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Diverging Overland Flow

1. Analytical Solutions

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Kinematic wave theory is utilized to investigate diverging overland flow which does not appear to have been previously investigated in watershed hydrology. This paper is the first in a sequel of two. It developes analytical solutions for a diverging geometry for two cases: (1) when infiltration is considered through rainfall-excess, and (2) when infiltration is treated concurrently with runoff; this gives rise to a free boundary problem. Part II discusses the application of the proposed model to natural watersheds.

Introduction

Kinematic modeling of watershed runoff requires specification of (1) geometry, (2) inflow, (3) kinematic equations, and (4) initial and boundary conditions. A natural watershed is often complex in its geometric configuration. Therefore, it is necessary to transform its complex geometry into a simpler one having a similar hydrologic response.

Depending upon the geometric complexity natural watersheds may be represented by a combination of four basic geometric elements (1) plane, (2) converging section, (3) diverging section, and (4) channel. These elements can be arranged in such a way as to provide an almost perfect representation of a given watershed regardless of its geometric complexity (Wooding 1965; Brakensiek 1967; Woolhiser 1969; Kibler and Woolhiser 1970; Singh 1976a, 1976b, 1978; Singh and Woolhiser 1976; Reid 1977).

It has been argued that the converging geometry can be employed to represent a watershed irrespective of its complexity. It is, however, implicit that the watershed geometry is simple and that it has an appearance resembling this configuration. Many upland watersheds in nature are fortunately such that they are amenable to such a simplified representation. These watersheds have more or less built-in convergence.

On the other hand, there are many watersheds in nature (Zernitz 1932; Howe, 1960) which diverge in shape. These have an appearance converse of that of converging watersheds. The flow on these watersheds is radially outward, converse of that on converging surfaces. There are also many watersheds, both rural and urban, which are roughly leaf-shaped. Their upland portions diverge and lower portions converge. It would then appear that it may be more appropriate to represent the diverging watershed by diverging geometry. If a watershed is diverging and converging then coupling of a diverging section with a converging section may be desirable to represent it. If a watershed has a complex geometry a portion of which is diverging then it would be desirable to include the diverging section in a network model to represent it.

There are two ways to specify inflow in runoff modeling. First, rainfall and infiltration can be combined to yield rainfall excess. This approach is more frequently used in hydrology. Second, infiltration can be considered in concurrence with runoff during and after the occurrence of rainfall. With a few exceptions, notably the work by Smith (1970), Smith and Woolhiser (1971), Sherman and Singh (1976b), this approach has not been pursued vigorously. Implications of this approach have been discussed by Singh (1978).

A survey of literature would indicate that overland flow on diverging surfaces has not been explicitly included in mathematical modeling of watershed runoff. This study proposes to investigate the hydrology of diverging overland flow in detail utilizing kinematic wave approximation. This paper is the first in a sequence of two. It developes analytical solutions treating infiltration through space-time invariant rainfall-excess and in concurrence with rainfall and runoff.

Hydrology of Diverging Overland Flow

Diverging Geometry

The diverging geometry is shown in Fig. 1 where R denotes the length of the section, a the parameter related to the degree of divergence, θ the interior angle, and S_o the ground slope. Therefore, $R(1\text{-}a)$ is the length of flow. Because of radial symmetry θ does not affect relative response characteristics; only the watershed area must be preseved. It is therefore dependent on R and a. The diverging section may possess some interesting features: (1) its discrete analog is a system composed of a cascade of unequal nonlinear reservoirs (a systems view). (2) Its response is similar to that of a cascade of planes of increasing size.

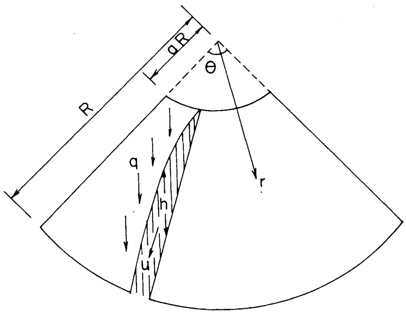


Fig. 1. Diverging geometry.

Kinematic Equations

The hydrology of flow over a diverging surface is described by kinematic wave theory that includes the equation of continuity and an approximation of the equation of momentum. For unsteady, nonuniform flow over a diverging surface these equations (Singh and Agiralioglu 1980) on a unit width basis are

continuity equation,

$$\frac{\partial h}{\partial t} + u\frac{\partial h}{\partial r} + h\frac{\partial u}{\partial r} + \frac{uh}{r} = q(r,t) - f(r,t) \quad (1)$$

and the kinematic approximation to the momentum equation,

$$Q = uh = \alpha(r,t)\,h^{n} \quad (2)$$

where h is local mean depth of flow, u local mean velocity, Q rate of outflow per unit width, $q(r,t)$ rate of lateral inflow or rainfall, f infiltration rate, r space coordinate, t time coordinate, and n and α are kinematic friction relationship parameters.

Combining Eqs. (1)-(2) with α constant,

$$\frac{\partial h}{\partial t} + \alpha h^{n-1}\frac{\partial h}{\partial r} = q(r,t) - f(r,t) - \frac{\alpha h^{n}}{r} \quad (3)$$

Eq. (3) is the basic governing equation.

Initial Conditions

The initial conditions imposed on h are

$$\begin{aligned} h(r,0) &= 0, \quad aR \le r \le R \\ h(aR,t) &= 0, \quad 0 \le t \le T \end{aligned} \quad (4)$$

It is implied here that the diverging surface is inclined on a finite slope. This condition may not be unduly restrictive in upland areas. Thus the problem reduces to solving Eq. (3) subject to Eq. (4).

Rainfall and Infiltration

The term $q(r,t)$ has two connotations. If $f(r,t)$ is accounted for through rainfall excess, then mathematically it is tantamount to saying that $f(r,t) = 0$ and $q(r,t)$ represents rainfall excess. If the concept of rainfall excess is abandoned, $q(r,t)$ will represent rainfall, and $f(r,t)$ will be finite.

We specify $q(r,t)$ as

$$q(r,t) > 0, \quad t < T; \quad q(r,t) = 0, \quad t \geq T \tag{5a}$$

where T is the duration of q. $f(r,t)$ depends on the depth of flow h in the following sense,

$$f(r,t) > 0 \quad \text{if} \quad h(r,t) > 0; \quad f(r,t) = 0 \quad \text{if} \quad h(r,t) = 0 \tag{5b}$$

We will further assume that

$$q(r,t) > f(r,t), \quad 0 \leq t \leq T \tag{5c}$$

Although α, q and f will vary in space and time (Sherman and Singh 1976a, 1976b), we will consider them to be constant in the ensuing analytical development. This assumption is necessary to obtain explicit solutions. It may be added that allowing them to vary in space will only slightly increase mathematical complexity but the essential features of the solution $h(r,t)$ or $Q(r,t)$ will not change.

Solution Domain

If q represents rainfall excess then Eq. (3) will hold in the region $S = \{aR < r < R, t > 0\}$. When q represents rainfall, the solution domain is somewhat more complicated. It is plausible on physical grounds that there will be a curve $t = t^{\circ}(r)$ in $[t \geqslant T; aR < r < R]$ starting at $r = aR$, $t = T$, such that $h(r,t^{\circ}(r)) = 0$. This curve gives the time history of the water edge as it advances from $r = aR$ to $r = R$. Eqs. (1)-(2) are satisfied in $S \doteq [0 < t < t^{\circ}(r), aR = r = R]$. Thus $t^{\circ}(r)$ is a free boundary, and Eqs. (1)-(2) and (4) constitute a free boundary problem; in the domain above the curve $t^{\circ}(r)$, $h(r,t) = 0$. The free boundary $t = t^{\circ}(r)$ is not known apriori and must be determined along with the solution. It may be pointed out that the free boundary will not arise if infiltration is treated through rainfall-excess.

Mathematical Solutions

We solve Eq. (3) using the method of characteristics. Its characteristics are given by

$$\frac{dt}{dr} = \frac{1}{n\alpha h^{n-1}} \tag{6}$$

$$\frac{dh}{dr} = \frac{(q-f)}{n\alpha h^{n-1}} - \frac{h}{nr} \tag{7}$$

The solution of Eq. (3) is the surface $h(r,t)$ formed by all the characteristics passing through the segment $t = 0$, $aR \leqslant r \leqslant R$ and the segment $r = aR$, $0 \leqslant t \leqslant T$.

The free boundary $t = t^{\circ}(r)$ is the locus $h(r,t) = 0$ in the (r,t) plane. The initial conditions then become

$$t(aR) = t_0, \quad h(aR) = 0, \quad 0 \leq t_0 \leq T \tag{8}$$

and

$$t(r_0) = 0, \quad h(r_0) = 0, \quad aR \leq r \leq R \tag{9}$$

We assume that the curves $t = t(r,t_o)$, which are the solutions of Eqs. (6)-(7) and (9) do not intersect for distinct values of t_0. Similarly, we assume that the curves $t = t(r,r_o)$ which are the solutions of Eqs. (6)-(7) and (9) do not intersect for distinct values of r_o. This can be shown to be true but we omit the proof.

We can distinguish three cases A, B_1 and B_2 which depend on the relative disposition of the three curves $t = t^{\circ}(r)$, $t = T$, and $t = t(r,aR)$ ($t = t(r,r^*)$ is the prolongation of $t = t(r,aR)$ to the right of $r = r^*$) as shown in Figs. 2-4. The case A is defined by $t^{\circ}(r) > T > t(r,aR)$, $aR \leqslant r \leqslant R$. This will give rise to equilibrium hydrograph. For case B_1, $t^{\circ}(r) > T$ and $t^{\circ}(r) > t(r, aR)$ but $t = T$ and $t = t(r,aR)$ intersect at $r = r^*$, i.e., $T = t(r^*,aR)$ and $aR < r^* < R$. Similarly, in the case B_2 $t^{\circ}(r) > T$ but $t = T$ and $t = t(r,aR)$ intersect at $r = r^*$, and $t = t^{\circ}(r)$ and $t = t(r,r^*)$ intersect at $r = \bar{r}$, i.e., $t^{\circ}(\bar{r}) = t(\bar{r},r^*)$ and $aR < \bar{r} < R$. Both cases B_1 and B_2 will give rise to partial equilibrium hydrographs.

Since $t^{\circ}(r)$ and $t(r,aR)$ are not known until we have solved the problem, it appears that we cannot distinguish the cases A, B_1, and B_2 beforehand. However, for space-time invariant q and f, these cases can be distinguished easily as seen from the discussion in the text. For simplicity let us introduce

$$q^* = q - f; \quad \beta = \left(\frac{q}{2}\right)^{\frac{1}{n}}; \quad \beta^* = \left(\frac{q^*}{2}\right)^{\frac{1}{n}}; \quad m = -\frac{(n-1)}{n};$$

$$\gamma = \frac{1}{n}\left(\frac{q}{2}\right)^m; \quad \gamma^* = \frac{1}{n}\left(\frac{q^*}{2}\right)^m; \quad \text{and} \quad \rho = \frac{f}{q}$$

Mathematical Solution for Case A

The solution region, as shown in Fig. 1, is divided into domains D_1, D_2, and D_3. We derive the solution for each domain.

Domain D_2. This is defined by $r = aR$, $t = T$ and $t = t(r,aR)$. We solve Eqs. (6)-(7) subject to Eq. (8). Solving Eq. (7) first,

$$h(r,t_0) = \beta^* \left(\frac{r^2 - a^2R^2}{\alpha\, r}\right)^{\frac{1}{n}} \tag{10}$$

Inserting Eq. (10) in Eq. (6) and then solving,

$$t(r,t_0) = t_0 + \gamma^* \left(\frac{1}{\alpha}\right)^{\frac{1}{n}} \int_{aR}^{r} \left(\frac{x^2 - a^2R^2}{x}\right)^m dx$$

Using Binomial theorem,

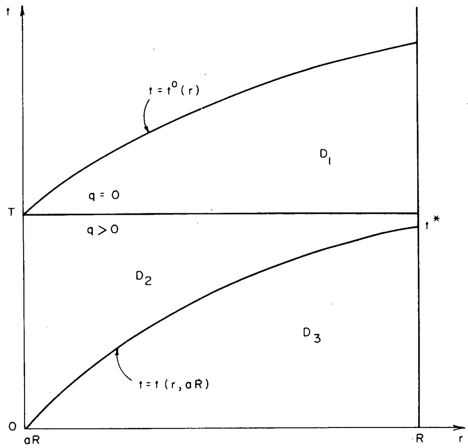


Fig. 2. Solution domain for case A.

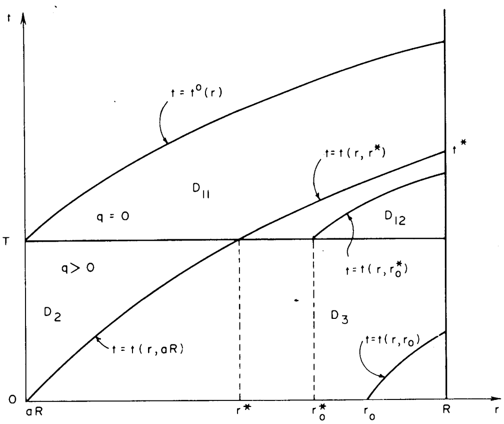


Fig. 3. Solution domain for case B_1.

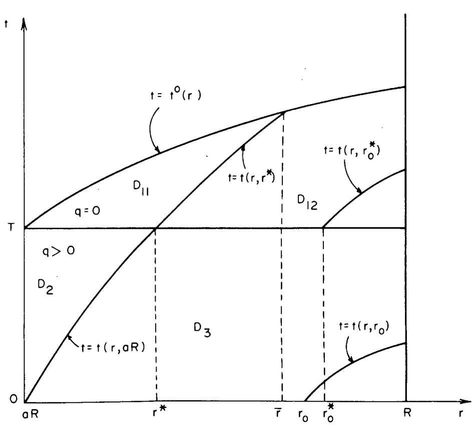


Fig. 4. Solution domain for case B_2.

$$t(r,t_0) = t_0 + \gamma^*(\frac{1}{\alpha})^{\frac{1}{n}} \sum_{j=0}^{\infty} \binom{m}{j} (-1)^j (aR)^{2j}$$
$$(\frac{r^{m-2j+1} - (aR)^{m-2j+1}}{m-2j+1}) \quad (11)$$

If $aR = 0$, which is true in many watersheds,

$$h(r,t_0) = \beta^*(\frac{r}{\alpha})^{\frac{1}{n}} \quad (12)$$
$$t(r,t_0) = t_0 + n\,\gamma^*(\frac{r}{\alpha})^{\frac{1}{n}} \quad (13)$$

On the other hand, if q represents rainfall excess, the solution is

$$h(r,t_0) = [\frac{q(r^2-a^2R^2)}{2r\alpha}]^{\frac{1}{n}} \quad (14)$$

Substituting Eq. (14) into Eq. (7) and solving,

$$t(r,t_0) = t_0 + \frac{1}{n}(\frac{1}{\alpha})^{\frac{1}{n}} (\frac{q}{2})^m \int_{aR}^{r} (\frac{x^2-a^2R^2}{x})^m \, dx$$

Using Binomial theorem,

$$t(r,t_0) = t_0 + \frac{1}{n}(\frac{1}{\alpha})^{\frac{1}{n}} (\frac{q}{2})^m \sum_{j=0}^{\infty} \binom{m}{j} (-1)^j (aR)^{2j}$$
$$(\frac{r^{m-2j+1} - (aR)^{m-2j+1}}{m-2j+1}) \quad (15)$$

if $aR = 0$

$$h(r,t_0) = (\frac{qr}{2\alpha})^{\frac{1}{n}} \quad (16)$$
$$t(r,t_0) = t_0 + (\frac{q}{2})^m (\frac{r}{\alpha})^{\frac{1}{n}} \quad (17)$$

It is seen from Eqs. (10)-(17) that the flow does not depend on t and is, therefore, steady but nonuniform.

Domain D_3. This is defined by $t = 0$, $r = R$, and $t = t(r,aR)$. Eqs. (6)-(7) are solved subject to Eq. (9),

$$h(r,r_0) = \beta^*(\frac{r^2-r_0^2}{r\alpha})^{\frac{1}{n}} \quad (18)$$

$$t(r,r_0) = \gamma^*(\frac{1}{\alpha})^{\frac{1}{n}} \int_{r_0}^{r} (\frac{x^2-r_0^2}{x})^m \, dx$$

Using Binomial expansion,

$$t(r,r_0) = \gamma^* \left(\frac{1}{\alpha}\right)^{\frac{1}{n}} \sum_{j=0}^{\infty} \binom{m}{j} (-1)^j r_0^{2j} \left(\frac{r^{m-2j+1} - r_0^{m-2j+1}}{m-2j+1}\right) \tag{19}$$

If q represents rainfall-excess, the solution follows,

$$h(r,r_0) = \left(\frac{q(r^2-r_0^2)}{2\alpha r}\right)^{\frac{1}{n}} \tag{20}$$

$$t(r,r_0) = \frac{1}{n}\left(\frac{1}{\alpha}\right)^{\frac{1}{n}} \left(\frac{q}{2}\right)^m \int_{r_0}^{r} \left(\frac{x}{x^2-r_0^2}\right)^m dx$$

Using Binomial theorem,

$$t(r,r_0) = \frac{1}{n}\left(\frac{1}{\alpha}\right)^{\frac{1}{n}} \left(\frac{q}{2}\right)^m \sum_{j=0}^{\infty} \binom{m}{j} (-1)^j r_0^{2j} \left(\frac{r^{m-2j+1} - r_0^{m-2j+1}}{m-2j+1}\right) \tag{21}$$

It is evident from Eqs. (18)-(21) that h, as a function of r and t, depends on both r and t implying that the flow is both unsteady and nonuniform.

Domain D_1. This is defined by $t = T$, $r = R$, and $t = t^o(r)$. W solve Eqs.(6)-(7) with $q = 0$ subject to the initial condition,

$$t(r_0^*) = T; \quad h(r_0^*) = h(r_0^*,T) = h_0 \tag{22}$$

where h_o is to be obtained by replacing r by r_o^* in Eq. (10). Therefore,

$$h(r_0^*) = \beta^* \left(\frac{r_0^{*2} - a^2R^2}{\alpha r_0^*}\right)^{\frac{1}{n}} \tag{23}$$

Solving Eq. (7) and inserting Eq. (23),

$$h(r,r_0^*) = \beta \left(\frac{(1-\rho)(r_0^{*2}-a^2R^2) - \rho(r^2-r_0^{*2})}{\alpha r}\right)^{\frac{1}{n}} \tag{24}$$

Substituting Eq, (24) into Eq. (6) and solving,

$$t(r,r_0^*) = T + \gamma\left(\frac{1}{\alpha}\right)^{\frac{1}{n}} \int_{r_0^*}^{r} \left(\frac{(1-\rho)(r_0^{*2}-a^2R^2) - \rho(x^2-r_0^{*2})}{x}\right)^m dx$$

which can be written as

$$t(r,r_0^*) = T + \gamma\left(\frac{1}{\alpha}\right)^{\frac{1}{n}} (r_0^* - a^2R^2(1-\rho))^m \sum_{j=0}^{\infty} \binom{m}{j} (-1)^j \left(\frac{\rho \cdot}{r_0^{*2} - a^2R^2(1-\rho)}\right)^j \left(\frac{r^{-m+2j+1} - r_0^{*\,-m+2j+1}}{-m+2j+1}\right) \tag{25}$$

When $aR = 0$, we obtain

$$h(r_0^*) = \beta^* \left(\frac{r_0^*}{\alpha}\right)^{\frac{1}{n}} \tag{26}$$

$$h(r, r_0^*) = \beta \left(\frac{r_0^{*2} - \rho r^2}{\alpha r}\right)^{\frac{1}{n}} \tag{27}$$

$$t(r, t_0^*) = T + \gamma \left(\frac{1}{\alpha}\right)^{\frac{1}{n}} \int_{r_0^*}^{r} \left(\frac{r_0^{*2} - \rho x^2}{x}\right)^m dx$$

which can be written as

$$t(r, r_0^*) = T + \gamma \left(\frac{1}{\alpha}\right)^{\frac{1}{n}} r_0^{*2m} \sum_{j=0}^{\infty} \binom{m}{j} (-1)^j \left(\frac{\rho}{r_0^{*2}}\right)^j \left(\frac{r^{-m+2j+1} - r_0^{*-m+2j+1}}{-m+2j+1}\right) \tag{28}$$

On the other hand, if rainfall excess constitutes inflow,

$$h_0 = \left(\frac{q(r_0^{*2} - a^2R^2)}{2\alpha r_0^*}\right)^{\frac{1}{n}} \tag{29}$$

$$h(r, r_0^*) = h_0 \left(\frac{r_0^*}{r}\right)^{\frac{1}{n}} = \left(\frac{q(r_0^{*2} - a^2R^2)}{2\alpha r}\right)^{\frac{1}{n}} \tag{30}$$

$$t(r, r_0^*) = T + \frac{1}{(2n-1)} \left(\frac{1}{\alpha}\right)^{\frac{1}{n}} \left(\frac{q(r_0^{*2} - a^2R^2)}{2}\right)^m \left(r^{(2n-1)/n} - r_0^{*(2n-1)/n}\right) \tag{31}$$

It is clear from Eqs. (23)-(31) that the flow is both unsteady and nonuniform. *Free Boundary of Domain D_I*. The free boundary $t = t^o(r)$ can be determined by equating Eq. (24) to zero,

$$(1-\rho)(r_0^{*2} - a^2R^2) - \rho(r^2 - r_0^{*2}) = 0 \tag{32}$$

and Eq. (25). By eliminating r_o^* between Eqs. (32) and (25),

$$r_0^* = \psi(r) = ((1-\rho)a^2R^2 + \rho r^2)^{\frac{1}{2}} \tag{33}$$

$$t^0(r) = T + \gamma \left(\frac{1}{\alpha}\right)^{\frac{1}{n}} \rho^m \int_{\psi(r)}^{r} \left(\frac{r^2 - x^2}{x}\right)^m dx$$

Using Binomial expansion,

$$t^0(r) = T + \gamma \left(\frac{1}{\alpha}\right)^{\frac{1}{n}} \rho^m r^{2m} \sum_{j=0}^{\infty} \binom{m}{j} (-1)^j r^{-2j} \left(\frac{r^{-m+2j+1} - (\psi(r))^{-m+2j+1}}{-m+2j+1} \right) \tag{34}$$

Hydrograph Characteristics. For fixed *r*, *h(r,t)* is an increasing function of *t* in domain D_3 independent of *t* in domain D_2 and a decreasing function of *t* in domain D_1 as shown in Fig. 5. It may be noted that the overland flow recession will last only for a finite period of time if infiltration is explicitly considered. The length of the hydrograph duration is explicitly known and will be given by Eq. (34) when *r* is replaced by *R*. This illustrates the effect of considering infiltration in concurrence with runoff. The hydrograph peak $h_{\max}(r)$ and its time can be obtained from Eqs. (10)-(14),

$$h_{\max}(r) = [h(r); \quad aR \leq r \leq R] \tag{35}$$

$$t_{\max}(r) = [t; \quad t(r,0) \leq t \leq T] \tag{36}$$

Time to and Depth at Equilibrium. In the case when *q* represents rainfall excess, the time to equilibrium t_e and the depth at equilibrium h_e at $r = R$ will be given when the characteristic passing through the origin $(aR,0)$ intersects the downstream boundary. In this case t_e will be the same as t^* as shown in Fig. 2, and will be independent of *T*. Thus we obtain

$$h_e = \left(\frac{qR(1-a^2)}{2\alpha}\right)^{\frac{1}{n}} \tag{37}$$

$$t_e = \left(\frac{1}{\alpha}\right)^{\frac{1}{n}} \left(\frac{q}{2}\right)^m \frac{1}{n} \int_{aR}^{R} \left(\frac{x^2 - a^2R^2}{x}\right)^m dx$$

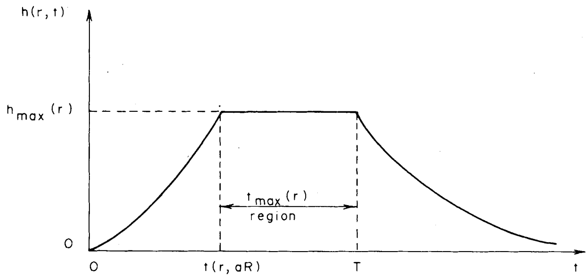
This can be written as

$$t_e = \left(\frac{1}{\alpha}\right)^{\frac{1}{n}} \frac{1}{n} \left(\frac{q}{2}\right)^m \sum_{j=0}^{\infty} \binom{m}{j} (-1)^j (aR)^{2j} \left(\frac{R^{m-2j+1} - (aR)^{m-2j+1}}{m-2j+1} \right) \tag{38}$$

If, however, $aR = 0$, we get closed-form solutions,

$$h_e = \left(\frac{qR}{2\alpha}\right)^{\frac{1}{n}} \tag{39}$$

$$t_e = \left(\frac{R}{\alpha}\right)^{\frac{1}{n}} \left(\frac{q}{2}\right)^m \tag{40}$$



(a) Case A $aR < r < R$
(b) Case B_1 and B_2 $aR < r < r^*$

Fig. 5. Depth of flow, *h (r,t)*, as a function of *t* for fixed *r*.

Mathematical Solutions for Cases B_1 and B_2

Domain D_2. This is defined by $x = 0$, $t = T$, and $t = t(r,aR)$. The solutions in both cases for this domain are given by Eqs. (10)-(17).

Domain D_3. This is defined by $t = 0$, $r = R$, $t = t(r,aR)$ and $t = T$. The solutions for this domain in both cases are given by Eqs. (18)-(21).

Domain D_{11}. This is defined by $t = T$, $t = t^o(r)$, and $t = t(r,r^*)$. The solutions for this domain in both cases are given by Eqs. (23)-(31).

Domain D_{12}. First, we consider the case when rainfall excess is used. This is defined by $t = T$, $t = t(r,r^*)$ and $r = R$. To obtain the solution let r_o^* be the solution of $T = t(r_o,r_o^*)$; that is the value of r when the curve $t = t(r,r_o)$ given by Eq. (21) intersects the line $t = T$. Then along the segment $r^* \leqslant r_o^* \leqslant R, t = T$ we have from Eq. (20),

$$h(r_0, r_0^*) = \left(\frac{q(r_0^{*2} - r_0^2)}{2\alpha r_0^*}\right)^{\frac{1}{n}} \tag{41}$$

We now solve Eqs. (6)-(7) with $q = 0$ subject to the initial condition,

$$t(r_0^*) = T, \quad h(r_0^*) = h(r_0, r_0^*), \quad r^* \le r_0^* \le R$$

Thus we get

$$h(r; r_0, r_0^*) = \left(\frac{q(r_0^{*2} - r_0^2)}{2r\alpha}\right)^{\frac{1}{n}} \tag{42}$$

$$t(r; r_0, r_0^*) = T + \frac{1}{2n-1}\left(\frac{1}{\alpha}\right)^{\frac{1}{n}}\left(\frac{q}{2}\right)^m (r_0^{*2} - r_0^2)^m \left(r^{(2n-1)/n} - r_0^{*(2n-1)/n}\right) \tag{43}$$

The parameters r_0 and r_o^* are bound by the relation,

$$T = \frac{1}{n}\left(\frac{1}{\alpha}\right)^{\frac{1}{n}}\left(\frac{q}{2}\right)^m \int_{r_0}^{r_0^*} \left(\frac{x}{x^2-r_0^2}\right)^m dx$$

Therefore,

$$T = \frac{1}{n}\left(\frac{1}{\alpha}\right)^{\frac{1}{n}}\left(\frac{q}{2}\right)^m \sum_{j=0}^{\infty}\binom{m}{j}(-1)^j r_0^{2j}\left(\frac{r_0^{*\,m-2j+1} - r_0^{m-2j+1}}{m-2j+1}\right) \quad (44)$$

Thus in Eqs. (42)-(44) we may think of r_0^* as parameter in which case we will have to replace r_0 which appears in these functions by its solution to Eq. (44) in terms of r_0^*. On the other hand, we may think of r_o as the parameter in Eqs. (42)-(43) in which case we will have to replace r_o^* in these equations by its solution to Eq. (44). Since in Eq. (44) r_0^* is an increasing function of r_o, the correspondence between them is one to one.

On the other hand, if f is included explicitly, D_{12} is defined by $t = T$, $r = R$, $t = t(r,r^*)$ and $t = t^o(r)$. Again we define r_o^* by $T = t(r_o,r_o^*)$; $r^* \leqslant r_o^* \leqslant R$. Thus from Eq. (19),

$$T = \gamma^*\left(\frac{1}{\alpha}\right)^{\frac{1}{n}} \int_{r_0}^{r_0^*} \left(\frac{x^2-r_0^2}{x}\right)^m dx$$

Using Binomial theorem,

$$T = \gamma^*\left(\frac{1}{\alpha}\right)^{\frac{1}{n}} \sum_{j=0}^{\infty}\binom{m}{j}(-1)^j r_0^{2j}\left(\frac{r^{m-2j+1} - r_0^{m-2j+1}}{m-2j+1}\right) \quad (45)$$

It is apparent that there is a unique correspondence between r_o and r_o^*. The initial condition will become

$$t(r_0^*) = T; \quad h(r_0^*) = h(r_0^*,T) = h_0 \quad (46)$$

h_o will be defined by Eq. (21). Solving Eq. (7) with $q = 0$ and Eq. (46),

$$h(r;r_0,r_0^*) = \beta\left(\frac{(1-\rho)(r_0^{*2}-r_0^2) - \rho(r_0^2-r_0^{*2})}{\alpha r}\right)^{\frac{1}{n}} \quad (47)$$

Substituting Eq. (47) into Eq. (6) and solving,

$$t(r;r_0,r_0^*) = T + \gamma\left(\frac{1}{\alpha}\right)^{\frac{1}{n}} \int_{r_0^*}^{r} \left(\frac{(1-\rho)(r_0^{*2}-r_0^2) - \rho(x^2-r_0^{*2})}{x}\right)^m dx$$

which can be written using Binomial theorem,

$$t(r;r_0,r_0^*) = T + \gamma\left(\frac{1}{\alpha}\right)^{\frac{1}{n}} (r_0^{*2}-r_0^2(\rho-1))^m \sum_{j=0}^{\infty}\binom{m}{j}(-1)^j \left(\frac{\rho}{r_0^{*2}-r_0(\rho-1)}\right)^j \left(\frac{r^{-m+2j+1} - r_0^{*\,-m+2j+1}}{-m+2j+1}\right) \quad (48)$$

Free Boundary of Domain D_{12}. In case B_2 part of the boundary of domain D_{12} is $t = t^o(r)$. This is obtained by eliminating r_o and r_o^* between Eqs. (45) and (46). Equating Eq. (47) to zero,

$$(1-\rho)(r_0^{*2}-r^2) - \rho(r^2-r_0^{*2}) = 0$$

Therefore,

$$\psi(r,r_0) = r_0^* = (\rho r^2+(1-\rho)r_0^2)^{\frac{1}{2}} \tag{49}$$

Substituting it into Eq. (45),

$$T = \gamma^* (\frac{1}{\alpha})^{\frac{1}{n}} \int_{r_0}^{\psi(r,r_0)} (\frac{x^2-r_0^2}{x})^m dx$$

whose series expansion is

$$T = \gamma^* (\frac{1}{\alpha})^{\frac{1}{n}} \sum_{j=0}^{\infty} \binom{m}{j} (-1)^j r_0^{2j} (\frac{(\psi(r,r_0))^{m-2j+1} - r_0^{m-2j+1}}{m-2j+1}) \tag{50}$$

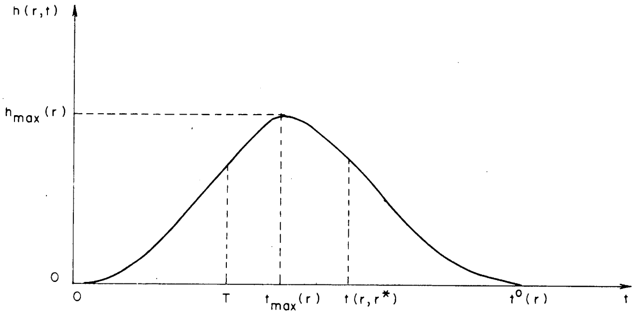
and into Eq. (48),

$$t^0(r) = T + \gamma(\frac{1}{\alpha})^{\frac{1}{n}} \rho^m \int_{\psi(r,r_0)}^{r} (\frac{r^2-x^2}{x})^m dx$$

It can be further written as

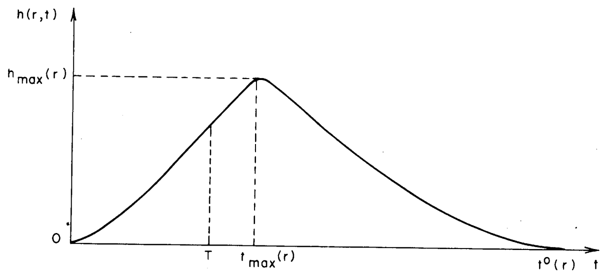
$$t^0(r) = T + \gamma(\frac{1}{\alpha})^{\frac{1}{n}} \rho^m r^{2m} \sum_{j=0}^{\infty} \binom{m}{j} (-1)^j r^{-2j} (\frac{r^{-m+2j+1} - (\psi(r))^{-m+2j+1}}{-m+2j+1}) \tag{51}$$

In Eqs. (50)-(51) $\bar{r} \leqslant r \leqslant R$; when $aR \leqslant r \leqslant \bar{r}$, $t = t^o(r)$ is defined by Eq. (34).
Hydrograph Characteristics. Here we only consider the case where q represents rainfall excess. As in the equilibrium case we consider h as a function of t for fixed r. In the partial equilibrium case this behavior of $h(r,t)$ is the same as in the equilibrium case for $aR \leqslant r \leqslant r^*$ as shown in Fig. 5. If $r^* < r \leqslant R$ then $h(r,t)$ is an increasing function of t if $(r,t) \varepsilon D_3$, and it is a decreasing function of t if $(r,t) \varepsilon D_{11}$. It remains to consider the behavior of $h(r,t)$ for fixed r when $(r,t) \varepsilon D_{12}$; the maximum of $h(r,t)$, for fixed r, will occur in D_{12}, possibly on $t = T$ or on $t = t(r,r^*)$. We omit the proof and refer to Singh and Agiralioglu (1980). It will suffice to point out that $\partial h(r;r_o^*(r_o),r_o)/\partial r_o < 0$ and $\partial t(r;r_o(r_o)r_o)/\partial r_o < 0$, and therefore, $h(r,t)$ is an increasing function of t for fixed r, $r^* \leqslant r \leqslant R$. The appearance of $h(r,t)$ in the partial equilibrium case will be as shown in Figs. 6-7. The maximum depth of flow and its time can then be specified as



(a) Case B_1 $r^* \leqslant r \leqslant R$
(b) Case B_2 $r^* \leqslant r \leqslant \bar{r}$

Fig. 6. Depth of flow, *h (r,t),* as a function of *t* for fixed *r*.



Case B_2 $\bar{r} < r \leqslant R$

Fig. 7. The depth of flow, *h (r,t),* as a function of *t* for fixed *r*.

$$h_{max}(r) = \left(\frac{q(r^2-r^{*2})}{2\alpha r} \right)^{\frac{1}{n}} \tag{52}$$

$$t_{max}(r) = T + \left(\frac{q(r^2-r^{*2})}{2} \right)^m \left(\frac{1}{\alpha} \right)^{\frac{1}{n}} \frac{1}{2n-1} \left(r^{2n-1/n} - r^{*(2n-1)/n} \right) \tag{53}$$

Time of Concentration, t^*. In the partial equilibrium case with q representing rainfall excess, t^* can be considered as the time of concentration. It is given by the intersection of $t = t(r,r^*)$ with $r = R$. It is thus seen that t^* is a function of T, and can be expressed as

$$t^* = T + \left(\frac{q(r^{*2}-a^2R^2)}{2} \right)^m \left(\frac{1}{\alpha} \right)^n \frac{1}{2n-1} \left(R^{2n-1/n} - r^{*(2n-1)/n} \right) \tag{54}$$

where r^* will be specified by

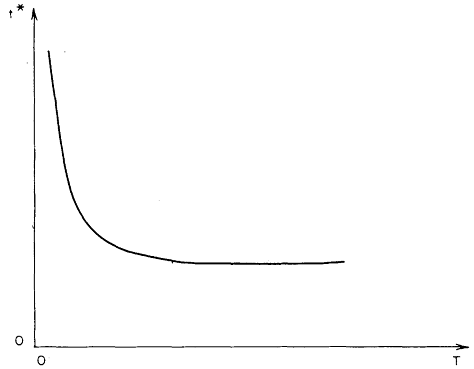


Fig. 8. t^* as a function of T, rainfall duration.

$$T = \frac{1}{n} \left(\frac{1}{\alpha} \right)^{\frac{1}{n}} \int_{aR}^{r^*} \left(\frac{x^2 - a^2 R^2}{x} \right)^m dx$$

Therefore,

$$T = \frac{1}{n} \left(\frac{1}{\alpha} \right)^{\frac{1}{n}} \left(\frac{q}{2} \right)^m \sum_{j=0}^{\infty} \binom{m}{j} (-1)^j (aR)^{2j} \left(\frac{r^{*\,m-2j+1} - (aR)^{m-2j+1}}{m-2j+1} \right) \tag{55}$$

If $aR = 0$, we get

$$r^* = \alpha T^n \left(\frac{q}{2} \right)^{n-1} \tag{56}$$

It is seen from Eq. (54) that $dt^*/dr^* < 0$. *Thus* $t^*(T)$ is a decreasing function of T as shown in Fig. 8. As $T \rightarrow 0$. $r^* \rightarrow aR$ and $t^* \rightarrow \infty$.

Criterion to Distinguish the Cases A, B_1 and B_2

The criterion to distinguish these cases is obtained from

$$T = \gamma^* \left(\frac{1}{\alpha} \right)^{\frac{1}{n}} \int_{aR}^{r} \left(\frac{x^2 - a^2 R^2}{x} \right)^m dx$$

which reduces to

$$T = \gamma^* \left(\frac{1}{\alpha} \right)^{\frac{1}{n}} \sum_{j=0}^{\infty} \binom{m}{j} (-1)^j (aR)^{2j} \left(\frac{r^{m-2j+1} - (aR)^{m-2j+1}}{m-2j+1} \right) \tag{57}$$

If $aR = 0$,

$$T = n\gamma^* \left(\frac{r}{\alpha} \right)^n \tag{58}$$

If Eq. (57) does not have a root between aR and r then we are in case A; if there is such a root r^* then we are in case B_1 and B_2. If we define $F(r)$ to be the right side of Eq. (33) then case A occurs if and only if $F(r) \leqslant T$, and cases B_1 and B_2 occur only if $F(r) > T$. To distinguish between cases B_1 and B_2 we note that, referring to Eq. (49),

$$(1-\rho)(r_0^{*2}-a^2R^2) - \rho(r^2-r^{*2}) = 0 \qquad (59)$$

does not have a root between aR and R in case B_1 but does have such a root $\bar{r}$ in case B_2. Let

$$F(r) = \left(\frac{r_0^{*2}}{\rho} - \left(\frac{1}{\rho} - 1\right)a^2R^2\right)^{\frac{1}{n}} \qquad (60)$$

then such a root exists only if $F(r) < R$, otherwise not. If this is true, we are in case B_2, otherwise in case B_1. The intersection of the curves $t = t(r,r^*)$ and $t^{\circ}(r)$ occurs in case B_2 at

$$\bar{r} = R - \left(\frac{r^{*2}}{\rho} - \left(\frac{1}{\rho} - 1\right)a^2R^2\right)^{\frac{1}{2}}$$

$$\bar{t} = T + \gamma\left(\frac{1}{\alpha}\right)^{\frac{1}{n}} \int_{r^*}^{\bar{r}} \left(\frac{\bar{r}^2 - x^2}{x}\right)^m dx$$

This can be simplified to

$$\bar{t} = T + \gamma\left(\frac{1}{\alpha}\right)^{\frac{1}{n}} (\bar{r})^{-2m} \sum_{j=0}^{\infty} \binom{m}{j} (-1)^j \bar{r}^{-2j} \left(\frac{\bar{r}^{-m+2j+1} - r^{*-m+2j+1}}{-m+2j+1}\right) \qquad (61)$$

Comparison with Converging Flow

In the introduction some geometric differences between converging and diverging surfaces were noted. These differences are reflected in their responses to rainfall and infiltration. To compare the experimental data of Langford and Turner (1973), which were observed on a rectangular plane, 4.572 m wide and, 22.86 m long, was used. It has impervious bitumen paved surface with a slope $S_0 = 0.01$. The rainfall intensity was 0.06 m/hr. and equilibrium discharge of 1,742 cm^3/sec. Utilizing Darcy-Weisbach relation α and n were estimated to be 2.16 and 1.70. For this data diverging and converging flow hydrographs are shown in Fig. 9 for the same resistance coefficient, taking $a = 0$ and convergence parameter as 0.01. It is evident that the converging flow model gives shorter time of concentration than does the diverging flow model. The diverging model recedes slower than does the converging model. These differences are compatible with their geometric differences.

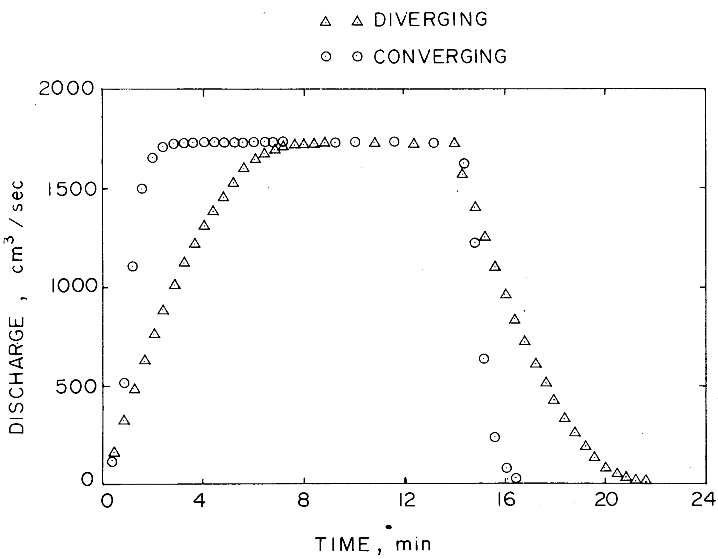


Fig. 9. Comparison of diverging and converging overland flow models.

Conclusions

Explicit, analytical solutions have been derived for diverging overland flow with constant rainfall, infiltration and friction parameter. These solutions can be derived by including spatial variability with only a slight increase in mathematical complexity, but the essential features of the solution are not changed. The solutions demonstrate that simultaneous consideration of rainfall and infiltration phases of the hydrologic cycle alters the character of overland flow hydrograph.

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