Note on the Modified Anomaly Equation of Nielsen and Schroer in Arbitrary Even Dimensions

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(Received February 3, 1989)

Applying the stochastic quantization method to massless, Euclidean Dirac theory formulated on a hypersphere $S^{2n}$, the Nielsen-Schroer modified anomaly equation is derived. It is pointed out that the zero-modes contribution precisely cancels the anomaly term for the previously considered vortex-like ($D=2$) and instanton ($D=4$) models, leaving the conserved axial-vector current.

1. The anomaly equation expresses an explicit breakdown of the chiral symmetry due to the presence of anomalies at one-loop level. If zero-modes also exist for massless fermions coupled to an external gauge field with an asymptotically "flat" (pure gauge) configuration, it takes the modified form of Nielsen and Schroer provided the divergence of the axial-vector current does not produce the so-called surface term. The absence of the latter is guaranteed in the hyperspherical formulation which introduces a subtlety in the normalization condition. The purpose of this note is to show that the modified anomaly equation is obtained by applying the stochastic quantization method with an infrared cutoff to massless, Euclidean Dirac theory in the hyperspherical formulation. A possibility of cancellation between the Nielsen-Schroer zero-modes contribution and the anomaly term is also pointed out.

2. We will first prove that the divergence of the axial-vector current produces no surface term if the externally-imposed gauge field approaches a pure gauge at infinity (asymptotically "flat") in Euclidean space with even dimensions $D=2n$ so that one-point compactification $R^{2n} \rightarrow S^{2n}$ is allowed. The proof is essentially the same as that in Ref. 2) but simpler than that and worth repeating here.

The hyperspherical formulation is achieved by projecting Euclidean space $R^{2n}$ onto a hypersphere $S^{2n}$ embedded in $R^{2n+1}$ through

\[ x_\mu = \kappa^{-1} r_\mu, \quad \kappa = 1 + r_{2n+1} = \frac{2}{1+x^2}, \]

\[ r_{2n+1} = \frac{1-x^2}{1+x^2}, \quad r^2 = r_1^2 + \cdots + r_{2n+1}^2 = 1, \]

where $x = (x_\mu) \in R^{2n}, \mu = 1, \cdots, 2n$, and $r = (r_a) \in S^{2n}, a = 1, \cdots, 2n+1$. Let $\int d\Omega$ be the integration over $S^{2n}$. Calculation of the Jacobian factor of the transformation (1) gives

\[ d\Omega = \kappa^{2n} d^{2n}x, \quad d^{2n}x = dx_1 \cdots dx_{2n}. \]

The angular momentum operator on $S^{2n}$

\[ l_{ab} = -i(r_a \tilde{\partial}_b - r_b \tilde{\partial}_a), \quad \tilde{\partial}_a = \partial/\partial r_a, \]
has components
\[ l_{\mu} = -i(x_\mu \partial_\nu - x_\nu \partial_\mu), \]
\[ l_{2n+1, \mu} = -i \left( \frac{1-x^2}{2} \partial_\mu + x_\mu x \cdot \partial \right), \quad \partial_\mu = \frac{\partial}{\partial x_\mu}. \] (4)

The massless Dirac action
\[ S = \int d^{2n+1}x \psi^\dagger \alpha \cdot D \psi, \] (5)
where \( \alpha \cdot D = \alpha_\mu D_\mu \) with \( D_\mu = \partial_\mu + A_\mu \), \( A_\mu \) denoting the prescribed external gauge field, and \( \dagger \) denotes the Hermitian conjugate, is converted into the hyperspherical one as follows. The Euclidean Dirac spinor \( \psi \) is transformed into
\[ \tilde{\psi}(r) = \kappa^{-1} + i\alpha \cdot x / \sqrt{2} \psi(x) \equiv \mathcal{P} \psi(x), \] (6)
and \( \tilde{\psi}^\dagger(r) = \psi^\dagger(x) \mathcal{P} \) with \( \alpha \cdot x = \alpha_\mu x_\mu \). The gauge field \((2n+1)\)-vector \( \tilde{A}_a \) with the constraint \( r \cdot \tilde{A} \equiv r_a \tilde{A}_a = 0 \) is defined on \( S^{2n} \) and related to \( A_\mu \) through \( \tilde{A}_a = \kappa^{-1} A_\mu - x \mu A_\mu, \tilde{A}_{2n+1} = -x \cdot A, x \cdot A \equiv x_\mu A_\mu \). Let \( \Gamma_\mu = \alpha_\mu \alpha_{2n+1}, \Gamma_{2n+1} = \alpha_{2n+1} = (-i)^a \alpha_1 \cdots \alpha_{2n}, \) and put
\[ j_{5n} = \frac{i}{2} \tilde{\psi}^\dagger [\Gamma \cdot r, \Gamma_a] \Gamma \cdot r \tilde{\psi}, \] (7a)
\[ j_{2n+1}^5 = \tilde{\psi}^\dagger \Gamma \cdot r \tilde{\psi}, \quad \Gamma \cdot r = \Gamma_a r_a. \] (7b)

It then follows from Eq. (6) that
\[ \kappa^{-2n+1} j_{5n} = j_{5\mu} - x_\mu j_{5n+1}^5, \quad \kappa^{-2n+1} j_{2n+1}^5 = j_{2n+1}^5, \] (8)
where \( j_{5\mu} = \psi^\dagger \alpha_{2n+1} A_\mu \psi \) and \( j_{2n+1} = \psi^\dagger \alpha_{2n+1} \psi \). Introducing the spin matrix \( S_{ab} = 1/4i [\Gamma_a, \Gamma_b] \), and the covariant angular momentum operator \( L_{ab} = l_{ab} |_{\delta_a - \delta_a + \delta_a} \), we can prove the identities
\[ P^\dagger (S \cdot L + n) P = \kappa^{2n} i \alpha \cdot D, \] (9a)
\[ P^\dagger (S \cdot L + n) P = i \alpha \cdot D \kappa^{-2n}, \] (9b)
\[ (S \cdot L + n) \Gamma \cdot r = -\Gamma \cdot r (S \cdot L + n), \] (9c)
where \( S \cdot L = S_{ab} L_{ab}, \bar{L} = L |_{\delta_a - \delta_a - \bar{\delta}_a} \) and \( \bar{D} = D |_{\delta_a - \bar{\delta}_a} \). Thus Eq. (5) is rewritten as
\[ S = -i \int d\Omega \tilde{\psi}^\dagger (S \cdot L + n) \tilde{\psi}, \] (10)
which explicitly displays \( O(2n+1) \) symmetry generated by \( l_{ab} + S_{ab} \). It also enjoys

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*) We choose Euclidean Dirac matrices to be Hermitian and satisfy \( (\alpha_\mu, \alpha_\nu) = 2 \delta_{\mu \nu}, \mu, \nu = 1, \ldots, 2n \). In the minimal dimensional representation, \( \alpha_\mu \) are matrices of dimensions \( 2^n \).

**) We shall follow the notations of Ref. 2) and Jackiw and Rebbi, Ref. 3), extending 2 and 4 dimensions to any even dimensions.
the chiral symmetry. The chiral transformation\(^{*}\) is given by
\[
\tilde{\phi} \rightarrow \tilde{\phi}' = e^{i\tilde{\beta} r} \tilde{\phi}, \quad \tilde{\phi}^\dagger \rightarrow \tilde{\phi}'^\dagger = e^{i\tilde{\beta} r} \tilde{\phi}^\dagger,
\]
where \(\tilde{\beta}\) is real. The chiral invariance is checked by Eq. (9c). The associated Noether current turns out to be given by Eq. (7a). Upon quantization it obeys the anomaly equation
\[
\langle \tilde{\partial}_a J_{5a}(r) \rangle = \tilde{G}_{2n}(r), \tag{12a}
\]
where \(\langle \cdot \rangle\) represents the vacuum expectation value in the background gauge field and
\[
\tilde{G}_{2n}(r) = \langle \tilde{\phi}'^\dagger(r) i[\Gamma \cdot r (S \cdot L + n) + (S \cdot \bar{L} + n) \Gamma \cdot r] \tilde{\phi}(r) \rangle. \tag{12b}
\]
This form will be used later on.

In order to translate Eq. (12) into Euclidean-space equation we observe that
\[
\tilde{\partial}_a J_{5a}(r) = \kappa^{-2n} (\partial_{\mu} a_{5\mu}(x)), \tag{13}
\]
which can be derived on the basis of the identities (9a)\(\sim\)(9c) and \((1 - i\alpha \cdot x) \Gamma \cdot r (1 + i \alpha \cdot x) = (1 + x^2) a_{2n+1} \). Hence we have
\[
\int d^2 x \langle \tilde{\partial}_5 J_{5}(x) \rangle = \int_{S^2} d\Omega \langle \tilde{\partial}_a J_{5a}(r) \rangle = 0, \tag{14}
\]
where the last equality follows from the fact \(\partial S^{2n} = 0\). This is what we promised to prove. Consequently, we obtain
\[
\int d\Omega \tilde{G}_{2n}(r) = \int d^2 x G_{2n}(x) = 0, \tag{15}
\]
where we have put \(G_{2n}(x) = \kappa^{2n} \tilde{G}_{2n}(r)\). As we shall see below, \(G_{2n}\) consists of two terms in the presence of zero modes, one being the familiar anomaly which is perturbative and the other the Nielsen-Schroer zero-modes contribution. Equation (15) is, therefore, equivalent to the well-known index theorem.

3. To compute the vacuum expectation value \(\tilde{G}_{2n}\) in the background gauge field, we have to quantize the theory with the action (10) regarding \(A\) as external. This we do by means of the stochastic quantization method of Parisi and Wu\(^{4,5}\). It is based on the Langevin equations for random, Grassmann spinor fields \(\tilde{\phi}(r, \tau)\) and \(\tilde{\phi}'^\dagger(r, \tau)\),
\[
\dot{\tilde{\phi}}(r, \tau) = i(S \cdot L + n + i\mu) \tilde{\phi}(r, \tau) + \tilde{\eta}(r, \tau), \tag{16a}\]
\[
\dot{\tilde{\phi}}^\dagger(r, \tau) = \tilde{\phi}^\dagger(r, \tau) i(S \cdot L + n + i\mu) + \tilde{\eta}^\dagger(r, \tau), \tag{16b}\]
where the dot stands for the derivative with respect to the extra time \(\tau\), \(\mu > 0\) is an infrared cutoff parameter, and \(\tilde{\eta}\) and \(\tilde{\eta}^\dagger\) are the Gaussian, Grassmann noise fields with \(\langle \tilde{\eta} \rangle_{\tilde{\eta}} = \langle \tilde{\eta}^\dagger \rangle_{\tilde{\eta}^\dagger} = 0\) and
\[
\langle \tilde{\eta}(r, \tau) \tilde{\eta}^\dagger(r', \tau') \rangle_{\tilde{\eta}} = 2\delta(\Omega - \Omega') \delta(\tau - \tau'). \tag{16c}\]

\(^{*}\) This is suggested by R. Jackiw and C. Rebbi, Phys. Rev. D14 (1976), 517.

\(^{**}\) A dimensionful constant necessary to balance the dimensions of both sides are set 1.
The fundamental relation in the Parisi-Wu method is
\[
\lim_{\tau \to \infty} \langle \hat{\phi}(r_1, \tau) \cdots \hat{\phi}(r_N, \tau) \hat{\phi}^\dagger(r_{N+1}, \tau) \cdots \hat{\phi}^\dagger(r_{2N}, \tau) \rangle_q \\
= \langle \hat{\phi}(r_1) \cdots \hat{\phi}(r_N) \hat{\phi}^\dagger(r_{N+1}) \cdots \hat{\phi}^\dagger(r_{2N}) \rangle_q,
\]
where the $\mu \to 0+$ limit should be taken after $\tau \to \infty$. According to Eq. (12b), we find that
\[
\tilde{G}_{2n}(r) = \lim_{\mu \to 0+} \lim_{\tau \to \infty} \langle 2\mu J_{2n+1}(r, \tau) \rangle_q \\
+ \langle \hat{\phi}^\dagger(r, \tau) [\hat{\Gamma} \cdot r i(S \cdot L + n + i\mu) + i(S \cdot L + n + i\mu) \hat{\Gamma} \cdot r] \hat{\phi}(r, \tau) \rangle_q,
\]
which becomes using the Langevin equations (16a, b) and noting the vanishing of the $\tau \to \infty$ limit of the $\hat{\eta}$-average including $\hat{\phi}$ and $\hat{\phi}^\dagger$.

\[
\tilde{B}_{2n}(r) = \lim_{\mu \to 0+} \lim_{\tau \to \infty} \langle 2\mu J_{2n+1}(r, \tau) \rangle_q, \quad \text{and} \quad \tilde{A}_{2n}(r) = \lim_{\mu \to 0+} \lim_{\tau \to \infty} \text{tr} \langle \hat{\phi}^\dagger(r, \tau) + \hat{\phi}(r, \tau) \hat{\eta}^\dagger(r, \tau) \rangle_q,
\]
tr denoting the trace over Dirac and/or internal symmetry matrices.

The next step consists of evaluating the correlation functions in Eq. (18). This is done by solving Eqs. (16a, b) in terms of Green's functions $\tilde{G}$ and $\tilde{G}^\dagger$ which obey
\[
\tilde{G}(r, r'; \tau, \tau') = K \tilde{G}(r, r'; \tau, \tau') + \delta(\Omega' - \Omega') \delta(\tau - \tau'),
\]
\[
\tilde{G}^\dagger(r', r; \tau, \tau') = K^\dagger \tilde{G}^\dagger(r', r; \tau, \tau') + \delta(\Omega' - \Omega') \delta(\tau - \tau'),
\]
where $K = i(S \cdot L + n + i\mu)$ and $\tilde{G} = \tilde{G}^\dagger = 0$ for $\tau < \tau'$. The solutions to Eqs. (19a, b) are given by
\[
\tilde{G}(r, r'; \tau, \tau') = \theta(\tau - \tau') \sum_k \tilde{u}_k(r) \tilde{u}_k^\dagger(r') e^{i(\Omega_k - \mu)(\tau - \tau')},
\]
\[
\tilde{G}^\dagger(r', r; \tau, \tau') = \theta(\tau - \tau') \sum_k \tilde{u}_k(r') \tilde{u}_k^\dagger(r) e^{i(\Omega_k - \mu)(\tau - \tau')},
\]
where
\[
(S \cdot L + n) \tilde{u}_k(r) = \lambda_k \tilde{u}_k(r), \quad \tilde{u}_k^\dagger(r) (S \cdot L + n) = \lambda_k \tilde{u}_k^\dagger(r),
\]
with $\lambda_k$ real and
\[
\int d\Omega \tilde{u}_k^\dagger(r) \tilde{u}_k(r) = \delta_{kl}, \quad \sum_k \tilde{u}_k(r) \tilde{u}_k^\dagger(r') = \delta(\Omega - \Omega').
\]
Substituting $\hat{\phi} = \tilde{G} \hat{\eta}$ and $\hat{\phi}^\dagger = \hat{\eta}^\dagger \tilde{G}^\dagger$, where the product is understood to include $\Omega$- and $\tau$-integrals, into Eq. (18) and making use of Eq. (16c) lead us to

*) For infrared singular Green's functions we need make a subtraction. The anomaly (12b) turns out, however, to be infrared regular. This is well known by the statement that the anomaly is mass-independent.

**) The initial conditions are chosen to be $\hat{\phi}(r, 0) = \hat{\phi}^\dagger(r, 0) = 0$. 

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July 1989  Progress Letters  43
The zero modes, if any, disappear from $G_{2n}$ in agreement with Eq. (15) provided the integration and the summation can be exchanged since

$$\int d\Omega \bar{u}_k(r) \Gamma \cdot r u_k(r) = 0 \quad \text{for} \quad \lambda_k \neq 0.$$  \hspace{1cm} (23)

To finally show that $B_{2n}$ represents the Nielsen-Schroer zero-modes contribution, while $\tilde{A}_{2n}$ is the familiar anomaly, we define following Ref. 2) $u_k(x) = \Gamma^{-1} \bar{u}_k(r)$, which satisfies the eigenvalue equations

$$i\alpha \cdot Du_k(x) = \lambda_k \kappa u_k(x), \quad u_k^\dagger(x) i\alpha \cdot \bar{D} = \lambda_k \kappa u_k^\dagger(x),$$  \hspace{1cm} (24a)

and the orthonormality conditions

$$\int d^{2n}x u_k^\dagger(x) u_l(x) = \delta_{kl}, \quad \sum_k u_k(x) u_k^\dagger(x') = \kappa^{-1} \delta(x-x'),$$  \hspace{1cm} (24b)

where we have used $\delta(Q-Q') = \kappa^{-2n} \delta(x-x')$. Equation (12) then becomes in the Euclidean $x$-space

$$\langle \partial^\mu j_\mu(x) \rangle = -2 \frac{2}{1+x^2} \sum_{l=0}^{2n} u_0^\dagger(x) a_{2n+1} u_0(x) + A_{2n}(x),$$  \hspace{1cm} (25*)

where the zero-modes contribution made explicit comes from $B_{2n}$, whereas

$$A_{2n}(x) = \kappa^{2n} \tilde{A}_{2n}(r) = 2 \sum_k \kappa u_k^\dagger(x) a_{2n+1} u_k(x)$$

$$= \lim_{\lambda_k \to 0} 2 \text{tr} \left[ a_{2n+1} \sum_k \kappa u_k(x) u_k^\dagger(x') \right]$$

$$= \lim_{\lambda_k \to 0} 2 \text{tr} \left[ a_{2n+1} \delta(x-x') \right]$$

$$= \lim_{\lambda_k \to 0} 2 \text{tr} \left[ a_{2n+1} f \left( \frac{(a \cdot D)^2}{\Lambda^2} \right) \delta(x-x') \right]$$  \hspace{1cm} (26)

is Fujikawa's formula of the anomaly with $f(x)$ Fujikawa's regularization function. Equation (25) is the modified anomaly equation of Nielsen and Schroer\textsuperscript{23} who subtracted off the zero modes from the propagator

$$\langle \hat{\phi}(r) \hat{\phi}^\dagger(r') \rangle = \lim_{\tau \to -\infty} \langle \hat{\phi}(r, \tau) \hat{\phi}^\dagger(r', \tau) \rangle_q = \sum_k \frac{\bar{u}_k(r) u_k^\dagger(r')}{\mu - i\lambda_k}.$$ 

*) We have also verified Eq. (25) by means of Fujikawa's path-integral method which greatly simplifies the calculation but seems to depend on a specification of the fermion measure. The stochastic quantization method, while equivalent to the path-integral one through Eq. (17), dispenses with the definition of the fermion measure, which suggests that anomalies are independent of the choice of the fermion measure in Fujikawa's method.
which is infrared singular as $\mu \to 0$, in the presence of zero modes. In our derivation the infrared singularity is canceled in $\bar{B}_{2n} = \lim_{\mu \to 0, 2\mu < f_{2n+1}^0(r)}$.

4. To conclude this paper we would like to point out a possibility of cancellation between the non-perturbative and perturbative anomalies in Eq. (25), where by the non-perturbative anomaly we mean the zero-modes contribution. Recall that $G_2 = (1/2\pi)\varepsilon_{\mu\nu}F_{\mu\nu}$, for $D=2$ Abelian gauge theory, while $G_4 = -(1/16\pi^2)\varepsilon_{\mu\nu\rho\sigma}trF_{\mu\nu}F_{\rho\sigma}$, for $D=4$ non-Abelian gauge theory. The vortex-like ($D=2$) model yields

$$G_2(x) = \frac{2}{\pi} \frac{\nu}{(1+x^2)^{\nu}}, \quad (27)$$

where $\nu$ is the winding number. In this model with massless fermions, there exist $|\nu|$ independent zero-energy solutions which turn out to give

$$2 \frac{2}{1+x^2} \sum_{\nu=\pm 1} \gamma_0^{(\nu)}(x)\gamma_0^{(\nu)}(x) = \frac{2}{\pi} \frac{\nu}{(1+x^2)^{\nu}}, \quad (28)$$

which precisely equals Eq. (27). On the other hand, the $\nu = \pm 1$ instanton ($D=4$) model with the size parameter $\rho = 1$ leads to

$$G_4(x) = \pm \frac{12}{\pi^2} \frac{1}{(1+x^2)^2}, \quad (29)$$

and allows for massless fermions a single zero mode\(^3\) with

$$2 \frac{2}{1+x^2} u_0^*(x)\gamma_0 u_0(x) = \pm \frac{12}{\pi^2} \frac{1}{(1+x^2)^2}, \quad (30)$$

which this time equals Eq. (29). The cancellation between the two terms in Eq. (25) really happens in these two models, leaving the conserved axial-vector current $\langle \partial \gamma_3 \rangle = 0$. This circumstance is entirely beyond perturbation theory.

Such a conspiracy would not happen if the extra factor $2/(1+x^2)$ in the non-perturbative anomaly were absent. That factor makes a natural appearance, however, in the hyperspherical formulation which is motivated\(^3\) by the fact that the index theorem demands the manifold to be compact.

The author is grateful to S. Aramaki, S. Fukuzumi and H. Kase for useful discussions.

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