index as a classical action of the fields \( Z_i \). Thus in order to quantize this model we have to fix a gauge of this topological symmetry. Before doing so, we note that in the manifold \( M = S^2 \), we can find instanton solutions with the index of negative values in the following equation,

\[
D_aZ_i + i\varepsilon_{ab}D_bZ_i = 0.
\]

(2)

In general manifold \( M \) such solutions would not necessarily exist. In this paper however we consider only such manifold \( M \) in which instanton solutions of Eq. (2) exist with the topological index of negative values. Thus we may adopt the gauge condition in Eq. (2), because by the use of the topological symmetry we can transform the fields \( Z_i \) into the instanton solution without the change of the index. We should also note that the transformation \( \delta Z_i = \varepsilon_i \) itself is invariant under the change of the parameter \( \varepsilon_i \); \( \delta \varepsilon_i = i\partial Z_i \). This is because \( Z_i \) is defined modulo \( U(1) \) transformation. After all, in quantum theory we need a real ghost \( \phi \) associated with the parameter \( \theta \) as well as complex ghosts \( \psi_i \) associated with \( \varepsilon_i \) (\( \psi_1 \) satisfy the equation, \( \psi \cdot \bar{Z} + Z \cdot \bar{\psi} = 0 \)). Taking the gauge condition in Eq. (2) and an additional gauge condition \( \psi \cdot Z = 0 \) for fixing the symmetry, \( \delta \varepsilon_i = i\partial Z_i \), we obtain the gauge fixed action.\(^3\)

\[
\varepsilon L = \delta_b(\bar{X}^a \cdot D_aZ + \lambda \bar{\psi} \cdot Z + i/2\bar{X}^a \cdot B_a) + \varepsilon(\text{h.c.})
\]

\[
= \varepsilon(i\bar{B}^a \cdot D_aZ - \bar{X}^a \cdot D_a\psi + \bar{X}^a \cdot Z(\bar{\psi} \cdot D_aZ + \bar{Z} \cdot D_a\psi) + \eta \bar{\psi} \cdot Z \\
+ \lambda(\psi - \bar{\psi}) - 1/2\bar{B}^a B_a + 1/2\phi \bar{X}^a \cdot X_a + \text{h.c.})
\]

(3)

where \( X^a(B^a, \eta) \) is an anti-ghost (Nakanishi-Lautrup fields) and \( \lambda \) is a real anti-ghost corresponding to the ghost \( \phi \). Both \( X^a \) and \( B^a \) satisfy the anti-self-dual equation.

\[
\bar{X}^a_i + i\varepsilon^{ab} \bar{X}_{b,i} = \bar{B}^a_i + i\varepsilon^{ab} \bar{B}_{b,i} = 0.
\]

(4)

BRS transformation \( \delta_b \) is defined by

\[
\delta_b Z_i = \varepsilon \psi_i, \quad \delta_b \psi_i = -\varepsilon \phi Z_i, \quad \delta_b \bar{X}^a_i = i\varepsilon \bar{B}^a_i, \quad \delta_b \bar{B}^a_i = -i\varepsilon \phi \bar{X}^a_i,
\]

\[
\delta_b \lambda = \varepsilon \eta, \quad \delta_b \eta = 0, \quad \delta_b \phi = 0,
\]

(5)

where \( \varepsilon \) is a BRS parameter \( (\varepsilon^t = -\varepsilon) \). It is easy to see that \( \delta_b(\varepsilon') \cdot \delta_b(\varepsilon)(\delta_b(\varepsilon) \) presents a BRS transformation with a parameter \( \varepsilon \) is equivalent to \( U(1) \) gauge transformation with the parameter \( i\varepsilon \phi \). This is the same structural of BRS nilpotency as in Witten's model\(^1\) where \( \delta_b \) is nilpotent up to \( SU(N) \) gauge transformation. Integrating over the auxiliary field \( B_a \), we obtain
\[ L' = \frac{1}{4} |D_a Z + i \epsilon_{ab} D^b Z|^2 - \bar{X}^a \cdot D_a \psi + X^a \cdot \bar{D}_a \psi + (\bar{X}^a \cdot Z + Z \cdot X^a)(\bar{\psi} \cdot D_a Z + \bar{Z} \cdot D_a \psi) + \phi \bar{X}^a \cdot X^a + \eta(\bar{\phi} \cdot Z - \bar{Z} \cdot \phi) + 2 \lambda (\phi - \bar{\phi} \cdot \phi), \]

where we have used the equation \( \bar{\phi} \cdot Z + \bar{Z} \cdot \phi = 0 \). BRS transformation of \( X^a \) is modified such that \( \delta_b X_i = 1/2 \epsilon (D^b Z_i - i \epsilon_{ab} D_b Z_i) \). Therefore, a quantum action is given by

\[ S_q = S + \int_M \sqrt{g} L' d^2 \sigma. \]  

From this action we can show\(^1\) that the energy-momentum tensor of this model is expressed as a commutator of the BRS charge \( Q \) and a tensor. Hence it turns out as in Witten's model that only physical states are topological; they are degenerate with vacuum.

Now we shall derive topological invariants of 2-manifold \( M \). These are described in terms of BRS invariant operators. In order to obtain such operators, we solve the following equations,

\[ dW_0 = \{ Q, W_i \}, \quad dW_i = \{ Q, W_0 \}, \]

\[ dW_0 = 0 \quad \text{and} \quad \{ Q, W_0 \} = 0, \]  

where \( W_a \) are a-form on \( M \). We note that \( \phi \) is BRS invariant and non-trivial (\( \phi \neq \{ Q, K \} \) for any \( K \)). Hence, setting \( W_0 = - i \phi \), we find solutions of Eq. (8),

\[ W_0 = (D_a Z \cdot \psi + D_a \psi \cdot Z) d \sigma^a \quad \text{and} \quad W_2 = 1/2 F_{ab} \, d \sigma^a \wedge d \sigma^b. \]  

Using these solutions, we obtain BRS invariant operators,

\[ \Gamma_0 = W_0, \quad \Gamma_1 = \int_M W_i \quad \text{and} \quad \Gamma_2 = \int_M W_2, \]  

where \( \gamma_a \) is a homology \( a \)-cycle on \( M \). We see that the functional average of \( \Gamma_q \), \( \langle \Gamma_q \rangle \) is topological invariant. \( \langle \Gamma_q \rangle \) is an analogue of Donaldson polynomials\(^1\) in Witten's model.

It should be mentioned that although the functional average is performed with the weight, \( \exp(-S_\alpha/\epsilon^2) \) (\( \epsilon \) is a coupling constant), the average, \( \langle \Gamma_q \rangle \) is independent of the coupling constant. This follows from the fact\(^1\) that except for the index \( S \), \( S_\alpha \) is a BRS commutator. Therefore \( \langle \Gamma_q \rangle \) may be calculated in the limit, \( \epsilon^2 \to 0 \) and hence \( \langle \Gamma_q \rangle \) is described in terms of instanton solutions, \( \bar{Z}^m \) and zero modes of \( \delta Z_i (\phi_i) \) around \( Z^m : \delta Z_i (\phi_i) \) satisfy the equations,

\[ D_a^{i a} W - \bar{Z}^m (\bar{W} \cdot D_a^{i a} Z^m + \bar{Z}^{i a} \cdot D_a^{i a} W) + \text{dual} = 0 \quad \text{and} \quad \bar{W} \cdot \bar{Z}^m = 0, \]

where \( W_i = \delta Z_i \) or \( \phi_i \) and \( D_a^{i a} = \partial_a - i \bar{Z}^{i a} \cdot \partial_a Z^m \). Non-zero components\(^1\) of \( \delta Z_i (\phi_i) \) do not contribute to \( \langle \Gamma_q \rangle \) in the limit, \( \epsilon^2 \to 0 \).

Second, we shall present a topological quantum mechanics whose base manifold is one-dimensional. Although the manifold is almost trivial, the model is instructive for gaining insight into TQFT.
A field variable we consider is a real function $q(t)$ on a simply connected manifold $\Sigma$ whose coordinate is given by time $t$. A classical action is assumed to be independent of $q(t)$. Obviously it is invariant under the arbitrary change of the variable, $\delta q = \epsilon$. To fix this arbitrariness, we take a gauge $V(q, \dot{q}) = 0$ where $\dot{q} = dq/dt$ and $V$ is a function of $q$ and $\dot{q}$ ($V$ is not invariant under $\delta \dot{q} = \epsilon$). Introducing a ghost $\phi$ and an anti-ghost $\lambda$ we derive a gauge fixed action,

$$i\epsilon L = \delta_b(\lambda V + k\lambda B)$$

$$= i\epsilon(BV + kB^2 + i\lambda \phi \frac{\delta V}{\delta \dot{q}}),$$

(12)

where $k$ is a gauge parameter and $B$ is an auxiliary field. The BRS transformation $\delta_b$ in Eq. (12) is given by

$$\delta_b \phi = \epsilon \phi, \quad \delta_b \lambda = 0, \quad \delta_b \lambda = i\epsilon B \quad \text{and} \quad \delta_b B = 0,$$

(13)

where $\epsilon$ is a BRS parameter ($\epsilon^* = -\epsilon$). All of these fields are real. Integrating the field $B$, we obtain

$$L' = 1/2 \dot{q}^2 + i\lambda \phi.$$  

(14)

Here we have taken explicitly the gauge $V = \dot{q}$ and $k = -1/2$. The BRS transformation of $\lambda$ is replaced by $\delta_{\lambda} \lambda = i\epsilon \dot{q}$. It then follows that Hamiltonian along with canonical commutation relations is given by

$$H = 1/2p^2, \quad [q, p] = i\{\phi, \Pi\} = i$$

(15)

with canonical conjugate variables $P(= \dot{q})$ and $\Pi(= i\lambda)$. Therefore we see that the model is so simple that all interesting quantities are calculable.

Let us see physical states defined in this simple model. In order to do so we construct the BRS charge.

$$Q = \phi P,$$

(16)

which generates the BRS transformation, e.g., $[i\epsilon Q, q] = \epsilon \phi$. Furthermore, if we define the operator $\bar{Q} = \lambda P$, Hamiltonian is written such that

$$H = 1/2\{Q, \bar{Q}\}.$$  

(17)

The existence of these operators ($Q, \bar{Q}$) is a typical feature of TQFT.

Writing quantum states such as $|\alpha\rangle = |P\rangle \otimes \uparrow\langle P| P\rangle = P\rangle |P\rangle$, $\phi|+\rangle = 0$ and $\lambda|+\rangle = |\uparrow\downarrow\rangle$. We can easily obtain the two physical states, $|P = 0\rangle \otimes |+\rangle$ and $|P = 0\rangle \otimes |\uparrow\downarrow\rangle$. (The physical states are defined such that $Q|\alpha\rangle = 0$ and $|\alpha\rangle \neq Q|\beta\rangle$ for any $\beta$. So, we may obtain other physical states by adding the states $Q|\beta\rangle$ to the above two states. But all of these states are not necessarily physically independent. Hence we may restrict the physical states as being taken above. This is a sort of “gauge fixing”, although the procedure is not standard in BRS formalism.) Obviously the energies of these states are zero, as expected from Eq. (17). One of the states is vacuum and the other one is an excited state (but it is energetically degenerate with the vacuum). It turns out that with the non-trivial physical operator $\phi + \lambda$, one of the states is
transformed into the other one \((\{Q, \phi+\lambda\}=P, \text{ but } P \text{ vanishes on the restricted physical space composed of the above two states. Hence } \phi+\lambda \text{ may be taken as a physical operator). This } \phi+\lambda \text{ is the only physical operator. It should be noted that although } P \text{ is apparently a physical operator, } P \text{ is trivial because } P \text{ vanishes on the physical space.}

This operator corresponds to \(I_0\) in Eq. (10). Hence the vacuum expectation value of \(\phi+\lambda\) in Heisenberg representation depends only on the homology group \(H_0(\Sigma)\). Namely it has no time dependence. In this model we do not have a corresponding physical operator \(I_1\) derived from \(I_0\) (\(I_1\) vanishes identically).

In this paper we have presented two topological models which have typical features of TQFT shown in Ref. 1). The topological \(CP^{n-1}\) model can be generalized easily to sigma models on homogeneous spaces \((G/H)\) : In the models we use a field variable \(g \in G\) as a homogeneous coordinate and impose the gauge symmetry of the gauge group \(H\). This way of construction is different from that in Ref. 2), where local coordinates of \(G/H\) were used instead of \(g\) and thus there were no gauge symmetries. Our formulation leads automatically to a BRS invariant bosonic field like \(\phi\) in this paper as a gauge parameter of the gauge transformation \((\delta g)^2\). (Note that \((\delta g)^2\) becomes a gauge transformation of the group \(H\) because the gauge ambiguity is not fixed in our formulation. See Eq. (5).) Using this bosonic field, we can easily obtain subsequent BRS invariants (see Eq. (8)). On the other hand, in order to obtain these invariants in Witten's formulation,\(^2\) we have to find an appropriate differential zero form on \(G/H\) corresponding to the above bosonic field. It is not trivial problem. Therefore our formulation of the sigma models on homogeneous spaces is more manageable than Witten's one.

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