Stability of the Friedmann Universe in the Poincaré Gauge Theory

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In the Poincaré Gauge Theory, solutions of the Friedmann universe which is stable against quantum corrections are obtained under linear approximations. These solutions are small oscillations around the standard Big-Bang solution of the General Relativity (GR). In GR, quantum effects due to vacuum polarization make the universe unstable. However, in these solutions we can choose parameters so that quantum effects do not break the stability.

§ 1. Introduction

Recently it has been clarified that quantum effects of matter fields play important roles in General Relativistic (GR) cosmologies of the early universe such as the origin of fluctuations,\(^1\) the inflationary universe,\(^2\) the damping of anisotropies,\(^3\) the matter generations by particle productions\(^4\) and so on.

Among them, there is the problem of instability of the universe.\(^5,6\) That is, if we add quantum effects due to vacuum polarization of quantized matter fields\(^7\) to the right-hand side (RHS) of the Einstein equation (semi-classical picture of GR)

\[ R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G (T^\text{cl}_{\mu\nu} + \langle 0 | T_{\mu\nu} | 0 \rangle), \quad (1.1) \]

then the Friedmann universe\(^5\) and the Minkowski spacetime\(^6\) become unstable. Here the first and second terms on the RHS are the classical and quantum parts of the energy-momentum tensor, respectively. This instability causes serious difficulties in cosmology as will be reviewed in Appendix A. One way to avoid these difficulties is to solve the problem according to the exact theory of quantum gravity. But at present we have no such theory; moreover, it is not clear whether such a theory indeed solves the stability problem. In this paper an alternative way is investigated; that is, we treat the stability problem within the semi-classical picture of the Poincaré Gauge Theory (PGT). (The universe is also unstable in semi-classical picture of the New General Relativity (NGR).\(^9\))

The Poincaré Gauge Theory\(^9\) is a gauge theory for extended gravity. Its gauge group is \(T \otimes L_{\text{internal}}\), where \(T\) is the translational gauge group and \(L_{\text{internal}}\) is the internal Lorentz gauge group. It contains GR and NGR as its special cases. The underlying spacetime manifold of this gravitational theory is the Riemann-Cartan spacetime characterized by curvature and torsion. The torsion couples with the intrinsic spin of matter. Because spinor fields are representations of the Lorentz group, they can easily be introduced into this theory.

Quantum effects due to vacuum polarization in PGT are investigated in Ref. 10): It is shown therein under the assumption of asymptotic freedom and multiplicative
renormalizability, that at high temperature the theory is asymptotically conformally invariant and that particles become massless. These results are then used to obtain the expression for \( \langle 0 | T_{\mu \nu} | 0 \rangle \).

This paper is based on this semi-classical picture of PGT, and is organized as follows. In § 2, we summarize classical equations of PGT for the scale parameter \( A(t) \) and the torsion \( S(t) \) for the homogeneous and isotropic universe. In § 3, under linear approximation of this equation for \( A(t) \), it is shown for the flat case \( k=0 \) and the radiation dominant universe (RDU) that there is a classical, stable solution of the universe if the parameters of PGT are chosen appropriately. This solution is a small oscillation around a minimum point with period of \( \sim 10^2 t_p \) (\( t_p \) is the Planck time), and corresponds to a small, oscillating deviation from the standard Big-Bang solution (SBBS) of GR. In § 4, conditions for the parameters of PGT are given, under which quantum effects of matter fields due to vacuum polarization do not lead to instability of the classical solution. In § 5, the case of the matter dominant universe (MDU) is considered in the same manner. The last section is dedicated to a summary and discussion. In Appendix A, we show the instability of the universe and the difficulties of cosmology in GR with quantum corrections. In Appendix B, we survey the Lagrangian and the free parameters in PGT.

§ 2. The classical equation of the Friedmann universe in PGT

In this section, we summarize the main results of Refs. 11) and 12) on the Friedmann universe in PGT; the former obtained nonezero components of torsion fields, while the latter derived the equation for the scale parameter and obtained a nonsingular bounce solution of the universe.

We consider a homogeneous and isotropic space with the metric

\[
ds^2 = -dt^2 + \frac{A^2(t)}{1 + \frac{k}{4} r^2} (dx^2 + dy^2 + dz^2),
\]

where \( A(t) \) is the scale parameter, \( r^2 = x^2 + y^2 + z^2 \) and \( k = 0, 1, -1 \) corresponding to the flat, closed, and open universe respectively. (Throughout this paper, we take the natural unit where \( c = \hbar = 1 \).) It is shown\(^{11}) \) that under the condition of homogeneity, isotropy and parity conservation only the following components of the torsion tensor remain nonvanishing:

\[
\begin{align*}
T_{10} & = T_{20} = T_{30} = S(t) \neq 0, \\
other components & = 0.
\end{align*}
\]

Here, \( 0 \) and \( i(=1, 2, 3) \) are the indices for time and space components, respectively.

The equation for the unknown function \( A(t) \) is\(^{12}) \)

\[
k + \left( \frac{\dot{A}}{3B} \right)^2 = \frac{\rho_0 + \frac{1}{3} f F^2 - 9 \beta A^{-2} (A^2 + k)}{6B},
\]

where \( F \) is the scalar curvature in the presence of torsion,
\[ F = \frac{1}{2b}(\rho - 3p - 18\beta A^{-2}(k + \dot{A}^2 + A\ddot{A})) \]  (2.4)

Here \( \rho \) and \( p \) are the classical energy density and pressure, respectively, and \( B \) is given by

\[ B = b + \frac{2}{3} fF \]  (2.5)

Because \( F \) contains \( \dot{A} \), the classical equation (2.3) is a third order differential equation for \( A \). The constants \( f \), \( b \) and \( a \) are given by

\[ f = \frac{1}{4} (a_s + 12a_6) \]  (2.6)

\[ b = \frac{a - \frac{3}{2} \beta}{\frac{1}{16\pi G}} \]  (2.7)

where \( a_s \), \( a_6 \) and \( \beta \) are three of the nine parameters of PGT. (See Appendix B, where we summarize the Lagrangian, free parameters in PGT and the derivations of (2.3) and (2.4).) The constant \( G(=m_p^{-2} \text{ with } m_p \text{ being the Planck mass } \sim 10^{19} \text{ GeV}) \) is the gravitational constant. Because the case \( f = 0 \) reduces to GR, we set \( f \neq 0 \). Because the case \( \beta = 0 \) leads to nonpropagating torsion, we take \( \beta \neq 0 \). Further the condition of the positive-definite energy and positive mass demands \( f > 0 \). Consequently we shall assume that

\[ f > 0 \]  (2.9)

and

\[ \beta \neq 0 \]  (2.10)

The torsion field is then given by

\[ S = -\frac{fF}{3B} \]  (2.11)

§ 3. The classical stable solution of the Friedmann universe in PGT

At the early universe the cases \( k = \pm 1 \) are effectively the same as the flat case \( k = 0 \). So we shall treat only the RDU with \( k = 0 \), in which case SBBS of GR is

\[ A(t) = \sqrt{2u_0 t} \]  (3.1)

where we define

\[ u_0 = \left( \frac{D_r}{6a} \right)^{1/2} \]  (3.2)

The constant \( D_r \) is related to the energy density of RDU by
\[ \rho \text{cl} = \frac{D_{r}}{A^4} \quad (3.3) \]

and its present value is estimated to be about \(10^{112}\) from observation.\(^2\) The value of \(D_{r}\) before the phase transition of GUT, however, is considered to be much smaller,\(^3\) since the phase transition can change the total entropy of the universe which is given by \(\sim D_{r}^{3/4}\).

Here we introduce a new function \(x = A \bar{A}\) that is equal to \(u_{0}\) when \(A\) is given by (3.1). The equation (2·3) is then transformed into

\[ \ddot{x} = \frac{b^2}{3f\beta} \left(x \pm \frac{1}{\sqrt{6b}} \left(D_{r} - 9\beta x^2 + \frac{27f\beta^2}{b^3} \cdot \dot{x}^2 \right)^{1/2} \right) \left(1 - \frac{6f\beta}{b^2 A^2} \cdot \dot{x} \right)^{1/2} \quad (3.4) \]

Since it is difficult to solve (3.4) exactly, we restrict ourselves to the case in which the \(\ddot{x}\)-terms are much smaller than the remaining terms: Consistency of this approximation will be justified later (see below (3.24)). The equation (3.4) then becomes

\[ \ddot{x} = \frac{b^2}{3f\beta} \left(x \pm \frac{1}{\sqrt{6b}} \left(D_{r} - 9\beta x^2 \right)^{1/2} \right) \frac{9\beta}{2\sqrt{6a}(D_{r} - 9\beta x^2)^{1/2}} \cdot \dot{x}^2 + \frac{(D_{r} - 9\beta x^2)^{1/2}}{\sqrt{6b} A^2} \cdot \dot{x} \quad (3.5) \]

This equation is interpreted as that for a particle moving in a potential \(V(x)\) with additional \(\dot{x}\)-dependent forces. The first term on the RHS of (3.5) expresses \(-dV(x)/dx\): We shall choose the minus sign in this term, because the equilibrium point \((dV(x)/dx = 0)\) then occurs at \(x = u_{0}\) which corresponds to (3.1).

To stabilize the universe, the point \(x = u_{0}\) must be a minimum point of the potential. So we demand \(f\beta < 0\). Combining this with (2·9), we get

\[ \beta < 0 \quad (3.6) \]

which ensures that the arguments in the square roots in (3.5) are positive.

Let us seek a small oscillative solution around \(u_{0}\). Putting

\[ x = A \bar{A} = u_{0} + \varepsilon, \quad (u_{0} \gg \varepsilon) \quad (3.7) \]

we get the linear approximation of (3.5)

\[ \ddot{\varepsilon} = \frac{ab}{3f\beta} \cdot \varepsilon + \frac{1}{2t} \cdot \dot{\varepsilon} + O \left(\frac{\varepsilon^2}{u_{0}^2}, \frac{\dot{\varepsilon}^2}{u_{0}^2}, \frac{\varepsilon \dot{\varepsilon}}{u_{0}^2} \right), \quad (3.8) \]

where we have used

\[ A^2(t) = 2 \int_{0}^{t} (u_{0} + \varepsilon) dt = 2u_{0} t + 2\int_{0}^{t} \varepsilon dt \quad (3.9) \]

If the second and third terms on the RHS of (3.8) can be ignored, \(\varepsilon\) is given by

\[ \varepsilon = \eta_{0} \cos \omega t \quad (\eta_{0} \ll u_{0}) \quad (3.10) \]

with
\[ x^2 = \frac{ab}{3f(-\beta)}. \]  

(3.11)

This is an approximate solution of (3.8) if the second term on the RHS is negligibly small compared to the first one, that is, if

\[ \left| \frac{ab}{3f\beta} \cdot \varepsilon \right| \gg \left| \frac{1}{2t} \cdot \dot{\varepsilon} \right| \]  

(3.12)

or equivalently

\[ xt \gg 1. \]  

(3.13)

Since \( x \) is smaller than \( \sim m_p/10^2 \) (see (4.18) and (4.20)), the condition (3.13) becomes

\[ t \gg \frac{10^2}{m_p} = 10^2 t_p, \]  

(3.14)

where \( t_p \) (\( \sim 10^{-44} \) sec) is the Planck time.

Putting (3.10) into (3.9), we get the following expression for the scale parameter:

\[ A(t) = \sqrt{2u_0 t} \left( 1 + \frac{1}{2} \cdot \frac{\eta_0}{u_0} \cdot \frac{1}{xt} \sin xt \right) \]  

(3.15)

which describes a small oscillation around SBBS of GR. Let us call this as the "trembling universe" (see § 6).

Let us prepare some relations for § 4. For later stages of the universe satisfying

\[ xt \gg \frac{u_0}{\eta_0}, \]  

(3.16a)

namely,

\[ t \gg 10^2 \frac{u_0}{\eta_0} t_p, \]  

(3.16b)

we can show

\[ AA' \approx -\eta_0 x \sin xt = \dot{\varepsilon}, \]  

(3.17)

\[ AA^{(3)} \approx -\eta_0 x^2 \cos xt = \ddot{\varepsilon}, \]  

(3.18)

\[ AA^{(4)} \approx \eta_0 x^3 \sin xt, \]  

(3.19)

\[ AA^{(5)} \approx \eta_0 x^4 \cos xt, \]  

(3.20)

where \( A^{(i)} \) (\( i = 3, 4, 5 \)) is the \( i \)-th time-derivative of \( A(t) \). To get (3.17), for example, we differentiate \( AA = u_0 + \eta_0 \cos xt \), getting

\[ AA' = -\eta_0 x \sin xt - A^2. \]  

(3.21)

Under the condition (3.16), we can show that the first oscillating term on the RHS of (3.21) is much larger than the second term. The three other relations (3.18) \( \sim \) (3.20) are obtained similarly.

In the era where \( xt \ll u_0/\eta_0 \), we cannot obtain the relations such as (3.17) \( \sim \) (3.20),
and then cannot estimate the quantum corrections (see § 4). So, in the following, we shall restrict ourselves to the era satisfying (3.16).

Then, the scalar curvature and its time-derivative, \( F \) and \( \dot{F} \), can be expressed by

\[
F = -\frac{9\beta}{b} \left( \frac{\dot{A}^2 + A\ddot{A}}{A^2} \right) \approx -\frac{9\beta}{b} \frac{\dot{A}}{A},
\]

\[
\dot{F} = -\frac{9\beta}{b} \left( \frac{\dot{A}^2 + A\ddot{A}}{A^2} \right) \approx -\frac{9\beta}{b} \frac{\ddot{A}}{A^2} \approx -\frac{9\beta}{b} A^{(3)},
\]

where we have used (3.17) and (3.18). (Note that in RDU, \( \rho_a = 3\rho_a \).) The use of (3.23) in (2.11) gives the following expression for the torsion:

\[
S(t) \approx \frac{a}{\beta} \frac{\eta_n}{u_0} \frac{\cos xt}{t}.
\]

Using (3.1), (3.16) and (3.17), we can confirm that the \( \dot{x} \)-terms in (3.4) are much smaller than other terms and hence that the approximation we used to obtain (3.15) is consistent.

§ 4. Quantum effects in RDU

Quantum effects due to vacuum polarization of matter fields at one loop level in a background gravitational field is investigated by many authors in GR.\(^7\) For PGT, it is shown by Buchbinder, Odintsov and Shapiro\(^{10}\) under the assumption of asymptotic freedom and multiplicative renormalizability that the theory is asymptotically conformally invariant and that matter fields become massless at high temperature. So there are no particle productions at high temperature.\(^4\) Further, they get the expression for \( \langle 0 \vert T_{\mu
u} \vert 0 \rangle \) at one loop approximation. It consists of two parts; the one is the same as GR and the other is “essentially” made of the axial-vector part of the torsion tensor \( a_i = (1/6) \varepsilon_{imn} T^{imn} \). However, in homogeneous and isotropic space \( a_i \) vanishes (see (2.2)), and hence quantum effects in PGT are the same as in GR. The explicit expressions for quantum effects are

\[
\rho_q = \frac{6\lambda \tilde{a}}{A^3} \left( A^2 \dot{A} A^{(3)} + AA^2 \ddot{A} - \frac{1}{2} A^2 \dot{A}^2 - \frac{3}{2} A^4 \right) + \frac{3\lambda \tilde{\beta}}{A^4} \cdot A^4,
\]

\[
\text{Tr}=\langle 0 \vert T_{\mu
u} \vert 0 \rangle = -\rho_q + 3 \rho_q
\]

\[
= \frac{6\lambda \tilde{a}}{A^3} \left( A^2 \dot{A} A^{(4)} + 3AA \ddot{A} A^{(3)} - 5A^2 \dot{A} + AA \ddot{A} \right) - \frac{12\lambda \tilde{\beta}}{A^4} \cdot A^2,
\]

where

\[
\lambda = \frac{1}{2880 \pi^2}, \quad \text{(natural unit)}
\]

\[
\tilde{a} = N_\phi + 6N_\phi + 12N_A,
\]

\[
\tilde{\beta} = N_\phi + 11N_\phi + 62N_A,
\]
and \( N_s, N_\phi \) and \( N_A \) are the numbers of scalar, spinor and vector field species, respectively. If temperature of the universe is not so high, then masses of matter fields cannot be ignored, and different treatments are needed. We shall discuss the low temperature limit in §5 as the case of MDU.

Due to these quantum effects, \( \rho_{cl} \) and \( P_{cl} \) are modified like

\[
\rho_{cl} \to \rho_{cl} + \rho_q, \quad \left( u_0 \to u_0 \left( 1 + \frac{\rho_q}{2 \rho_{cl}} \right) \right) \tag{4.6}
\]

\[
P_{cl} \to P_{cl} + \rho_q \tag{4.7}
\]

(see (3.2) and (3.3)). Accordingly, the scalar curvature \( F \) of (2.4) and its time-derivative \( \dot{F} \) are changed like

\[
F \to F + \Delta F = -\frac{9\beta}{b} \left( \frac{\dot{A}^2 + A \ddot{A}}{A^2} \right) - \frac{1}{2b} \cdot \text{Tr} = F \left( 1 + \frac{\Delta F}{F} \right), \tag{4.8}
\]

\[
\dot{F} \to \dot{F} + \Delta \dot{F} = -\frac{9\beta}{b} \left( \frac{\dot{A}^2 + A \ddot{A}}{A^2} \right) - \frac{1}{2b} \cdot \dot{\text{Tr}} = \dot{F} \left( 1 + \frac{\Delta \dot{F}}{\dot{F}} \right). \tag{4.9}
\]

The equation for \( A \) with quantum corrections are then obtained from (2.3) by making the above replacements of \( \rho_{cl}, \rho_{c_0}, F \) and \( \dot{F} \). Since \( \dot{\text{Tr}} \) involves \( A^{(5)} \), the resultant equation is a 5th-order differential equation. From this equation, we can get an equation for \( \epsilon \) of (3.7) with quantum correction. It has the form of (3.8) with the following replacements of \( \epsilon, \dot{\epsilon} \) and \( \ddot{\epsilon} \),

\[
\epsilon = A \dot{A} - u_0 \to \epsilon - u_0 \cdot \frac{\rho_q}{2 \rho_{cl}}, \tag{4.10}
\]

\[
\dot{\epsilon} = -\frac{b}{9\beta} \cdot A^2 \cdot F \to \dot{\epsilon} + \frac{b}{9\beta} \cdot A^2 \Delta F, \tag{4.11}
\]

\[
\ddot{\epsilon} \approx -\frac{b}{9\beta} \cdot A^2 \cdot \dot{F} \to \ddot{\epsilon} + \frac{b}{9\beta} \cdot A^2 \Delta \dot{F}, \tag{4.12}
\]

respectively. The second terms on the RHS of (4.10) \~ (4.12) represent the quantum corrections.

In the following, we shall show that if we choose the parameters \( f \) and \( \beta \) properly within the conditions (2.9) and (3.6), these quantum corrections are all absorbed into the second and third negligible terms on the RHS of (3.8); therefore, the classical stable solution of the universe (3.10) is maintained.

First, we can show that the effect of \( \rho_q \) is absorbed into the third term \( O(\epsilon^2) \) on the RHS of (3.8), if we demand

\[
\frac{\rho_q}{\rho_{cl}} \leq \frac{\eta_0^2}{u_0^2}. \tag{4.13}
\]

Secondly, the effect of \( \Delta F \) is the same order as the second term on the RHS of (3.8), if we demand
\[
\frac{\Delta F}{F} \lesssim 1. \quad (4.14)
\]

Lastly, the effect of $\Delta F$ is $O(\varepsilon^2)$; if we demand
\[
\frac{\Delta F}{F} \lesssim \frac{\eta_0}{u_0}. \quad (4.15)
\]

This is shown as follows: The term of $\varepsilon$ in (3.8) is modified by (4.9) as $\tilde{\varepsilon} \to \varepsilon(1 + \Delta F/F)$, and quantum effects $\varepsilon \Delta F/F$ is $O(\varepsilon^2)$ when (4.15) is satisfied.

Using the expressions for $\rho_q$ and $\text{Tr}$ given by (4.1) and (4.2) respectively, and employing the relations (3.17)~(3.20), we can show after tedious calculations that the three inequalities (4.13)~(4.15) are realized if the following conditions are satisfied:

\[
f \gtrsim \lambda \tilde{a} \cdot \frac{u_0}{\eta_0}, \quad (4.16)
\]

\[
- \beta \gtrsim a \quad (4.17)
\]

and

\[
x \tau \gg \frac{u_0}{\eta_0} \quad (3.16a)
\]

or equivalently

\[
t \gg 4\sqrt{2\pi} \cdot \sqrt{f} \cdot \frac{u_0}{\eta_0} \cdot t_p. \quad (3.16b')
\]

Here we have used

\[
\chi^2 \sim \frac{m_p^2}{32\pi f} \quad (4.18)
\]

which is obtained when (4.17) is satisfied.

As an example, let us briefly outline the arguments leading to (4.16). Taking the first term on the RHS of (4.1) as $\rho_q$, the inequality (4.13) becomes

\[
\frac{\rho_q}{\rho_{ct}} \lesssim \frac{6\lambda \tilde{a}}{A^4} \cdot \frac{A^2 A A^{(3)}}{D_r/A^4} \sim \lambda \tilde{a} \cdot \frac{\eta_0}{u_0} \cdot \frac{1}{f} \lesssim \frac{\eta_0^2}{u_0^2}, \quad (4.19)
\]

where we have used (3.1), (3.2) and (3.18). From this relation we get (4.16). Repeating similar analyses for each term in $\rho_q$, we get (3.16) and (4.16) as the sufficient conditions for (4.13).

Taking $\tilde{a} \sim 10^2$ and, for example, $\eta_0/u_0 \sim 10^{-3}$, the conditions (4.16) and (3.16) become

\[
f \gtrsim 10^2 \quad (4.20)
\]

and

\[
t \gg 10^7 \cdot t_p \quad (4.21)
\]
respectively: The latter condition (4.21) corresponds to
\[ T \ll 10^{16.5} \text{ GeV}, \tag{4.22} \]
where average energy of particles is much less than \( m_p \approx 10^{19} \text{ GeV}, \) therefore the one loop approximations for \( \rho_q \) and \( \text{Tr} \) are justified.

§ 5. The case of the matter dominant universe (MDU)

In MDU, temperature of the universe is sufficiently low in comparison with masses of leptons, quarks, Higgs, massive gauge bosons and so on. So, it is plausible to assume that the effects of these massive particles disappear from vacuum polarization because of the decoupling theorem.\textsuperscript{163} As for the effect of particle productions, however, we must pay due attentions (see § 6).

Therefore, the case of MDU can be treated by repeating almost the same analysis as for RDU. In the following, we only point out main differences between the two cases.

The standard Big-Bang solution of GR for MDU and \( k=0 \) is
\[ A(t) = \left( \frac{9}{4} v_0 \right)^{1/3} t^{2/3}, \tag{5.1} \]
where
\[ v_0 = \frac{D_m}{6a}, \tag{5.2} \]
and \( D_m \) is the constant which is related with mass density of MDU (= \( \rho_d \)) as
\[ \rho_d = \frac{D_m}{A^3}, \tag{5.3} \]
and its value is given by the observation\textsuperscript{149} as
\[ \frac{D_m}{m_p} \sim 10^{58}. \tag{5.4} \]

Corresponding to (5.1), we introduce a new function \( y \) instead of \( x \)
\[ y = A A^2 \tag{5.5} \]
which become \( v_0 \) when \( A \) is given by (5.1).

We can show that if we choose the parameters as \( f > 0 \) and \( \beta < 0 \), then the potential \( U(y) \) has a minimum at \( y = v_0 \); namely, the universe is stable. In the process of getting \( U(y) \), we have used the inequality
\[ \frac{f D_m}{m_p A^3} < 1 \tag{5.6} \]
which is satisfied in MDU. Next we get a small oscillating solution around the minimum point \( v_0 \).
\[ y = v_0 + \xi_0 \cos xt \quad (v_0 \gg \xi_0), \]  
(5.7)

from which we obtain

\[ A(t) = \left( \frac{9}{4} \right)^{1/3} \cdot t^{2/3} \left( 1 + \frac{1}{3} \cdot \frac{\xi_0}{v_0} \cdot \frac{1}{xt} \sin xt \right). \]  
(5.8)

In MDU, where inequality

\[ xt \gg \frac{v_0}{\xi_0} \]  
(5.9a)

is satisfied obviously (see similar inequality (3.16)), we have the following relations instead of (3.17) \sim (3.20):

\[ A \dot{A} \dot{A} \approx -\frac{x^2 \xi_0}{2} \cdot \sin xt = \frac{\xi}{2}, \]  
(5.10)

\[ A \dot{A} \dot{A}^{(3)} \approx -\frac{x^2 \xi_0}{2} \cdot \cos xt = -\frac{\xi}{2}, \]  
(5.11)

\[ A \dot{A} \dot{A}^{(4)} \approx \frac{x^3 \xi_0}{2} \cdot \sin xt \]  
(5.12)

and

\[ A \dot{A} \dot{A}^{(5)} \approx \frac{x^4 \xi_0}{2} \cdot \cos xt. \]  
(5.13)

Using (5.1), (5.4) and (5.10) \sim (5.13), we can show that the condition under which the quantum effects do not break the stability of the universe is almost the same as that in RDU; namely, (4.16), (4.17) and (5.9) instead of (3.16).

We have then \( F \), \( \dot{F} \) and torsion in MDU as

\[ F = \frac{9 \beta x^2 \xi_0}{bA^2} \cdot \sin xt, \]  
(5.14)

\[ \dot{F} \approx \frac{9 \beta x^2 \xi_0}{bA^2} \cdot \cos xt, \]  
(5.15)

\[ S(t) \approx \frac{\alpha}{\beta} \cdot \frac{\xi_0}{v_0} \cdot \frac{\cos xt}{t}, \]  
(5.16)

respectively. Here, we have used (4.17) to get (5.16).

\section*{§ 6. Summary and discussion}

First, we have shown at the classical level for the case of RDU and MDU with \( k = 0 \) that the Friedmann universe is stable under linear approximation if the parameters of PGT satisfy \( f > 0 \) and \( \beta < 0 \).

Secondly, we have shown for both cases of high temperature (RDU) and low temperature (MDU) that the quantum effects due to vacuum polarization at one loop
Table I. The order of the differential equation and the stability.

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</tr>
<tr>
<td>PGT</td>
<td>3rd-order stable</td>
<td>5th-order (effectively 3rd-order) stable at least in certain parameter domain</td>
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</table>

level do not break this classical stability of the universe if we choose the parameters of PGT to satisfy

\[ f \simeq \lambda \tilde{a} \cdot \frac{u_0}{\eta_0}, \]

\[ -\beta \geq \alpha, \]

and if we restrict the era of the universe as

\[ t \geq 4\sqrt{\frac{2\pi}{\sqrt{f}} \cdot \frac{u_0}{\eta_0} \cdot t_p} \quad \text{(for RDU)} \quad (3.16b) \]

or

\[ t \geq 4\sqrt{\frac{2\pi}{\sqrt{f}} \cdot \frac{u_0}{\xi_0} \cdot t_p} \quad \text{(for MDU)} \quad (5.9b) \]

Because the coupling between the torsion field and the fermion field is given by \( \sim 1/\sqrt{f} \) in PGT, the condition \( f \geq 10^5 \) (4.20) means that this coupling can be treated by perturbation method.

To justify the linear approximation (see (3.8)), it is plausible to assume, for example, \( \eta_0 / u_0 \ll 10^{-5} \). However, the smaller the value \( \eta_0 / u_0 \) becomes, the later the era in which the stability of the universe can be shown by our methods becomes (see (3.16b)). (See the estimations of (4.20) \( \sim (4.22) \).)

In Table I, we show the comparison between GR and PGT in the classical theory and in the theory that includes quantum effects of one loop approximation. Why the universe can be stable in PGT? To see this, let us compare the equation of GR which contains quantum effects (see (A·1) and (A·2) or (A·5)) with the equation of PGT (see (3·5)). In GR with quantum effects, due to the positive sign of \( \alpha \lambda \) of \( A^3 \)-term on the RHS of (A·1), the potential has a maximum and the universe becomes unstable. However, in PGT, we can choose the parameters \( f \) and \( \beta \) freely under the restriction of (2·9) and (2·10), and the plus or minus sign in the first term on the RHS of (3·5) so that the potential has a minimum. The universe remains stable.

The other cases \( k = \pm 1 \), the de Sitter universe and the universe at intermediate temperature, at which the effects of masses cannot be ignored, are now being investigated.

It is argued that, though charged particles in the trembling universe radiate photons whose energy is as high as \( \sim 10^{-2} \) \( \hbar \nu \), the oscillation of the universe has damped out by these radiations at early stages of the universe. More quantitative analysis on this point and on the effects of particle productions on the present universe is needed for our choice of the parameters.

If the oscillation survives at the present universe, the torsion is given by (5·13) and its value is about \( \leq 10^{-37} \) [eV]; So it seems difficult to detect it. If the oscillation has damped out, the value of torsion is estimated to be \( \sim 10^{-154} \) [eV].
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Appendix A

In this appendix, we shall show the instability of the Friedmann universe with quantum effects in GR\(^5\) for the case \(k=0\), RDU and serious difficulties of cosmology which arise from it. In the following, we owe Ruizmaikina and Ruizmaikin\(^5\) for mathematics.

In GR, the equation of the scale parameter \(A(t)\) for the case under consideration is

\[
A^2 \dot{A}^2 = u_0^2 + \frac{\ddot{a}}{2a} (2A^2 \dot{A}^2 + 3\dot{A}^2 A^3 - 3\dot{A}^2 - 2A^2 - 3\dot{A}^4) + \frac{\ddot{\beta}}{2a} A^4, \tag{A\cdot1}
\]

where \(u_0, a, \ddot{a}, \ddot{\beta}\) and \(\lambda\) are given by \((3\cdot2), (2\cdot8), (4\cdot4), (4\cdot5)\) and \((4\cdot3)\) respectively. With \(x = A \dot{A}\), \((A\cdot1)\) is rewritten as

\[
\dot{x} = \frac{a(x^2 - u_0)}{a\lambda x} + \left(\frac{\ddot{\beta}}{2\ddot{a}} - 1\right) \frac{x^3 + x\dot{x} + \dot{x}^2}{A^4}, \tag{A\cdot2}
\]

Further, using “particle position” \(z\) and “time” \(\xi\) defined as

\[
z = (A \dot{A})^{3/2} = x^{3/2}, \tag{A\cdot3}
\]

\[
\xi = 12^{-3/4} A^{3/2}, \tag{A\cdot4}
\]

respectively, \((A\cdot2)\) is rewritten in a form without velocity term as follows:

\[
\frac{d^2 z}{d\xi^2} = \frac{2a}{3\ddot{a} \lambda} \xi^{-2/3}(z^{-1/3} - u_0^2 \cdot z^{-5/3}) - \frac{\ddot{\beta}}{12 \ddot{a}} \xi^{-2} z. \tag{A\cdot5}
\]

This is the equation of a particle moving in a potential. When \(m_p t \gg 1\) and \(z = O(u_0^{3/2})\), we can show that the second term is much smaller than the first term in \((A\cdot5)\). Then, because the sign of \(\ddot{a}\lambda\) is positive (this has origin in the positive sign of \(A^{(3)}\)-term on the RHS of \((A\cdot1)\)), the potential has a maximum at \(z = u_0^{3/2}\). This corresponds to SBBS of GR which is denoted by \(A_0\). So the particle must roll down the potential sooner or later, and evolution of the universe becomes much different from SBBS.

When \(z \ll u_0^{3/2}\), we can show that \(\dot{A}(\xi) \ll \dot{A}_0(\xi)\), so that the Hubble constant becomes much smaller than the value of SBBS which is in agreement with the observation.\(^{14}\) Further, the helium mass fraction which is very sensitive to the expansion rate when the universe is a few minutes old\(^{15}\) becomes much smaller than the value of SBBS. When \(z \gg u_0^{3/2}\), we can get opposite results based on the relation \(\dot{A}(\xi) \gg \dot{A}_0(\xi)\). The other cosmological observables which are sensitive to \(\dot{A}\) such as
the fluctuations, etc. will be much different from the observations too. These are serious difficulties in cosmologies of GR with quantum effects.

**Appendix B**

In this appendix, we summarize the Lagrangian and the free parameters of PGT. The gravitational part of the Lagrangian in PGT consists of three parts,

\[
L_G = L_T + L_F + aF,
\]

where

\[
L_T = \alpha(t_{ij} \dot{u}^{ij}) + \beta(u_i \dot{v}^i) + \gamma(\ddot{a}_i \dot{a}^i),
\]

\[
L_F = a_1(A_{ijmn}A_{ijmn}) + a_2(B_{ijmn}B_{ijmn}) + a_3(C_{ijmn}C_{ijmn}) + a_4(E_{ij}E^{ij}) + a_5(I_{ij}I^{ij}) + a_6 F.
\]

Here, \(\alpha, \beta, \gamma\) and \(a_6 (q-1 \sim 6)\) are parameters of PGT which are restricted by the positive definite mass and energy conditions (see (2.9) and (2.10)), and \(t_{ij}, v_i\) and \(\ddot{a}_i\) are the irreducible components of the translational gauge field strength \(T_{ij}, \dot{u}^{ij}\) and \(A_{ijmn}, B_{ijmn}, C_{ijmn}, E_{ij}, I_{ij}\) and \(F\) are the irreducible components of the Lorentz gauge field strength \(F_{ijmn}\) and the suffixes are the arbitrary coordinate indexes.

When space-time is homogeneous and isotropic and the parity is conserved (see (2.1) and (2.2)), we can show after lengthy calculations that Lagrangian (B.1) reduce to

\[
L = \beta(-9 \dot{S}^2) + a_3(3(\ddot{A} - \ddot{B})^2) + a_6(36(\ddot{A} + \ddot{B})^2) + a(6(\ddot{A} + \ddot{B})),
\]

and the other components in (B.1) vanish, where

\[
\ddot{A} = A^{-1}(\dddot{A} - \dot{A} \dot{S} - A \dddot{S}),
\]

\[
\ddot{B} = A^{-2}((\dddot{A} - A \dddot{S})^2 - k).
\]

We can derive two independent equations by taking functional derivatives about \(A(t)\) and \(S(t)\) against the action made from the total Lagrangian \(L_G + L_M\), where \(L_M\) is the matter Lagrangian, and from these two equations together with (B.5) and (B.6) we get (2.3) and (2.4).^{12}

**References**

7) N. D. Birrell and P. C. W. Davies, Quantum Fields in Curved Space (Cambridge University Press, 1982).